QUANTUM SYMMETRIES AND K-THEORY

GREGORY MOORE
RUTGERS

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NOTES AVAILABLE AT

www.physics.rutgers.edu/~gmoore/

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PRELIMINARY REMARKS

THE AIM OF THESE LECTURES IS TO EXPLAIN SOME MATHEMATICAL IDEAS WHICH HAVE PROVEN TO BE USEFUL BOTH IN RECENT ADVANCES IN CONDENSED MATTER PHYSICS AS WELL AS IN STRING THEORY, AND IN "PHYSICAL MATHEMATICS" MORE GENERALLY.

THE IDEAS I WILL EXPLAIN ARE KEY IDEAS IN THE APPLICATION OF "TWISTED EQUIVARIANT K-THEORY" TO THE ABOVE SUBJECTS.

UNFORTUNATELY WE WON'T HAVE TIME TO GET TO TWISTED K-THEORY, BUT THE FOLLOWING MATERIAL IS IMPORTANT TO UNDERSTAND THAT TOPIC.

MY OWN INTEREST IN THE TOPIC ORIGINATED IN THE APPLICATION OF K-THEORY TO "RR FIELDS" & "D-BRANES" IN STRING THEORY. IN AN EXTENSIVE PROJECT WITH J. DISTLER & D. FREED WE HAVE INVESTIGATED HOW "TWISTED EQUIVARIANT K-THEORY" IS A VERY USEFUL TOOL IN THE THEORY OF ORBIFOLDS AND ORIENTIFOLDS.

THE FAMOUS FREED-HOPKINS-TELEMAN THEOREM ALSO SHOWED A DEEP CONNECTION WITH CONFORMAL FIELD THEORY.

HOWEVER, RECENTLY THERE HAS BEEN A VERY INTERESTING RELATED NEW DEVELOPMENT IN C.M.T.

THIS IS THE SUBJECT OF "TOPOLOGICAL PHASES OF MATTER."

A. KITAEV AND A. LUDWIG et. al. SHOWED THAT K-THEORY IS RELEVANT TO THE CLASSIFICATION OF PHASES OF MATTER WELL-DESCRIBED BY FREE FERMIONS.

RECALL THAT THE BASIC BAND THEORY OF METALS & INSULATORS IS A THEORY OF FREE FERMIONS, AND INDEED IN THAT CONTEXT THERE HAVE BEEN SPECTACULAR EXPERIMENTAL CONFIRMATIONS OF THE IDEAS OF FU, KANE, MELE, QI, ZHANG, AND MANY OTHERS.

D. FREED AND I GOT INTRIGUED BY ALL THIS AND — BEING INCLINED TO FORMAL MATHEMATICS — HAVE WRITTEN A PAPER PUTTING DOWN SOME MATHEMATICAL FOUNDATIONS FOR THE SUBJECT:

D. Freed & G. Moore,
"TWISTED EQUIVARIANT MATTER"

IT SHOULD APPEAR ON THE arXiv IN THE "NEAR FUTURE."

WHAT WE FOUND WAS THAT HERE TOO "TWISTED EQUIVARIANT K-THEORY" IS A VERY USEFUL TOOL FOR ORGANIZING TOPOLOG. PHASES OF MATTER.

WHILE THE FOLLOWING NOTES DO NOT QUITE GET TO K-THEORY THEY DO GIVE A PEDAGOGICAL INTRODUCTION TO THE FIRST HALF OF OUR PAPER, EXPLAINING SEVERAL CONCEPTS + CONSTRUCTIONS ESSENTIAL TO THE K-THEORY APPLICATION.

I HAVE TRIED TO MOTIVATE THINGS IN A SELF-CONTAINED WAY WITH PHYSICAL RESULTS:

WIGNER'S THEOREM

DYSON'S 3-FOLD WAY

ALTLAND-ZIRNBAUER CLASSIFICATION.

PLAN

PART I:

WIGNER'S THEOREM

GROUP EXTENSIONS

DYSON'S 3-FOLD WAY

CLIFFORD ALGEBRAS

FREE FERMIONS

10-FOLD WAYS & SYMMETRIC SPACES

PART II:

- GROTHENDIECK GROUPS, REP. RINGS AND TWISTED K-THEORY
- GROUPOIDS AND TWISTINGS
- BAND STRUCTURE + CRYST. GROUPS
- CANONICAL TWISTING
- TWISTED EQUIVARIANT K-THEORY AND BAND STRUCTURE

DETAILED OUTLINE FOR PART I

1. WIGNER'S THEOREM & QUANTUM SYMMETRY

2. A CRASH COURSE ON GROUP EXTENSIONS

3. RETURN TO WIGNER'S THEOREM

4. SYMMETRIES OF QUANTUM SYSTEMS AND THE PULLBACK CONSTRUCTION

5. CO-REPRESENTATIONS

6. REAL, COMPLEX, & QUATERNIONIC VECTOR SPACES

7. DYSON'S 3-FOLD WAY

a.) DIVISION ALGEBRAS
b.) SCHUR'S LEMMA FOR IRRED. COREPS.
c.) DYSON'S PROBLEM: THE ENSEMBLE OF HAMILTONIANS COMPATIBLE WITH A FIXED SYMMETRY TYPE.

8. SYMMETRIES & TIME REVERSAL

9. GAPPED HAMILTONIANS & TOPOLOGICAL PHASES OF MATTER

10. SUPER-LINEAR ALGEBRA

11. THE 10 CT-GROUPS

12. CLIFFORD ALGEBRAS AND THE 10 SUPER-DIVISION ALGEBRAS

13. $\mathbb{Z}_2$-GRADED COREPS OF THE CT-GROUPS

14. FINITE-DIMENSIONAL FREE FERMIONS WITH SYMMETRY

15. THE FREE FERMION DYSON PROBLEM & THE ALTLAND-ZIRNBAUER CLASSIFICATION

16. BOTT PERIODICITY & CARTAN SYMMETRIC SPACES

17. EXAMPLE: SYMMETRIES OF A SINGLE FERMIONIC OSCILLATOR.

1. WIGNER'S THEOREM

A. RAYS IN HILBERT SPACE

THE DIRAC-VON NEUMANN AXIOMS OF QUANTUM MECHANICS POSIT THAT TO A PHYSICAL SYSTEM WE ASSOCIATE A HILBERT SPACE $\mathcal{H}$ (AND A "$*$-ALGEBRA" OF OBSERVABLES).

(PURE) QUANTUM STATES ARE ASSOCIATED NOT WITH VECTORS

$$\psi \in \mathcal{H}$$

BUT RATHER WITH COMPLEX LINES IN $\mathcal{H}$:

ψ AND $z \cdot \psi$, $z \in \mathbb{C}^*$

REPRESENT THE SAME QUANTUM STATE. EVEN IF WE NORMALIZE TO

$$\|\psi\| = \sqrt{\langle \psi | \psi \rangle} = 1$$

THERE IS STILL AN IDENTIFICATION

$$\psi \sim e^{i\theta} \psi$$

WE DENOTE THE COMPLEX LINE THROUGH A VECTOR ψ BY $[\psi]$: IT IS THE EQUIVALENCE CLASS $\psi \sim z \cdot \psi$, $z \in \mathbb{C}^*$.

THE SPACE OF SUCH LINES IS PROJECTIVE HILBERT SPACE

$$\mathbb{P}\mathcal{H} := (\mathcal{H} - \{0\}) / \mathbb{C}^*$$

1B. OVERLAPS

PHYSICAL MEASUREMENTS ARE DESCRIBED BY "OVERLAPS" OR "TRANSITION PROBABILITIES":

$$P_{12} = \frac{|\langle \psi_1 | \psi_2 \rangle|^2}{\langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle}$$

NOTE THIS IS WELL-DEFINED FOR $\psi_1, \psi_2 \neq 0$ AND MOREOVER DOES NOT DEPEND ON THE NORMALIZATION SO IT IS A FUNCTION

$$P([\psi_1],[\psi_2])$$

THAT IS, IT DEFINES A UNIVERSAL FUNCTION

$$\gamma : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \longrightarrow [0,1]$$

IN FACT, IF $\mathcal{H}$ IS FINITE DIMENSIONAL, SO $\mathcal{H} \cong \mathbb{C}^N$ THEN

$$\mathbb{P}\mathcal{H} = \mathbb{C}P^{N-1}$$

AND THERE IS A WELL-KNOWN FUBINI-STUDY METRIC ON $\mathbb{C}P^{N-1}$

EXAMPLE: IF $N=2$ THEN

$$\mathbb{P}\mathbb{C}^2 = \mathbb{C}P^1 \cong S^2$$

HAS THE USUAL ROUND METRIC

IN GENERAL p IS RELATED TO THE FUBINI-STUDY DISTANCE BY:

$$\boxed{p(1,2) = \left(\cos \frac{d_{12}}{2}\right)^2}$$

PROOF: ANY TWO LINEARLY INDEPENDENT VECTORS ψ_1, ψ_2 DEFINE A PLANE $\mathbb{C}^2 \subset \mathcal{H}$.

THE FS METRIC ALLOWS US TO REDUCE TO THIS $\mathbb{C}P^1$.

WLOG WE MAY TAKE

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_2 = \begin{pmatrix} e^{i(\psi+\phi)/2} \cos \frac{\theta}{2} \\ e^{i(\psi-\phi)/2} \sin \frac{\theta}{2} \end{pmatrix}$$

θ = GEODESIC DISTANCE ON S^2 ■

1C. QUANTUM SYMMETRY

DEF: A QUANTUM SYMMETRY IS A MAP OF PHYSICAL STATES

$$\boxed{s : \mathbb{P}\mathcal{H} \longrightarrow \mathbb{P}\mathcal{H}}$$

WHICH PRESERVES OVERLAPS:

$$\boxed{p\left(s[\psi_1], s[\psi_2]\right) = p\left([\psi_1], [\psi_2]\right)}$$

EXAMPLE: IF $\mathcal{H} = \mathbb{C}^2$ THEN

A QUANTUM SYMMETRY IS JUST AN ISOMETRY OF S^2 WITH THE STANDARD ROUND METRIC.

QUANTUM SYMMETRIES FORM A GROUP. WE DENOTE IT BY

$$\mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$$

FOR EXAMPLE, IF $\mathcal{H} \cong \mathbb{C}^2$ THEN IT IS THE ISOMETRY GROUP OF S^2. THIS GROUP IS JUST $O(3)$.

Wigner asks: How can we make such transformations?

If $u \in U(\mathcal{H})$ is a unitary transformation then $\psi \mapsto u\psi$ defines a map of lines

$$u : [\psi] \longmapsto [u \cdot \psi]$$

and clearly this preserves p.

BUT! THERE ARE ALSO ANTI-UNITARY TRANSFORMATIONS!

RECALL THAT $a : \mathcal{H} \to \mathcal{H}$ IS <u>ANTI-LINEAR</u> IF

$$a(\psi_1 + \psi_2) = a(\psi_1) + a(\psi_2)$$

<u>BUT</u>: $a(z\psi) = z^* a(\psi)$

$$\forall \ \psi \in \mathcal{H}, \quad z \in \mathbb{C}$$

IT IS CALLED <u>ANTI-UNITARY</u> IF IN ADDITION

$$\|a(\psi)\|^2 = \|\psi\|^2$$

Such an anti-unitary u also takes lines to lines:

$$a : [\psi] \longmapsto [a(\psi)]$$

and this map preserves p.

Are there other maps

$s: \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$

preserving probabilities ?

NO!

That is the content of Wigner's theorem: Every $s \in \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$ is induced, as above, by a unitary or antiunitary map $u: \mathcal{H} \to \mathcal{H}$ or $a: \mathcal{H} \longrightarrow \mathcal{H}$, as above.

PROOF: SEE WEINBERG, QFT VOL. I, sec. 1.

ALTERNATIVE: FREED, arXiv: 1211.2133

We are now going to make a mathematical excursion so that we can restate Wigner's theorem in terms of <u>group extensions</u>.

2. A CRASH COURSE ON GROUP EXTENSIONS

- A MAP BETWEEN GROUPS, $\varphi: G_1 \to G_2$ IS A <u>HOMOMORPHISM</u> IF

$$\varphi(g_1 g_1') = \varphi(g_1)\varphi(g_1') \quad \forall g_1, g_1'$$

$$\ker \varphi = \{ g_1 \in G_1 \mid \varphi(g_1) = 1 \}$$

$$\operatorname{im} \varphi = \{ g_2 \in G_2 \mid \exists g_1 \ \varphi(g_1) = g_2 \}$$

- A <u>GROUP EXTENSION</u> IS AN "EXACT SEQUENCE" OF HOMOMORPHISMS

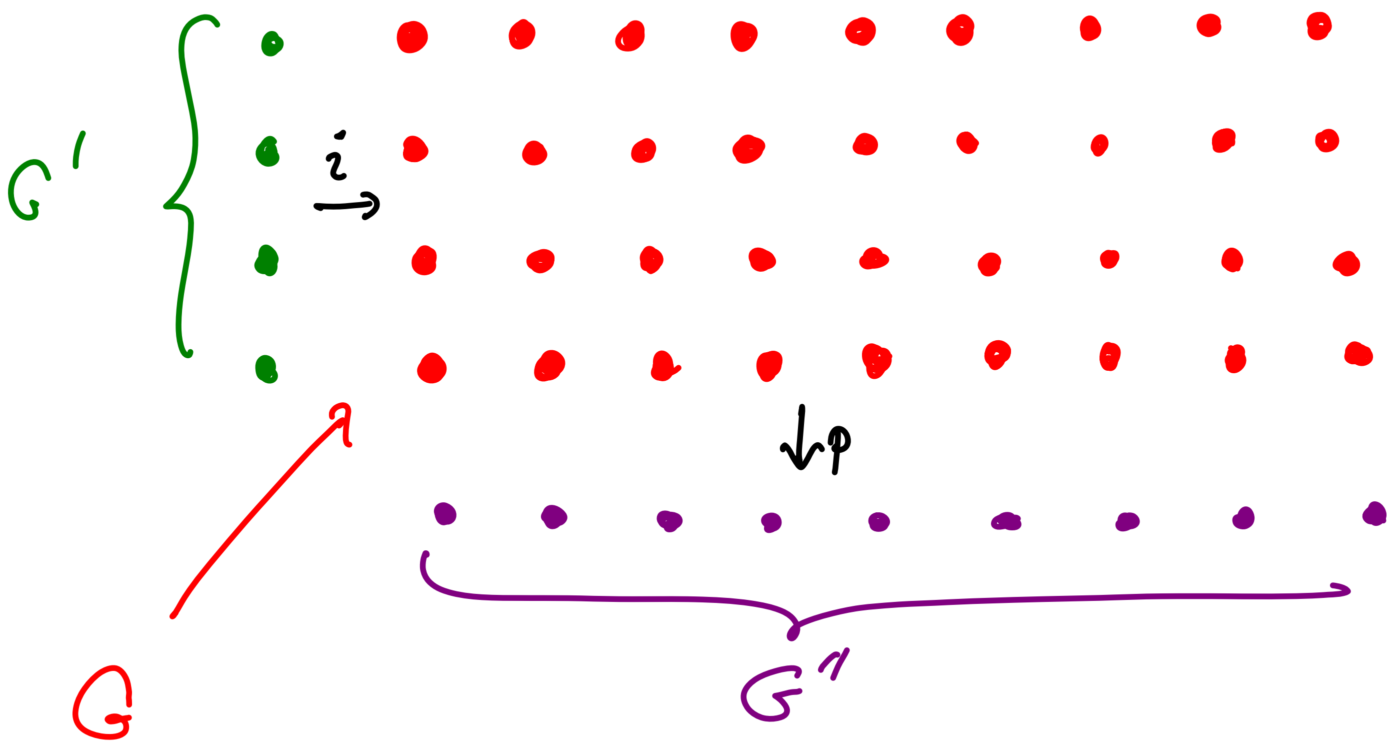
$$1 \to G' \xrightarrow{i} G \xrightarrow{p} G'' \longrightarrow 1$$

MEANING THAT

- p is <u>onto</u>: $p(G) = G''$
- i is <u>into</u> $\ker(i) = 1$
- $\operatorname{im}(i) = \ker(p)$

Remarks:

1. Later on it will be useful to think geometrically. What we have is an example of a principal G' bundle over G''



Even when G', G'' are finite groups the bundle might not be equivariantly trivial, i.e. the group structure might not be the simple direct product $G' \times G''$

② When G' is abelian and

$$i : G' \hookrightarrow Z(G)$$

maps G' into the center of G

we have a central extension.

③ We can define a notion of isomorphic group extensions and then the isom. classes of G' central extensions of G'' are classified by group cohomology:

$$H^2(G''; G')$$

EXAMPLE 1: THEORY OF SPIN

THERE IS A STANDARD HOMOMORPHISM

$$p: SU(2) \longrightarrow SO(3)$$
$$u \longmapsto R$$

$$u \, \vec{x} \cdot \vec{\sigma} \, u^{-1} = R\vec{x} \cdot \vec{\sigma}$$

ONE CAN SHOW:

(α.) Every R comes from some u

(β.) $\ker(p) = \{\pm 1\}$

So:

$$1 \to \mathbb{Z}_2 \xrightarrow{i} SU(2) \xrightarrow{p} SO(3) \to 1$$

Moreover:

(γ.) There is <u>NO</u> CONTINUOUS HOMOM

$s: SO(3) \longrightarrow SU(2)$ inverting p:

$$ps = \text{Identity}$$

EXAMPLE 2: EXTENSIONS OF $\mathbb{Z}_2$ BY $\mathbb{Z}_2$

? QUESTION: WHICH G FIT INTO

$$1 \to \mathbb{Z}_2 \xrightarrow{i} G \xrightarrow{p} \mathbb{Z}_2 \to 1 \quad ?$$

2a.) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

$= \langle \sigma_1, \sigma_2 \rangle$

$$\sigma_1^2 = 1, \quad \sigma_2^2 = 1 \quad \sigma_1 \sigma_2 = \sigma_2 \sigma_1$$

TAKE $\quad i(\sigma_1) = \sigma_1 \quad p(\sigma_1) = 1$

$p(\sigma_2) = \sigma_2$

NOTE: THERE IS A HOMOMORPHISM

$s : \mathbb{Z}_2 = \langle \sigma_2 \rangle \longrightarrow G$ SO THAT

$$p \circ s = \text{Identity}$$

2b.) NOW CONSIDER $G \cong \mathbb{Z}/4\mathbb{Z}$.
LET $\omega = i = e^{i\pi/2}$ AND TAKE

$$G = \langle \omega \rangle = \{1, \omega, \omega^2, \omega^3\} = \{\pm 1, \pm i\}$$

DEFINE $p : G \longrightarrow \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$
BY ITS VALUE ON A GENERATOR

$$p(\omega) = \omega^2 = -1$$

$\ker p = \{1, \omega^2\} \cong \mathbb{Z}_2$, SO:

$$1 \to \mathbb{Z}_2 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

N.B.

1. $G \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$
2. THERE IS NO $s : \mathbb{Z}_2 \to G$ WITH $p \circ s = \mathrm{Id}$. PROOF: $s(\sigma) = \omega^j$ SO $p \circ s(\sigma) = \sigma \Rightarrow \omega^{2j} = -1 \Longrightarrow$

$$1 = s(1) = s(\sigma^2) = s(\sigma)s(\sigma) = \omega^{2j} = -1$$

$\rightarrow\leftarrow$

Def: WE SAY AN EXTENSION

$$1 \longrightarrow G' \xrightarrow{i} G \xrightarrow{p} G'' \longrightarrow 1$$

SPLITS IF $\exists$ A HOMOMORPHISM OF GROUPS $s : G'' \longrightarrow G$ SUCH THAT $p \circ s = Id$.

WE USUALLY WRITE

$$1 \longrightarrow G' \xrightarrow{i} G \underset{\overset{\dashleftarrow}{s}}{\xrightarrow{p}} G'' \longrightarrow 1$$

Remarks:

1. Geometrically, s is a cross-section of the principal G'-bundle which is a homomorphism of groups.

2. When the extension splits G is isomorphic to the <u>semidirect product</u> $G' \ltimes G''$ because if $P(g) = g''$ then we have

$$g = g' \cdot s(g'')$$

for some $g' \in G'$ and the multiplication law is:

$$g_1' s(g_1'')\, g_2' s(g_2'') = \left(g_1' s(g_1'') g_2' s(g_1'')^{-1}\right) \cdot S(g_1'' g_2'')$$
$$= \left(g_1' \cdot \alpha_{g_1''}(g_2')\right) \cdot s(g_1'' g_2'')$$

$$\alpha_{g_1''} : g_2' \longmapsto s(g_1'') g_2' s(g_1'')^{-1}$$
$$G' \longmapsto Aut(G'')$$
$$= \left(g_1' \alpha_{g_1''}(g_2')\right) \cdot s(g_1'' g_2'')$$

EXAMPLE 3: THE ISOMETRY GROUP OF AFFINE EUCLIDEAN SPACE $\mathbb{E}^d$ IS THE GROUP OF DISTANCE-PRESERV. TRANSFORMATIONS

$$\| f(p_1) - f(p_2) \| = \| p_1 - p_2 \| \quad \forall p_1, p_2$$

IT IS DENOTED $Euc(d)$

ONE CAN SHOW THAT

$$1 \to \mathbb{R}^d \longrightarrow Euc(d) \longrightarrow O(d) \longrightarrow 1$$

AND, IF WE CHOOSE AN ORIGIN FOR $\mathbb{E}^d$ THEN WE CAN WRITE

$$\{R \mid v\} \cdot x = Rx + v$$

FROM THIS WE CAN COMPUTE

$$\{R_1 \mid v_1\} \cdot \{R_2 \mid v_2\} = \{R_1 R_2 \mid v_1 + R_1 v_2\}$$

NOTE: $p : \{R|v\} \longmapsto \{R|0\}$ IS A SURJECTIVE HOMOMORPHISM.

A SPLITTING IS GIVEN BY

$$s : O(d) \longrightarrow \mathrm{Euc}(d)$$
$$R \longmapsto \{R|0\}$$

AND INDEED

$$\mathrm{Euc}(d) \cong \mathbb{R}^d \rtimes O(d)$$

(BUT THE ISOMORPHISM <u>DEPENDS</u> ON A CHOICE OF ORIGIN.)

CRYSTALLOGRAPHY

DEF: (a.) A CRYSTAL IS A SUBSET $C \subset \mathbb{E}^d$ INVARIANT UNDER TRANSLATIONS BY A FULL LATTICE $\Pi \subset \mathbb{R}^d \subset Euc(d)$

(b.) THE SPACE GROUP $G(C)$ OF C IS THE SUBGROUP OF $Euc(d)$ TAKING $C \to C$.

(c.) THE POINT GROUP $P(C)$ IS THE PROJECTION OF $G(C)$ TO $O(d)$

(d.) THE GROUP EXTENSION

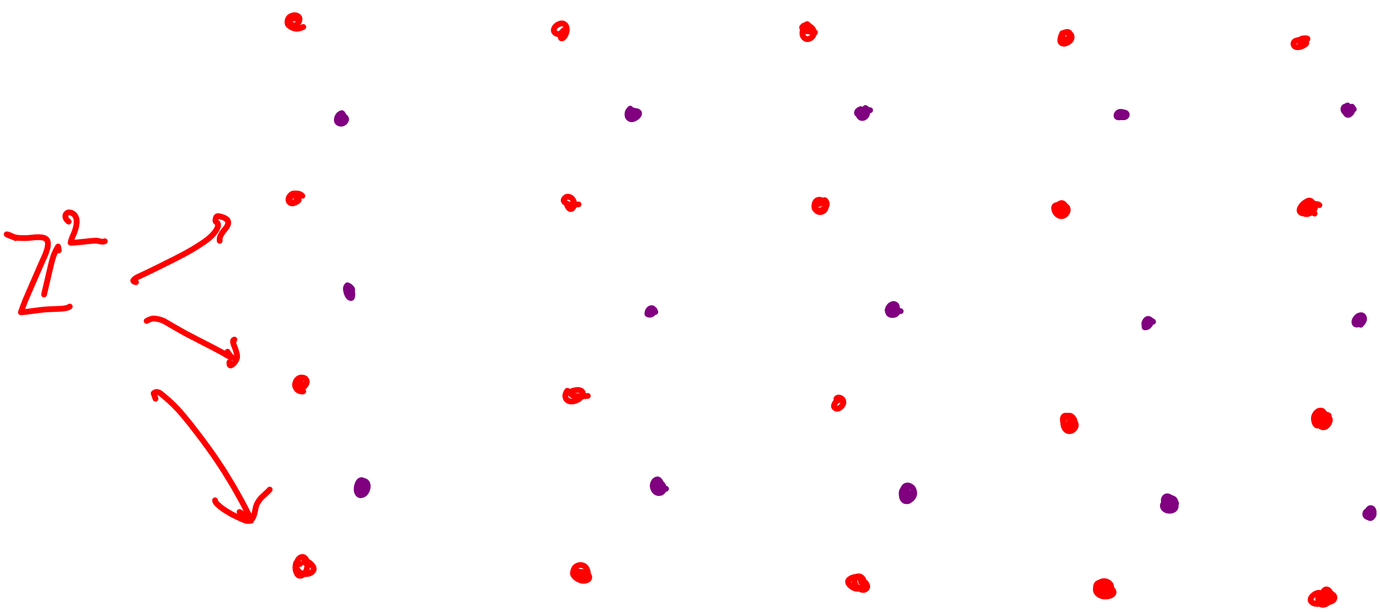
$$1 \to \Pi \to G(C) \to P(C) \to 1$$

NEED NOT SPLIT. WHEN IT DOES THE CRYSTAL IS SAID TO BE "SYMMORPHIC".

Example 4

$$C = \mathbb{Z}^2 \cup \mathbb{Z}^2 + \vec{\delta}$$

$$\vec{\delta} = (\delta, 1/2), \quad 0 < \delta < 1/2$$



$$1 \to \mathbb{Z}^2 \to G(C) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \sigma_1, \sigma_2 \rangle$$

$$\hat{\sigma}_1 : (x_1, x_2) \longmapsto (-x_1 + \delta, x_2 + \tfrac{1}{2})$$

$$\hat{\sigma}_2 : (x_1, x_2) \longmapsto (x_1, -x_2)$$

Nonsplit = "nonsymmorphic"

3. RETURN TO WIGNER'S THEOREM

3A. A NEW GROUP

WE HAVE ALREADY DEFINED THE GROUP $\mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$ OF QUANTUM SYMMETRIES.

WE NOW INTRODUCE A NEW GROUP, $\mathrm{Aut}_{qtm}(\mathcal{H})$.

THIS IS DEFINED TO BE THE GROUP OF ALL UNITARY AND ANTIUNITRY TRANSFORMATIONS

$$u, a : \mathcal{H} \to \mathcal{H}$$

NOTE THAT IT IS A GROUP BECAUSE

$$u \, u' = u''$$
$$u \, a = a'$$
$$a \, u = a''$$
$$a \, a' = u$$

THIS ALSO SHOWS THERE IS A HOMOMORPHISM:

$$\phi : \mathrm{Aut}_{qtm}(\mathcal{H}) \longrightarrow \{\pm 1\} \cong \mathbb{Z}_2$$

$$\phi(S) := \begin{cases} +1 & S \text{ is unitary} \\ -1 & S \text{ is antiunitary} \end{cases}$$

EXERCISE: SHOW THAT THE SEQUENCE

$$1 \to U(\mathcal{H}) \longrightarrow \mathrm{Aut}_{qtm}(\mathcal{H}) \xrightarrow{\phi} \{\pm 1\} \to 1$$

SPLITS.

3B. WIGNER'S THEOREM

WE SAW ABOVE THAT BOTH KINDS OF TRANSFORMATIONS $u, a \in \mathrm{Aut}_{qtm}(\mathcal{H})$ TAKE LINES TO LINES AND DEFINE QUANTUM SYMMETRIES; e.g.,

$$p(a): [\psi] \longrightarrow [a(\psi)]$$

$$p: \mathrm{Aut}_{qtm}(\mathcal{H}) \longrightarrow \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$$

IS A HOMOMORPHISM.

WIGNER'S THEOREM IS THE STATEMENT THAT p IS ONTO.

$$\Longrightarrow$$

$$1 \to \mathbb{T} \to \mathrm{Aut}_{qtm}(\mathcal{H}) \xrightarrow{p} \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) \to 1$$

WHAT IS THE KERNEL OF p?

IF $\lambda \in U(1)$ IS A PHASE, $|\lambda|=1$, THEN $\psi \longmapsto \lambda \cdot \psi$ IS CERTAINLY A UNITARY TMN.

CLEARLY, $[\lambda\psi] = [\psi]$,

SO $U(1) := \mathbb{T} \subset \ker(p)$

BUT ANYTHING IN $\ker(p)$ MUST BE DIAGONAL IN EVERY BASIS, SO $\ker(p) = \mathbb{T}$.

IN WHAT FOLLOWS, $\lambda \in \mathbb{C}$ s.t. $|\lambda| = 1$ WILL USUALLY DENOTE THE TMN

$$\psi \longrightarrow \lambda \cdot \psi$$

REMARKS

(1) FOR $S \in \mathrm{Aut}_{qtm}(\mathcal{H})$

$$S \cdot \lambda = \lambda^{\phi(S)} \cdot S$$

$$= \begin{cases} \lambda S & \phi(S) = +1 \\ \bar{\lambda} S & \phi(S) = -1 \end{cases}$$

THEREFORE $\mathbb{T} \subset \mathrm{Aut}_{qtm}(\mathcal{H})$

IS <u>NOT</u> <u>CENTRAL</u>.

THE RESTRICTION TO $\ker(\phi)$:

$$1 \longrightarrow \mathbb{T} \longrightarrow U(N) \longrightarrow PU(N) = \frac{SU(N)}{\mathbb{Z}_N} \longrightarrow 1$$

IS A CENTRAL EXTENSION.

② THE CENTRAL EXTENSION

$$1 \to \mathbb{T} \to U(N) \to SU(N)/\mathbb{Z}_N \to 1$$

IS <u>NOT</u> SPLIT!

THIS IS THE SOURCE OF INTERESTING EXTENSIONS AND ANOMALIES IN QUANTUM MECHANICS.

EXAMPLE: AGAIN TAKE $\mathcal{H} = \mathbb{C}^2$.

THEN $\mathbb{P}\mathcal{H} = \mathbb{C}P^1 = S^2$

$$\mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) = O(3) = SO(3) \amalg P \cdot SO(3)$$

P IS ANY REFLECTION

$$\mathrm{Aut}_{qtm}(\mathcal{H}) = U(2) \amalg K \cdot U(2)$$

K IS ANY ELEMENT SO THAT

$Ku = u^* K$ (DOES NOT HAVE A 2×2 MATRIX REPRESENTATION.)

ALREADY

$$1 \to \mathbb{T} \to U(2) \longrightarrow PU(2) \longrightarrow 1$$

DOES NOT SPLIT.

TO PROVE THAT NOTE

$$PU(2) = U(2)/U(1) \cong SU(2)/\mathbb{Z}_2 \cong SO(3)$$

BUT A CONTINUOUS CROSS-SECTION $\quad s: SO(3) \longrightarrow U(2)$

WOULD INDUCE

$$s_*: \pi_1(SO(3)) \longrightarrow \pi_1(U(2))$$
$$\mathbb{Z}_2 \longrightarrow \mathbb{Z}$$

ONLY SUCH HOMOM. IS ZERO

BUT $p \circ s = \mathrm{Id} \implies p_* s_* = 1$

$$\Rightarrow\Leftarrow$$

(A SPLITTING OF $U(N) \to PU(N)$ FOR $N > 2$ WOULD RESTRICT TO ONE FOR $N=2$. SO $\nexists$ SPLITTING FOR $N>2$.)

4. SYMMETRIES OF QUANTUM SYSTEMS AND THE PULLBACK CONSTRUCTION

SUPPOSE A GROUP G IS A SYMMETRY OF SOME PHYSICAL SYSTEM

MATHEMATICALLY, WE POSIT A HOMOMORPHISM

$$\rho : G \longrightarrow \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$$

AND SO WE HAVE:

$$\boxed{\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \downarrow \rho & & \\ 1 & \to & \mathbb{T} & \to & \mathrm{Aut}_{qtm}(\mathcal{H}) & \longrightarrow & \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) & \to & 1 \end{array}}$$

For each $g \in G$ there is an entire circle $\mathbb{T}$ of possible lifts of $\rho(g)$. These are possible phases when realizing $\rho(g)$ as a linear/antilinear operator on $\mathcal{H}$.

Quite generally in Q.M. if we have a naive group of symmetries, what actually acts on the Hilbert space is an <u>extension of G</u>.

1.) Theory of spin.
2.) Anomalies in QFT

What extension do we get?

We need some more group theory

MATH CONSTRUCTION: "PULLBACK OF A GROUP EXT."

WE ARE GIVEN:

$$\begin{tikzcd} & & & G'' \arrow[d, "\rho"] & \\ 1 \arrow[r] & H' \arrow[r, "i"] & H \arrow[r, "p"] & H'' \arrow[r] & 1 \end{tikzcd}$$

DEFINE A SUBGROUP:

$$G \subset H \times G''$$

$$G := \{ (h, g'') \mid p(h) = \rho(g'') \}$$

CLAIM: G FITS INTO AN EXTENSION (THE "PULLED BACK EXT.")

$$1 \to H' \xrightarrow{i} G \xrightarrow{p_2} G'' \to 1$$

$$p_2(h, g'') = g''$$

Indeed we have a commutative diagram:

$$
\begin{tikzcd}
1 \arrow[r] & H' \arrow[r, "i"] \arrow[d, equal] & G \arrow[r, "p_2"] \arrow[d, "p_1"] & G'' \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & H' \arrow[r, "i"] & H \arrow[r, "p"] & H'' \arrow[r] & 1
\end{tikzcd}
$$

WHAT WE LEARN IS THAT FOR ANY GROUP OF QUANTUM SYMMETRIES $\rho : G \to \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H})$ WHAT ACTS ON THE QUANTUM HILBERT SPACE IS THE PULLED BACK EXTENSION:

$$
\boxed{
\begin{tikzcd}
1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[d, equal] & G^{\tau} \arrow[r] \arrow[d, "\rho^{\tau}"] & G \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & \mathbb{T} \arrow[r] & \mathrm{Aut}_{qtm}(\mathcal{H}) \arrow[r, "p"'] & \mathrm{Aut}_{qtm}(\mathbb{P}\mathcal{H}) \arrow[r] & 1
\end{tikzcd}
}
$$

LET US ABSTRACT THIS STRUCTURE:

DEF: GIVEN A $\mathbb{Z}_2$-GRADED GROUP (G, ϕ), i.e. $\phi : G \to \{\pm 1\}$, WE DEFINE A

ϕ-TWISTED EXTENSION OF G

TO BE AN EXTENSION OF THE FORM:

$$1 \to \mathbb{T} \xrightarrow{i} G^\tau \xrightarrow{p} G \to 1$$

WHERE G^τ IS A GROUP SUCH THAT

$$g\lambda = \lambda^{\phi(g)} \cdot g \quad \forall \begin{array}{l} \lambda \in \mathbb{T} \\ g \in G^\tau \end{array}$$

REMARKS:

1. SOMETIMES WE JUST WRITE τ FOR THE ENTIRE ϕ-TWISTED EXTENSION.

2. FOR $\phi = 1$ GET A <u>CENTRAL</u> <u>EXT</u>.

3. ϕ-TWISTED EXT'S AGAIN CLASSIFIED BY GROUP-COHOMOLOGY (TWISTED $H^{2+w}(G, G')$)

4. FOR GIVEN (G, ϕ) THERE CAN BE SEVERAL DIFFERENT EXTENSIONS

<u>EXAMPLE:</u> TAKE $G = \mathbb{Z}_2$

AND $\phi : \{\pm 1\} \longrightarrow \{\pm 1\}$ THE IDENTITY.

IT WILL BE CONVENIENT TO DENOTE $M_2 = \{1, \bar{T}\}$, $\bar{T}^2 = 1$.

OF COURSE $M_2 \cong \mathbb{Z}_2$.

TAKE $\phi(\bar{T}) = -1$.

THERE ARE TWO INEQUIVALENT ϕ-TWISTED EXTENSIONS:

$$1 \to \mathbb{T} \longrightarrow M_2^\tau \xrightarrow{p} M_2 \longrightarrow 1$$

CHOOSE A LIFT, T, OF $\bar{T}$.

THEN $p(T^2) = 1$ SO

$T^2 = \lambda \in \mathbb{T}$ BUT, THEN

$T\lambda = T \cdot T^2 = T^2 \cdot T = \lambda T$

ON THE OTHER HAND, $\phi(\bar{T}) = -1$

SO

$$T\lambda = \lambda^{-1} T$$

$$\implies \lambda^2 = 1 \implies \lambda = \pm 1$$

$$\implies T^2 = \pm 1$$

THUS THE TWO GROUPS ARE

$$M_2^{\pm} = \{\lambda T \mid \lambda T = T\lambda^{-1} \text{ \& } T^2 = \pm 1\}$$

THESE POSSIBILITIES ARE REALLY DISTINCT: IF T' IS ANOTHER LIFT OF $\bar{T}$ THEN $T' = \mu T$ FOR SOME $\mu \in \mathbb{T}$ AND SO

$$(T')^2 = (\mu T)^2 = \mu \bar{\mu} T^2 = T^2 \quad \blacksquare$$

REMARK: JUST AS THERE IS A $\mathbb{Z}_2$ COVER $\mathrm{Spin}(N) \longrightarrow SO(N)$ THERE ARE TWO $\mathbb{Z}_2$ COVERS

$$\mathrm{Pin}^{\pm}(N) \longrightarrow O(N)$$

(DEPENDING ON WHETHER THE LIFT OF A REFLECTION SQUARES TO ± 1)

HERE $M_2^{\pm} = \mathrm{Pin}^{\pm}(2)$.

PIN GROUPS WILL REAPPEAR (AND BE DEFINED) LATER IN SECTION 16.

5. "COREPRESENTATIONS"

DEF: a) A <u>COREPRESENTATION</u> of a $\mathbb{Z}_2$-graded group (G,ϕ) is a complex Hilbert space $\mathcal{H}$ and a homomorphism:

$$\rho : G \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathcal{H})$$

so that

$$\rho(g) = \begin{cases} \text{linear op.} & \phi(g) = +1 \\ \text{anti-linear op.} & \phi(g) = -1 \end{cases}$$

b.) IT IS "UNITARY" IF $\rho : G \to \mathrm{Aut}_{qtm}(\mathcal{H})$

c.) A (ϕ,τ)-TWISTED REPRESENTATION OF G IS A COREPRESENTATION OF (G^{τ},ϕ).

MANY OF THE STANDARD NOTIONS OF REP$^{\underline{N}}$ THEORY GENERALIZE IN A STRAIGHTFORWARD WAY:

<u>DEF</u>: A COREP. $(G, \phi, \rho, \mathcal{H})$ IS <u>IRREDUCIBLE</u> IF THERE IS NO NONTRIVIAL SUB-COREP.

i.e. NO SUBSPACE, $0 \subsetneq \mathcal{H}_1 \subsetneq \mathcal{H}$, INVARIANT UNDER $\rho(g)$, $\forall g \in G$.

<u>THEOREM</u> (COMPLETE REDUCIBILITY)

IF G IS <u>COMPACT</u> THEN ANY FINITE-DIMENSIONAL COREP $\mathcal{H}$ IS EQUIVALENT TO:

$$\mathcal{H} \cong \bigoplus_{\lambda \in (G,\phi)^\vee} N_\lambda \otimes_{\mathbb{R}} V_\lambda$$

(CALLED AN "ISOTYPICAL DECOMPOSITION"[64])

NOTATION:

1. $(G,\phi)^\vee$ = SET OF NON-ISOMORPHIC IRRED. COREPS.

2. N_λ IS A REAL VECTOR SPACE OF MULTIPLICITIES, e.g.

$$V_\lambda \oplus V_\lambda \oplus V_\lambda \cong \mathbb{R}^3 \otimes V_\lambda$$

3. $\otimes$ IS TAKEN FOR <u>REAL</u> VECTOR SPACES (SEE BELOW)

4. PUT SIMPLY THIS MEANS WE CAN BLOCK DIAGONALIZE THE ACTION OF $\rho(g)$:

$$\rho(g) \sim \begin{pmatrix} \ddots & & & & \\ & \boxed{} & & & \\ & & \boxed{\mathbb{1}_{N_\lambda} \otimes \rho_\lambda(g)} & & \\ & & & \boxed{} & \\ & & & & \ddots \end{pmatrix}$$

6. REAL, COMPLEX, & QUATERNIONIC VECTOR SPACES

RECALL THAT A VECTOR SPACE V OVER A FIELD k HAS OPERATIONS:

VECTOR SUM $\quad v_1 + v_2 \quad v_1, v_2 \in V$

SCALAR MULT. $\quad \alpha \cdot v \quad \alpha \in k, v \in V$

SATISFYING:

$$\alpha \cdot (v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2 \quad \text{ETC.}$$

IF WE REPLACE "FIELD k" BY "RING R" WE GET A "MODULE OVER A RING R"

VECTOR SPACES OVER $k=\mathbb{R}$ AND $k=\mathbb{C}$ ARE DIFFERENT, BUT THEY CAN BE RELATED:

- IF V IS A VECTOR SPACE OVER $\mathbb{C}$, IT IS ALSO A VECTOR SPACE $V_{\mathbb{R}}$ OVER $\mathbb{R}$. NOW FOR $v \in V_{\mathbb{R}}$, $\sqrt{-1}\, v \in V_{\mathbb{R}}$ IS ANOTHER VECTOR AND THESE ARE LINEARLY INDEPENDENT OVER $\mathbb{R}$, SO

$$\dim_{\mathbb{R}} V_{\mathbb{R}} = 2 \cdot \dim_{\mathbb{C}} V$$

- IN GENERAL WE CANNOT GO THE OTHER WAY, BUT IF W IS A V.S. OVER $\mathbb{R}$ AND THERE IS AN $\mathbb{R}$-LINEAR MAP

$$I : W \longrightarrow W$$
$$I^2 = -1$$

THEN WE CAN MAKE W A VECTOR SPACE OVER $\mathbb{C}$:

$$\sqrt{-1} \cdot w := I(w)$$

SUCH A LINEAR MAP IS CALLED A COMPLEX STRUCTURE ON W.

THE SET OF VECTORS IS THE SAME, BUT NOW THERE IS A SCALAR MULTIPLICATION BY $z \in \mathbb{C}$. WE SOMETIMES DENOTE THE COMPLEX VECTOR SPACE BY $\overline{V} = (\overline{W}, I)$.

NOW $w \in \overline{W}$ AND $I(w) \in \overline{W}$ ARE LINEARLY <u>DEPENDENT</u> OVER $\mathbb{C}$ SO

$$\dim_{\mathbb{C}} (\overline{W}, I) = \frac{1}{2} \dim_{\mathbb{R}} \overline{W}$$

EXAMPLE:

$$W = \mathbb{R}^2, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$(W, I) \cong \mathbb{C}$$

REMARKS:

1. THE SPACE OF ALL CPLX STRS ON $\mathbb{R}^{2N}$ (PRESERVING LENGTH) IS $O(2N)/U(N)$. THE REASON IS THAT ANY $I \in \mathrm{End}(\mathbb{R}^{2N})$ WITH $I^2 = -1$ IS OF THE FORM

$$I = g I_0 g^{-1} \qquad g \in O(2N)$$

$$I_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}$$

2 IF I IS A COMPLEX STRUCTURE SO IS $-I$. THE CORRESPONDING COMPLEX VECTOR SPACES ARE CONJUGATE:

$$\overline{V} = V$$

AS SETS OF VECTORS, BUT THE DEFINITION OF SCALAR MULTIPLICATION IS:

$$z\,\overline{v} := \overline{\bar{z} \cdot v}$$

- ANOTHER WAY TO TURN A REAL VECTOR SPACE W INTO A COMPLEX V.S. IS SIMPLY TO COMPLEXIFY:

$$W \longrightarrow W^{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$$

NOTE THAT $W^{\mathbb{C}}$ HAS AN <u>ANTI-LINEAR OPERATOR</u>: $K : W^{\mathbb{C}} \to W^{\mathbb{C}}$

$$K : \quad w \otimes z \longmapsto w \otimes \bar{z}$$

WHICH SQUARES TO $+1$

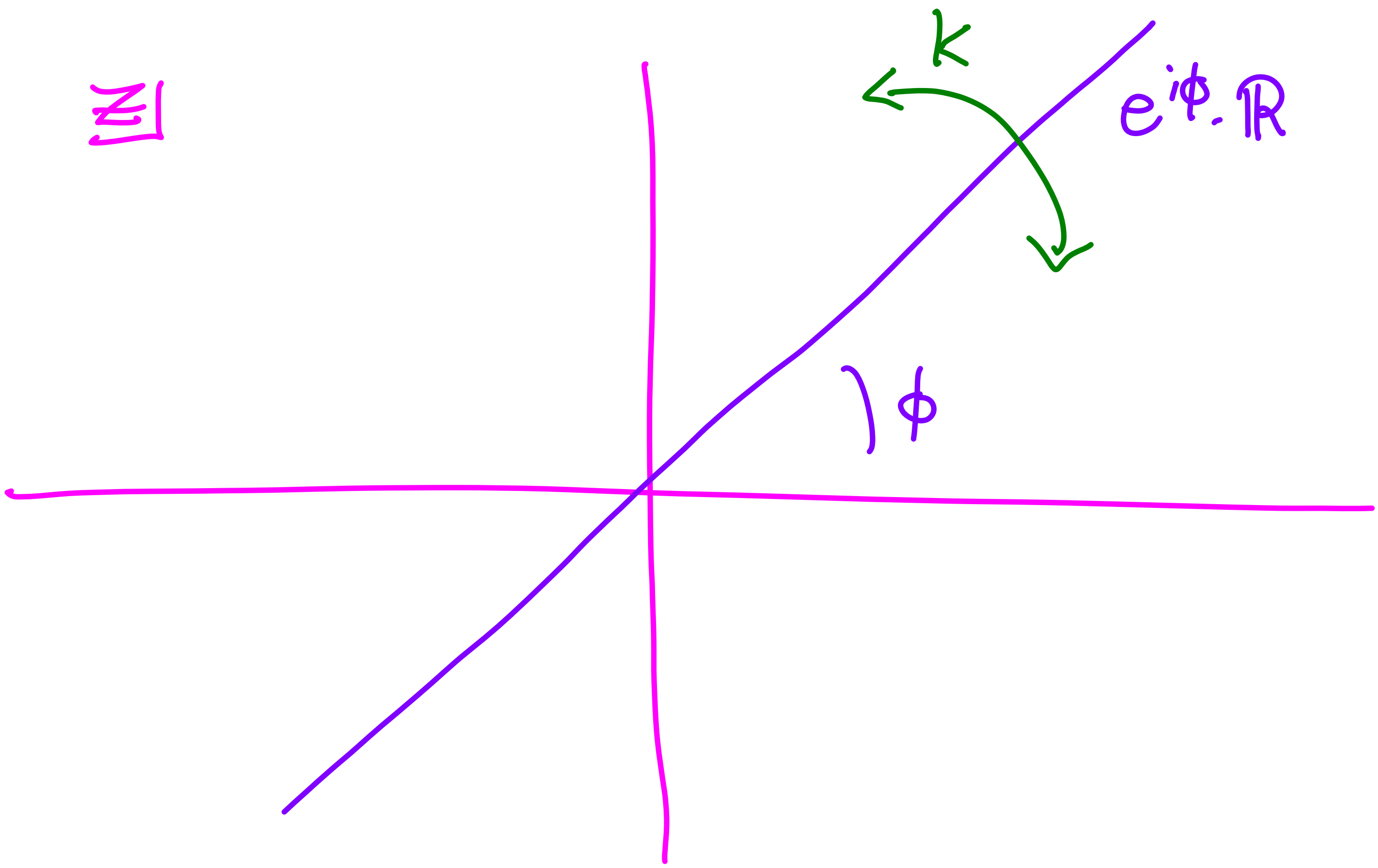
(NOTE THAT W = FIXED POINTS OF K.)

<u>DEF</u>: A <u>REAL</u> <u>STRUCTURE</u> ON A COMPLEX VECTOR SPACE V IS AN ANTI-LINEAR OPERATOR

$$K \in \mathrm{End}_{\mathbb{R}}(V)$$

SUCH THAT $K^2 = +1$

EXAMPLE: $V = \mathbb{C}$, K = REFLECTION THROUGH THE LINE $e^{i\phi}.\mathbb{R}$,



$$K: \; z \longrightarrow e^{2i\phi}\bar{z}$$

$$W = \mathrm{FIX}(K) = e^{i\phi}\mathbb{R}$$

AS A VECTOR SPACE,

$$W \cong \mathbb{R}.$$

EXERCISE: SUPPOSE $(W, I) \cong V$ IS A COMPLEX VECTOR SPACE.

SHOW THAT

$$W \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}$$

BY INTRODUCING PROJECTION OPERATORS $P_\pm = \frac{1}{2}(1 \pm I \otimes i)$.

(RECALL $\overline{V}$ IS THE CONJUGATE V.S.

$$\bar{z} \cdot \bar{v} = \overline{z \cdot v}$$)

AN EXTENDED EXAMPLE

SHOWING RELATION TO COREPS

NOW, FOR PRACTICE, LET US CONSIDER (ϕ, τ)-TWISTED REPS OF $M_2 = \{1, \overline{T}\}$, $\overline{T}^2 = +1$, $\phi(\overline{T}) = -1$.

WE SHOWED THERE ARE TWO INEQ. ϕ-TWISTED EXTENSIONS

$$1 \to \mathbb{T} \to M_2^+ \longrightarrow M_2 \longrightarrow 1$$

$$M_2^+ = \left\{ \lambda \cdot T \;\middle|\; T^2 = +1, \; T\lambda = \bar{\lambda}^{-1} T \right\}$$

$$1 \to \mathbb{T} \to M_2^- \longrightarrow M_2 \longrightarrow 1$$

$$M_2^- = \left\{ \lambda \cdot T \;\middle|\; T^2 = -1, \; T\lambda = \bar{\lambda}^{-1} T \right\}$$

FIRST SUPPOSE $\mathcal{H}$ IS A (ϕ,τ)-TWISTED REP OF M_2^+. THEN

$$K = \rho(T)$$

IS ANTI-LINEAR AND SQUARES TO +1. SO, A COREP of M_2^+ ON $\mathcal{H}$ IS NOTHING OTHER THAN A <u>REAL STRUCTURE</u>.

SECOND, WHAT CAN WE SAY IF $\mathcal{H}$ IS A COREP OF M_2^- ?

THEN, $K = \rho(T)$ IS ANTI-LINEAR AND NOW

$$K^2 = -1$$

SO IT IS <u>ANOTHER</u> COMPLEX STR. !

IN FACT, WE HAVE 3 ANTICOMMUTING COMPLEX STRUCTURES ON $\mathcal{H}$:

$$I: \quad v \longmapsto \sqrt{-1}\, v$$

$$K: \quad v \longmapsto \rho(T)(v)$$

$$I^2 = -1 \qquad K^2 = -1 \qquad KI = -IK$$

BUT THEN, DEFINE $J := KI$ AND CHECK:

$$J^2 = -1, \quad JI = -IJ, \quad JK = -KJ$$

WHAT WE HAVE DISCOVERED IS THAT $\mathcal{H}$ IS A <u>MODULE</u> <u>FOR</u> <u>THE</u> <u>QUATERNIONS</u> $\underline{\mathbb{H}}$.

DEF. THE QUATERNION ALGEBRA $\mathbb{H}$ IS THE ALGEBRA OVER $\mathbb{R}$ OF DIM = 4 SPANNED BY $1, \underline{i}, \underline{j}, \underline{k}$

$$\underline{i}^2 = \underline{j}^2 = \underline{k}^2 = -1, \quad \underline{i}\,\underline{j}\,\underline{k} = -1$$

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\underline{i} \oplus \mathbb{R}\underline{j} \oplus \mathbb{R}\underline{k}$$

REMARKS:

1. $\mathbb{H}$ IS ASSOCIATIVE BUT NON-COMMUTATIVE.

2. IF $q = x_4 + x_1\underline{i} + x_2\underline{j} + x_3\underline{k}$

   THEN $\bar{q} = x_4 - x_1\underline{i} - x_2\underline{j} - x_3\underline{k}$

DEFINE $\|q\|^2 = q\bar{q} = x_\mu x_\mu$

3. UNITARY MATRICES WITH ENTRIES IN $\mathbb{H}$ FORM A GROUP:

$$U(n, \mathbb{H})$$

THIS TURNS OUT TO BE ISOMORPHIC TO THE COMPACT SYMPLECTIC GROUP,

$$USp(2n) := U(2n, \mathbb{C}) \cap Sp(2n, \mathbb{C}).$$

<u>EXERCISE</u>:

$$U(1, \mathbb{H}) \approx SU(2).$$

EXERCISES:

① SHOW THAT THE IMAGINARY PAULI MATRICES SATISFY THE QUAT. RELATIONS:

$$\underline{i} \longrightarrow \sqrt{-1}\,\sigma^1, \quad \underline{j} \longrightarrow \sqrt{-1}\,\sigma^2, \quad \underline{k} \rightarrow \sqrt{-1}\,\sigma^3$$

AND THEREFORE WE CAN IDENTIFY

$$q \longrightarrow \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \qquad z, w \in \mathbb{C}$$

② (a.) SHOW THAT THE COMPLEX STRUCTURES ON $\mathbb{R}^4$ CAN BE IDENTIFIED AS

$$q \longrightarrow n\, q \qquad n^2 = -1$$

<u>OR</u>

$$q \longrightarrow q\, n \qquad n^2 = -1$$

(b.) SHOW THAT THE SPACE OF CPLX STRUCTURES IS THUS $\approx S^2 \amalg S^2$.

3. SHOW THAT A CO-REP OF (G, ϕ) CAN BE DEFINED AS A REAL VECTOR SPACE W WITH COMPLEX STRUCTURE I AND $\rho : G \longrightarrow \mathrm{End}(W)$ SO THAT

$$\rho(g) I = \phi(g) I \rho(g)$$

SUMMARY

1. A COMPLEX STRUCTURE ON A REAL VECTOR SPACE W IS A LINEAR MAP $I \in \mathrm{END}(W)$

$$I^2 = -1$$

2. A REAL STRUCTURE ON A COMPLEX VECTOR SPACE V IS AN ANTI-LINEAR MAP $K \in \mathrm{End}_{\mathbb{R}}(V)$

$$K^2 = +1$$

3. A QUATERNIONIC STRUCTURE ON A COMPLEX VECTOR SPACE V IS AN ANTI-LINEAR MAP $K \in \mathrm{End}_{\mathbb{R}}(V)$

$$K^2 = -1$$

7. DYSON'S 3-FOLD WAY

A. FROBENIUS THM ON REAL, ASSOCIATIVE DIVISION ALGEBRAS

B. SCHUR'S LEMMA FOR CO-REPS

C. DYSON'S PROBLEM

D. THE 3-FOLD WAY

7A.) FROBENIUS THEOREM

DEF: LET $\mathcal{A}$ BE A UNITAL, ASSOC. ALGEBRA / $\mathbb{R}$. THAT MEANS:

- $\mathcal{A}$ IS A V.S. / $\mathbb{R}$
- $\exists$ MULTIPLICATION:

$$a, b \in \mathcal{A} \longmapsto a \cdot b \in \mathcal{A}$$

SUCH THAT

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

ETC.

THEN, $\mathcal{A}$ IS A DIVISION ALGEBRA IF $\forall\ a \neq 0,\ \exists\ a^{-1} \in \mathcal{A}$

$$a \cdot a^{-1} = 1$$

THEOREM (FROBENIUS):

IF $\mathcal{A}$ IS AN ASSOCIATIVE REAL DIVISION ALGEBRA THEN:

- $\mathcal{A} \cong \mathbb{R}$ OR,
- $\mathcal{A} \cong \mathbb{C}$ OR,
- $\mathcal{A} \cong \mathbb{H}$

(REMARK: IF WE DROP THE "ASSOCIATIVE" CONDITION A THM. OF HURWITZ SAYS THERE IS ONLY ONE MORE: $\mathcal{A} \cong \mathbb{O}$.

NOTE THE DIM'S $/\mathbb{R} = 1, 2, 4, 8$.

COMPARE THE DIM'S IN WHICH MINIMAL SUPERSYMMETRIC YANGMILLS EXIST ARE $3, 4, 6, 10$.)

PROOF OF FROBENIUS THEOREM:

D = REAL, ASSOC. DIVISION ALG.

$$a \in D \implies L(a) \in \mathrm{End}(D)$$

$$L(a): b \longmapsto a \cdot b$$

Let $V := \{ a \mid \mathrm{Tr}(L(a)) = 0 \}$

$$D \cong \mathbb{R} \oplus V$$

__Lemma:__

$$V = \{ a \in D \mid a^2 \leq 0 \}$$

__proof of lemma:__ if $a \neq 0$

$$p_a(x) = \det(x - L(a))$$

must factor:

not real

$$p_a(x) = \prod_{i=1}^{r} (x - r_i) \prod_{j=1}^{m} (x - z_j)(x - \bar{z}_j)$$

$$= \prod_i (x - r_i) \prod_j \left(x^2 - 2\mathrm{Re}(z_j)\, x + |z_j|^2 \right)$$

But $p_a(a) = 0$. Since D is a division algebra

$a - r_i = 0$ OR $a^2 - 2\mathrm{Re}(z_j)a + |z_j|^2 = 0$

$\uparrow$

$\mathrm{Tr}(L(a)) \neq 0$

$\Downarrow$

$$\underbrace{p_a(x) = \left(x^2 - 2\mathrm{Re}(z)a + |z|^2\right)^m}$$

Coeff of x^{2m-1} is $\mathrm{Tr}\, L(a)$

So $\mathrm{Tr}\, L(a) = 0 \iff \mathrm{Re}(z) = 0$

$\iff a^2 = -|z|^2 < 0$ ∎

Now $Q(a,b) := -ab - ba$ is a positive definite form on V:

$$Q(a,b) = a^2 + b^2 - (a+b)^2$$

$$Q(a,a) = -2a^2 \geq 0, \text{ on } V.$$

NOW CHOOSE A MINIMAL SET OF GENERATORS $e_i \in \overline{V}$ FOR D

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

$$i,j = 1, \dots, n$$

THESE DEFINE A <u>CLIFFORD ALGEBRA</u> $C\ell_{-n}$. (ABOUT WHICH, MUCH MORE LATER ON.)

FOR $n = 0, 1, 2$ CHECK

$$C\ell_{-n} = \mathbb{R}, \mathbb{C}, \mathbb{H}$$

FOR $n > 2$

ASSOC. USED HERE.

$$(1 + e_1 e_2 e_3)(1 - e_1 e_2 e_3) = 0$$

$\Longrightarrow$ $e_1 e_2 e_3 = \pm 1$ (D a division alg.)

$\Longrightarrow$ $e_3 = \pm e_1 e_2$; e_i <u>MINIMAL</u> $\Rightarrow\!\Leftarrow$

$\blacksquare$

7B.) SCHUR'S LEMMA FOR COREPS

<u>LEMMA</u>:

SUPPOSE (G, ϕ, ρ, V) IS AN <u>IRREUCIBLE</u> COREP OF (G, ϕ) ON A COMPLEX VECTOR SPACE V. THEN THE <u>COMMUTANT</u>:

$$Z(\rho(G)) := \{ A \in \mathrm{End}(V) \mid A\rho(g) = \rho(g)A \}$$

IS A <u>REAL</u> <u>ASSOCIATIVE</u> <u>DIVISION</u> <u>ALGEBRA</u>.

PROOF: SUPPOSE $A \in Z$. THEN

$\ker(A) \subset V$

IS A SUB-COREP. SINCE V IS

IRREDUCIBLE EITHER

- $\ker(A) = 0$
- $\ker(A) = V$

$\ker(A) = V \iff A = 0$. SO, IF

$A \neq 0$ THEN $\ker(A) = 0$, SO

A IS INVERTIBLE.

SINCE Z IS ASSOC. Z IS A

DIVISION ALGEBRA OVER $\mathbb{R}$. $\blacksquare$

COROLLARY: $Z \cong \mathbb{R}, \mathbb{C}$ OR $\mathbb{H}$

REMARKS

1. COMPARE WITH SCHUR FOR IRREPS / $\mathbb{C}$. THEN $Z \cong \mathbb{C}$.

2. FOR EXAMPLES:

2a: $G = \mathbb{Z}_2$, $\phi(\sigma) = -1$

$V = \mathbb{C}$, $\rho(\sigma) = K \in End_{\mathbb{R}}(V)$

$$K(z) = \bar{z}$$

THEN $D \cong \mathbb{R}$

2b: $G = M_2^-$, $\phi(T) = -1$,

$V = \mathbb{C}^2$ $\quad \rho(\lambda) \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda z_1 \\ \lambda z_2 \end{pmatrix}$

$$\rho(T) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix}$$

$D \cong \mathbb{H}$

TO PROVE THIS WORK OVER $\mathbb{R}$

AND LET $W = \mathbb{H}$, $I = L(\underline{i})$,

$$\rho(e^{i\theta}) = \cos\theta + \sin\theta \, L(\underline{i})$$

$$\rho(T) = L(\underline{j})$$

$L(\underline{i}), L(\underline{j})$ = LEFT MULTIPLICATION BY $\underline{i}, \underline{j}$.

CLEARLY RIGHT-MULTIPLICATION BY ANY $q \in \mathbb{H}$ COMMUTES.

3 WE WILL LET $\{V_\lambda\}$ DENOTE A CHOICE OF A COMPLETE SET OF NONISOMORPHIC IRRED. COREPS FOR (G, ϕ).

THE DIVISION ALGEBRA CORRESPONDING TO V_λ WILL BE DENOTED D_λ.

4 WE CAN REFINE THE ISOTYPICAL DECOMPOSITION OF A COREP:

$$V = \bigoplus_{\lambda \in (G,\phi)^\vee} \mathrm{End}_G(V_\lambda, V) \otimes_{D_\lambda} V_\lambda$$

PUT DIFFERENTLY, THE <u>GROUP ALGEBRA</u> $\mathbb{R}[G]$ GENERATED BY $\rho(G)$ HAS A DECOMPOSITION

$$\bigoplus_\lambda \left(D_\lambda(t_\lambda)\right)^{\oplus s_\lambda}$$

AND THE COMMUTANT - IN THE SAME BASIS - IS

$$\bigoplus_\lambda \left(\bar{D}_\lambda(s_\lambda)\right)^{\oplus t_\lambda}$$

IN PLAIN ENGLISH, THE GENERAL ELEMENT IN $\rho(\mathbb{R}[G])$ CAN BE PUT IN THE FORM

$$\lambda\left\{\begin{pmatrix} \ddots & & \\ & \overbrace{\left.\begin{array}{ccc} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{array}\right\}}^{\lambda} s_\lambda \text{ BLOCKS} & \\ & & \ddots \end{pmatrix}\right.$$

A = LINEAR COMBINATION OF MATRICES IN THE IRREDUCIBLE REP. V_λ.

= SOME $t_\lambda \times t_\lambda$ MATRIX WITH ELEMENTS IN D_λ,

REPEATED s_λ TIMES.

NOW, THE COMMUTANT CONSISTS OF MATRICES OF THE FORM:

$$
\begin{bmatrix}
\ddots & & \\
& \underbrace{\begin{array}{c|c|c}
z_{11}\mathbb{1}_{t_\lambda} & z_{12}\mathbb{1}_{t_\lambda} & z_{1s_\lambda}\mathbb{1}_{t_\lambda} \\ \hline
z_{21}\mathbb{1}_{t_\lambda} & z_{22}\mathbb{1}_{t_\lambda} & z_{2s_\lambda}\mathbb{1}_{t_\lambda} \\ \hline
z_{s_\lambda 1}\mathbb{1}_{t_\lambda} & z_{s_\lambda 2}\mathbb{1}_{t_\lambda} & z_{s_\lambda s_\lambda}\mathbb{1}_{t_\lambda}
\end{array}}_{\substack{\text{AN } s_\lambda \times s_\lambda \text{ MATRIX} \\ \text{REPEATED } t_\lambda \text{ TIMES}}} & \\
& & \ddots
\end{bmatrix}
$$

7C.) THE DYSON PROBLEM

- OFTEN IN PHYSICS WE BEGIN WITH A HAMILTONIAN (OR ACTION) AND THEN FIND THE SYMMETRIES OF THE PHYSICAL SYSTEM.

- BUT THERE ARE CASES WHEN THE DYNAMICS ARE VERY COMPLICATED. A GOOD EXAMPLE IS IN THE THEORY OF NUCLEAR INTERACTIONS, OR SYSTEMS WITH MANY D.O.F.

- WIGNER HAD THE BEAUTIFUL IDEA THAT BY ASSUMING OUR HAMILTONIAN IS RANDOMLY SELECTED FROM AN ENSEMBLE WE CAN STILL MAKE USEFUL PREDICTIONS BASED ON AVERAGES OVER THE ENSEMBLE

$$\langle \mathcal{O} \rangle = \int_{\mathcal{E}} d\mu(H) \cdot \mathcal{O}$$

$\mathcal{E}$ = ENSEMBLE OF HAMILTONIANS

$d\mu(H)$ = PROBABILITY MEASURE

- SOMETIMES WE KNOW *a priori* THAT THE SYSTEM UNDER STUDY HAS A CERTAIN KIND OF SYMMETRY. DYSON POINTED OUT THAT THIS CAN CONSTRAIN THE ENSEMBLE (AND THEREBY CHANGE THE STATISTICS.)

- <u>DYSON'S PROBLEM</u>: GIVEN A $\mathbb{Z}_2$-GRADED GROUP (G, ϕ), AND A COREP $\mathcal{H}$, WHAT IS THE ENSEMBLE OF COMMUTING HAMILTONIANS?

7D.) SOLUTION OF DYSON'S PROBLEM: THE 3-FOLD WAY

GIVEN THE GROUNDWORK WE HAVE SPELLED OUT, THE SOLUTION IS IMMEDIATE:

- $(G, \phi, \rho, \mathcal{H})$ A COREP

  LET $V_\lambda \quad \lambda \in (G,\phi)^\vee$

DENOTE THE DISTINCT IRRED. COREPS. THEN WE HAVE COMPLETE REDUCIBILITY:

$$\mathcal{H} \cong \bigoplus_{\lambda} N_\lambda \otimes_{\mathbb{R}} V_\lambda$$

AS CO-REPS.

N_λ = REAL VECTORSPACE OF MULTIPLICITIES

- MOREOVER, THERE IS AN HERMITIAN STRUCTURE ON $N_\lambda \otimes_{\mathbb{R}} V_\lambda$, AND WE HAVE AN ISOM. AS UNITARY COREPS.

- NOW, IF WE ASSUME THAT THE HAMILTONIAN H MUST COMMUTE WITH $\rho(g)$, $g \in G$ (WE WILL RE-EXAMINE THIS HYPOTH. LATER) THEN $H \in Z(\mathcal{H})$.

BUT

$$Z(\mathcal{H}) = \bigoplus_\lambda \mathrm{End}(N_\lambda) \otimes_{\mathbb{R}} Z(V_\lambda)$$

- BUT WE HAVE SEEN THAT FOR EACH λ, $Z(V_\lambda) \cong \mathbb{R}, \mathbb{C}$, OR $\mathbb{H}$

- SO

$$Z(\mathcal{H}) = \bigoplus_{\lambda} \underbrace{\mathrm{End}(N_\lambda) \otimes D_\lambda}_{\text{MATRICES OVER } \mathbb{R},\ \mathbb{C},\ \text{OR } \mathbb{H}}$$

FINALLY H MUST BE HERMITIAN SO

$$\mathcal{E} = \prod_{\lambda} \mathcal{H}(N_\lambda, D_\lambda)$$

$$\mathcal{H}(N_\lambda, D_\lambda) = \begin{cases} \text{REAL SYMMETRIC} & D_\lambda = \mathbb{R} \\ \text{COMPLEX HERMITIAN} & D_\lambda = \mathbb{C} \\ \text{QUATERNION HERMITIAN} & D_\lambda = \mathbb{H} \end{cases}$$

(EACH ENSEMBLE HAS [illegible] PROB. DISTRIB.)

8. SYMMETRIES & TIME REVERSAL

PHYSICS TAKES PLACE IN SPACE & TIME, AND HAMILTONIAN EVOLUTION PRESUPPOSES A NOTION OF TIME.

SO IF WE HAVE A SYMMETRY GROUP G OF A Q.M. SYSTEM WITH A HAMILTONIAN IT SHOULD COME WITH A HOMOMORPHISM

$$t : G \longrightarrow \{\pm 1\}$$

WHICH TELLS WHETHER $t(g)$ PRESERVES OR REVERSES THE DIRECTION OF TIME.

ON THE OTHER HAND, WIGNER'S THEOREM ALSO GIVES $\phi : G \to \{\pm 1\}$

HOW ARE THESE RELATED?

$$U(\tau) := \exp\left(-\frac{i\tau}{\hbar} H\right)$$

TIME-EVOLUTION OPERATOR: $\mathcal{H} \to \mathcal{H}$

$$\boxed{\begin{aligned} \rho(g)\, U(\tau)\, \rho(g)^{-1} &= U(\tau)^{t(g)} \\ &= U(t(g)\tau) \end{aligned}}$$

$$\Longrightarrow$$

$$\exp\left(-\frac{\tau}{\hbar}\rho(g)(iH)\rho(g)^{-1}\right) = \exp\left(-\frac{i\, t(g)\tau}{\hbar} H\right)$$

$$\Longrightarrow$$

$$\phi(g)\rho(g) H \rho(g)^{-1} = t(g) H$$

$$\Longrightarrow$$

$$\rho(g) H \rho(g)^{-1} = \phi(g) t(g)\, H$$

SO THE ANSWER TO OUR QUESTION IS: ϕ AND t ARE <u>UNRELATED</u>, IN GENERAL.

DEFINE

$$c(g) = \phi(g)t(g) \in \{\pm 1\}$$

IT IS ANOTHER HOMOMORPHISM

$$c : G \longrightarrow \mathbb{Z}_2$$

BY DEFINITION

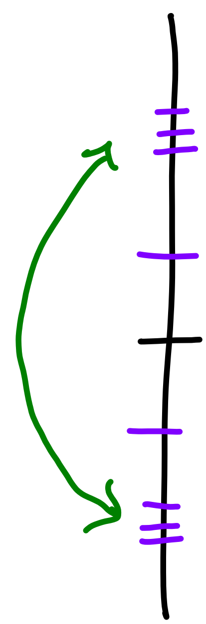
$$\phi \cdot c \cdot t = 1.$$

OFTEN $c(g)$ IS SET TO 1:

$$\rho(g) H \rho(g)^{-1} = c(g) \cdot H \implies$$

IF $c(g) = -1$ FOR ANY $g \in G$ THEN THE SPECTRUM OF H IS SYMMETRIC AROUND ZERO.

$\rho(g)$ WITH $c(g) = -1$



IN MANY PROBLEMS (e.g. REL. Q.F.T.) H IS BOUNDED BELOW, BUT NOT ABOVE. THEN WE MUST HAVE $c(g) = +1 \ \forall g \in G$

THAT IS, IF $c(g)=1$ THEN

$$\phi(g) = t(g),$$

SO ONLY TIME-ORIENTATION REVERSING TRANSFORMATIONS ARE ANTI-UNITRY.

HOWEVER, THERE <u>ARE</u> PHYSICAL EXAMPLES WHERE $c(g)$ CAN BE NONTRIVIAL, THAT IS, THERE CAN BE SYMMETRIES WHICH ARE BOTH ANTI-UNITARY AND TIME ORIENTATION PRESERVING.

<u>EXAMPLE</u>: PARTICLE-HOLE SYMMETRIES IN FREE FERMI SYSTEMS. MORE ON THAT LATER.

REMARKS

2 1 THE CMT LITERATURE IS INCONSISTENT ABOUT WHETHER WE SHOULD ALLOW "SYMMETRY GROUPS" WITH $c \neq 1$. 2

2 KRAMER'S THEOREM.

IF $c=1$ & THERE IS A TIME REVERSING SYMMETRY $T^2 = -1$ (e.g. ON ELECTRONS) THEN $\mathcal{H}$ HAS A QUATERNIONIC STRUCTURE, WHICH COMMUTES WITH H. IT FOLLOWS THAT ENERGY EIGENVALUES HAVE EVEN DEGENERACY.

9. GAPPED HAMILTONIANS AND PHASES OF MATTER

AN IMPORTANT TOPIC OF CURRENT RESEARCH IN CMT IS THE CLASSIFICATION OF "PHASES OF MATTER."

THERE ARE NEW PHASES – QUANTUM HALL FLUIDS & TOPOLOGICAL SUPERCONDUCTORS WHICH ARE SOMEHOW "TOPOLOGICALLY DISTINCT" FROM "ORDINARY PHASES."

THIS IS THE CONTEXT WHERE KITAEV SUGGESTED THE APPLICATION OF K-THEORY TO THE CLASSIFICATION PROBLEM.

ONE WAY TO MAKE "PHASES" PRECISE IS TO CONSIDER <u>GAPPED HAMILTONIANS</u>, THOSE WITH A GAP IN THE ENERGY SPECTRUM BETWEEN THE GROUND & FIRST EXCITED STATE.

(SO, IF WE SHIFT THE ZERO OF ENERGY INTO THIS GAP H IS INVERTIBLE.)

WE CAN THEN DEFINE A <u>TOPOLOGY</u> ON FAMILIES OF QUANTUM SYSTEMS WITH A GAPPED HAMILTONIAN.

THEN...

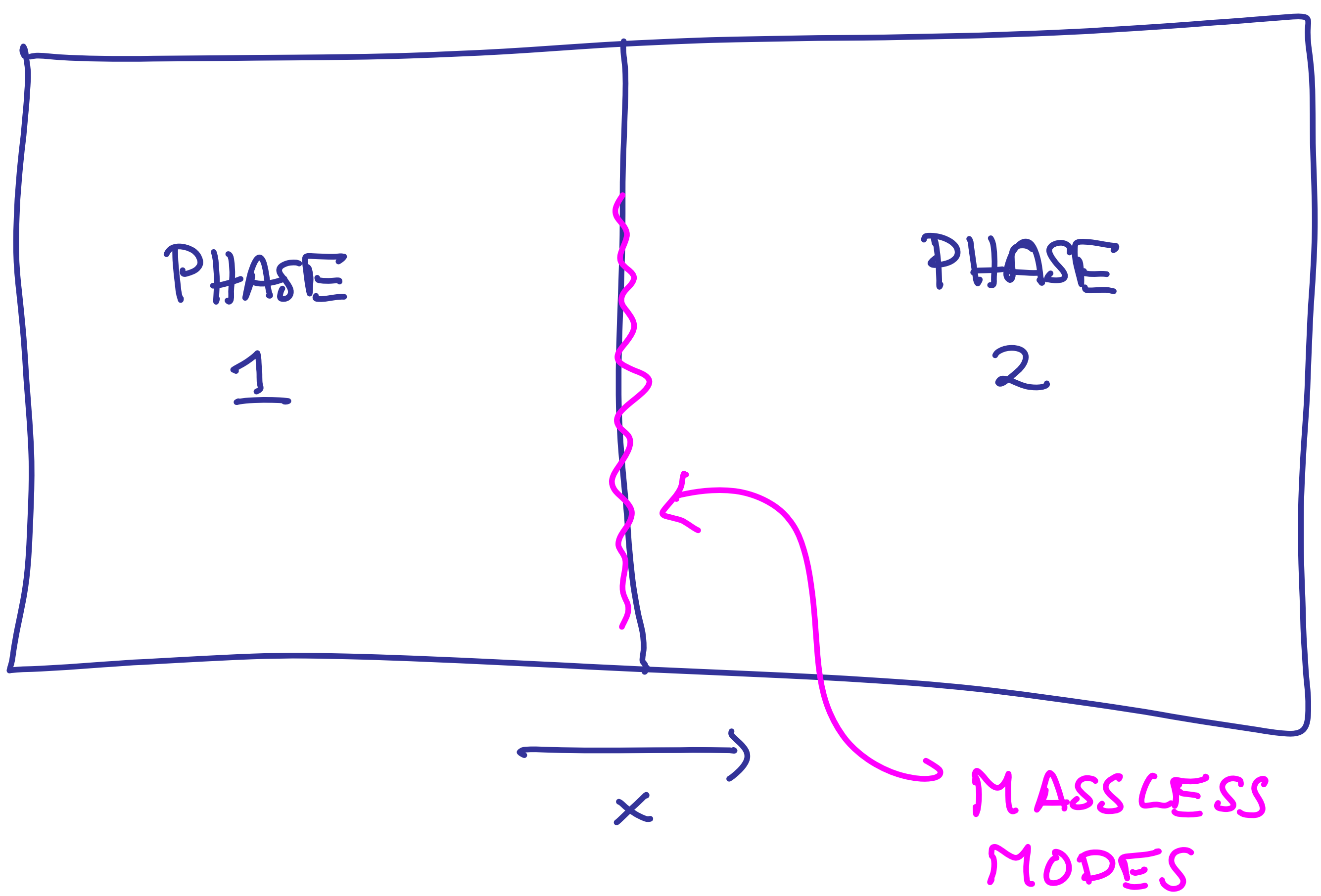
$(\mathcal{H}_0, H_0)$ AND $(\mathcal{H}_1, H_1)$

ARE IN THE SAME PHASE IF THERE IS A CONTINUOUS FAMILY OF SYSTEMS $(\mathcal{H}_s, H_s)$ $s \in [0,1]$ INTERPOLATING.

3 REMARKS

① IT IS NONTRIVIAL TO SAY WHAT "CONTINUOUS" MEANS WHEN $\mathcal{H}$ IS INFINITE DIMENSIONAL. THERE ARE DIFFERENT NOTIONS OF CONTINUITY. WE USE THE COMPACT-OPEN TOPOLOGY FOR OUR HILBERT BUNDLES AND GROUP REPS.

② DOMAIN WALLS BETWEEN TWO PHASES MUST CONTAIN MASSLESS MODES



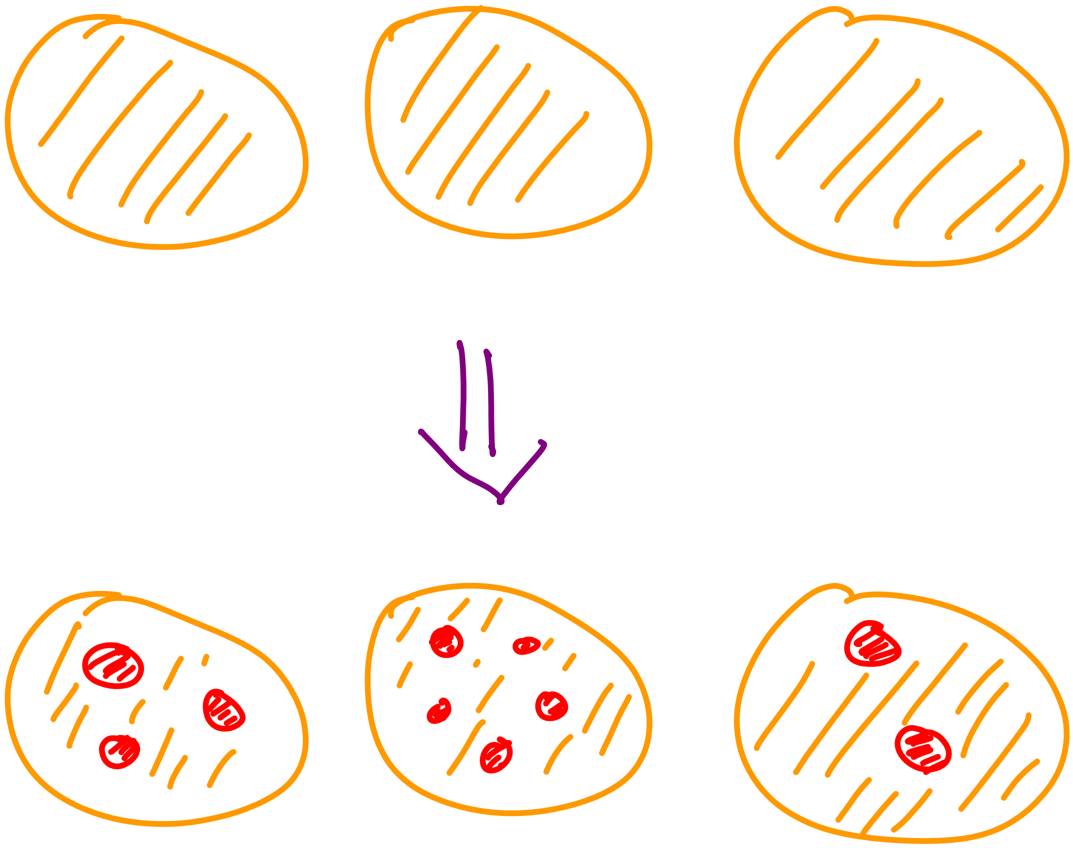
③ PHASES ARE THE CONNECTED COMPONENTS OF THE SPACE OF GAPPED HAMILTONIANS

NOW, IF WE ADD SYMMETRY TO THE STORY WE CAN GET A REFINED NOTION OF TOPOLOGICAL PHASES:

SUPPOSE G IS A GROUP WITH TWO HOMOMORPHISMS $c, \phi : G \to \{\pm 1\}$

WE CAN DEMAND THAT $\mathcal{H}$ BE A COREP FOR (G, ϕ) AND $\rho(g) H = c(g) H \rho(g)$.

THEN WE CAN DEFINE EQUIVARIANT PHASES BY CONSIDERING $(G, \phi, c, \mathcal{H}, H)_{0,1}$ TO BE CONTINUOUSLY CONNECTED ONLY IF THERE IS A CONTINUOUS GAPPED FAMILY $(G, \phi, c, \mathcal{H}, H)_s \; s \in [0,1]$



FOR EXAMPLE, WE CAN APPLY THIS IDEA TO BAND STRUCTURES WITH A FIXED CRYSTALLOGRAPHIC SYMMETRY.

THE FREED-MOORE PAPER ABOVE EXPLAINS HOW KITAEV'S CLASSIF. OF PHASES (OF FREE FERMIONS) BY K-THRY IS REFINED TO A CLASSIFICATION BY TWISTED EQUIVARIANT K-THEORY.

10. $\mathbb{Z}_2$-GRADINGS & SUPER-LINEAR ALGEBRA

WHEN WE HAVE A GAPPED HAMILTONIAN WE CAN DEFINE $\mathrm{sign}(H) \in \{\pm 1\}$. THIS SPLITS THE HILBERT SPACE

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

INTO EVEN AND ODD PIECES, GIVEN BY THE $\mathrm{sign}(H) = +1/-1$ EIGENSPACES.

REMARK: IF $\{H_s\}$ IS A CONTINUOUS FAMILY OF GAPPED HAMILTONIANS THEN $\mathrm{sign}(H_s)$ IS CONSTANT AS A FUNCTION OF s.

DEF: IN GENERAL, A VECTOR SPACE WITH A DECOMPOSITION

$$V = V^0 \oplus V^1$$

IS CALLED A $\mathbb{Z}_2$-GRADED, OR SUPER-VECTOR SPACE.
ITS GRADED DIMENSION IS THE PAIR OF INTEGERS:

$$(m|n) = (\dim V^0 \mid \dim V^1)$$

ON A $\overbrace{\text{SUPERVECTORSPACE}}$ THE LINER TRANSFORMATIONS

ALSO DECOMPOSE INTO EVEN/ODD

$$\vartheta \in \mathrm{End}(\mathcal{H})$$

IS <u>EVEN</u> IF IN THE

BLOCK DECOMP. WRT $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}'$

$$\vartheta = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

AND ϑ IS <u>ODD</u> IF

$$\vartheta = \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$$

NOTE THAT $\mathrm{End}(\mathcal{H})$ IS

ITSELF A SUPER-VECTOR SPACE

$$\mathrm{End}(\mathcal{H})^0 = \mathrm{End}(\mathcal{H}^0, \mathcal{H}^0) \oplus \mathrm{End}(\mathcal{H}', \mathcal{H}')$$

$$\mathrm{End}(\mathcal{H})^1 = \mathrm{End}(\mathcal{H}^0, \mathcal{H}') \oplus \mathrm{End}(\mathcal{H}', \mathcal{H}^0)$$

NOW SUPPOSE $\mathcal{H}$ IS A COREP OF (G, ϕ) AND

WE HAVE A HOMOMORPHISM

$$c : G \longrightarrow \{\pm 1\}$$

$$H \rho(g) = c(g) \rho(g) H$$

THEN

$$\rho(g) \in \mathrm{End}(\mathcal{H})^0 \quad \text{IF} \quad c(g) = +1$$

$$\rho(g) \in \mathrm{End}(\mathcal{H})^1 \quad \text{IF} \quad c(g) = -1$$

THIS IS AN EXAMPLE OF

A "$\mathbb{Z}_2$-GRADED REPRESENTATION."

DEF. IN GENERAL IF (G, c) IS A $\mathbb{Z}_2$-GRADED GROUP THEN A $\mathbb{Z}_2$-GRADED REPRESENTATION IS A SUPER-VECTORSPACE $V^{m|n}$ AND

$$\rho : G \to \mathrm{End}(V^{m|n})$$

SO THAT

$\rho(g)$ IS EVEN $\sim \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$

FOR $c(g) = +1$

$\rho(g)$ IS ODD $\sim \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$

FOR $c(g) = -1$

FINALLY, RECALL THAT IF V IS A VECTOR SPACE THEN $\mathrm{End}(V)$ IS AN <u>ALGEBRA</u>:

WE CAN ADD LINEAR OPERATORS $T_1 + T_2$ AND WE CAN MULTIPLY THEM BY SCALARS, $T \to \alpha T$, BUT WE CAN ALSO COMPOSE THEM $T_1 \circ T_2$ AND THIS DEFINES A (NONCOMMUTATIVE) ALGEBRA STRUCTURE ON $\mathrm{End}(V)$

WHAT HAPPENS WHEN $\mathcal{V}$ IS A SUPER VECTOR SPACE?

$$\mathcal{V} = \mathcal{V}^0 \oplus \mathcal{V}^1$$

AS WE SAW, $\mathrm{End}(\mathcal{V})$ IS ALSO A SUPER-VECTOR-SPACE

$$\mathrm{End}(\mathcal{V})^0 \sim \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

$$\mathrm{End}(\mathcal{V})^1 \sim \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$$

BUT IT IS ALSO A SUPERALGEBRA

$$\mathcal{O}_1 \circ \mathcal{O}_2 \in \mathrm{End}(\mathcal{H})^{|\mathcal{O}_1|+|\mathcal{O}_2|}$$

$$|\mathcal{O}| = \mathbb{Z}_2\text{-grading of } \mathcal{O} \in \mathbb{Z}/2\mathbb{Z}$$

IN GENERAL

DEF: $\mathcal{A}$ IS A SUPER-ALGEBRA IF $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ IS A SUPER VECTOR SPACE AND THE MULT. RESPECTS THE $\mathbb{Z}_2$-GRADING:

IF a, a' ARE HOMOGENEOUS ELTS THEN

$$\boxed{a \cdot a' \in \mathcal{A}^{|a|+|a'|}}$$

KOSZUL SIGN RULE

SUPER-LINEAR ALGEBRA INTRODUCES SOME IMPORTANT SIGNS: EXCHANGING THE ORDER OF ANY TWO ODD OBJECTS INTRODUCES AN "EXTRA" SIGN.

EXAMPLE ①: THE GRADED TENSOR PRODUCT OF SUPER-ALGEBRAS $\mathcal{A} \hat{\otimes} \mathcal{B}$ HAS

$$\boxed{(a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = (-1)^{|a_2|\cdot|b_1|}\, a_1 a_2 \otimes b_1 b_2}$$

FOR HOMOGENEOUS a_2, b_1.

EXAMPLE ②: IF $\mathcal{A}, \mathcal{B}$ HAVE $\mathbb{Z}_2$-GRADED REP'S V, W THEN THE PRODUCT $V \otimes W$ IS

$$\boxed{\rho(a \otimes b)\, v \otimes w = (-1)^{|b|\cdot|v|}\, \rho(a)v \otimes \rho(b)w}$$

FOR HOMOGENEOUS $b \in \mathcal{B}$, $v \in V$.

12. THE TEN CT GROUPS

2

MOTIVATION: IN SOME "DISORDERED" SYSTEMS OF FREE FERMIONS THE ONLY SYMMETRIES WE MIGHT KNOW ABOUT a priori ARE THE PRESENCE (OR ABSENCE) OF "TIME REVERSAL" & "PARTICLE-HOLE" SYMMETRY.

THUS, IT IS INTERESTING TO LOOK AT ϕ_s-TWISTED EXTENSIONS OF SUBGROUPS

OF $M_{2,2} = \{1, \overline{T}, \overline{C}, \overline{T}\overline{C}\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

WHERE $\overline{T}^2 = \overline{C}^2 = 1$, $\overline{T}\,\overline{C} = \overline{C}\,\overline{T}$

AND $\phi_s(\overline{T}) = \phi_s(\overline{C}) = -1$.

MATH QUESTION: WHAT ARE THE ϕ_s-TWISTED EXTENSIONS OF SUBGROUPS OF $M_{2,2}$

THIS GENERALIZES OUR PREVIOUS DISCUSSION OF FINDING ϕ-TWISTED EXTENSIONS OF $M_2 = \{1, \overline{T}\}$;

WE NOW HAVE TWO GENERATORS $\overline{T}$, $\overline{C}$ AND WE CONSIDER <u>SUBGROUPS</u>.

THERE ARE 5 SUBGROUPS, DEPENDING ON WHETHER $\overline{T}$, $\overline{C}$, OR $\overline{T}\,\overline{C}$ IS IN THE SUBGROUP:

$$\begin{tikzcd}[column sep=small]
 & M_{22} \arrow[d, phantom, "\shortparallel"] & \\
 & \{1, \overline{C}, \overline{T}, \overline{C}\,\overline{T}\} & \\
\{1, \overline{C}\} \arrow[ur] & \{1, \overline{T}\} \arrow[u] & \{1, \overline{C}\,\overline{T}\} \arrow[ul] \\
 & \{1\} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

THEN THE EXTENSION SIMPLY DEPENDS ON WHETHER THE LIFT T, C SQUARES TO ± 1.

EXERCISE 1: SHOW THAT WE MAY ALWAYS CHOOSE LIFTS T, C SO THAT $TC = CT$.

(WE WILL ALWAYS DO SO.)

EXERCISE 2: IF $\bar{S} = \bar{T}\bar{C}$ AND

$$\underline{\text{III}} = \{1, \bar{S}\} \subset M_{2,2}$$

THEN WE MAY ALWAYS CHOOSE A LIFT WITH $S^2 = 1$ (OR $S^2 = -1$, IF WE LIKE)

NOW WE LIST THE TEN CASES

WE WILL CALL THEM THE 10 CT GROUPS.

THE 10 CT GROUPS $\underline{III}^{c}$

SUBGROUP $\underline{III} \subset M_{2,2}$	T^2	C^2	
$\{1\}$			
$\{1, \bar{S}\}$			
$\{1, \bar{T}\}$	$+1$		
$M_{2,2}$	$+1$	-1	
$\{1, \bar{C}\}$		-1	
$M_{2,2}$	-1	-1	
$\{1, \bar{T}\}$	-1		
$M_{2,2}$	-1	$+1$	
$\{1, \bar{C}\}$		$+1$	
$M_{2,2}$	$+1$	$+1$	

REMARK: WE MOTIVATED THE STUDY OF $\mathcal{M}_{2,2}$ AND ITS SUBGROUPS USING THE EXAMPLE OF DISORDERED FERMIONS.

UNFORTUNATELY, IN THE LITERATURE ON THIS SUBJECT IT IS OFTEN ASSUMED THAT GIVEN HOMOMORPHISMS:

$$G \xrightarrow{(t,c)} \underline{III} \subset M_{2,2}$$

WE HAVE

$$G \approx G_0 \times \underline{III}$$

BUT THIS IS NOT TRUE IN GENERAL!

AS WE HAVE LEARNED ABOVE, WHAT WE HAVE IS AN EXTENSION

$$1 \to G_0 \longrightarrow G \longrightarrow \underline{111} \longrightarrow 1$$

WITH $G_0 = \ker(t) \cap \ker(c)$,

BUT IT NEED NOT SPLIT, LET ALONE BE A DIRECT PRODUCT.

FORTUNATELY, THE MOST IMPORTANT RESULTS DO NOT RELY ON SUCH A PRODUCT STRUCTURE.

2 11. CLIFFORD ALGEBRAS

WHEN I TEACH GROUP THEORY AT RUTGERS I TAKE 2 OR 3 90-MIN. LECTURES TO EXPLAIN THE SUBJECT. HERE WE WILL CUT CORNERS AND GIVE A LIGHTNING REVIEW.

<u>DEF.</u> FOR $n \in \mathbb{Z}$, Cl_n IS THE $\mathbb{Z}_2$-GRADED ALGEBRA / $\mathbb{R}$ WITH $|n|$ <u>ODD</u> GENERATORS S.T.

$$e_i e_j + e_j e_i = \begin{cases} 2\delta_{ij} & n > 0 \\ -2\delta_{ij} & n < 0 \end{cases}$$

$i, j = 1, \cdots, n$

Exercise: What is the superdimension of Cl_n ?

CLIFFORD ALGEBRAS / $\mathbb{R}$ ARE CLOSELY RELATED TO SUPER-DIVISION ALGEBRAS / $\mathbb{R}$

RECALL WE HAD $\mathbb{R}, \mathbb{C}, \mathbb{H}$.
THESE ARE PURELY EVEN SUPER-DIVISION ALGEBRAS.

<u>DEF.</u> AN ASSOCIATIVE UNITAL SUPER-ALGEBRA OVER A FIELD k IS AN ASSOCIATIVE <u>SUPER DIVISION ALGEBRA</u> IF EVERY NONZERO <u>HOMOGENEOUS</u> ELEMENT IS INVERTIBLE.

THE SUPER-ALGEBRAS $C\ell_n$ FOR SMALL $|n|$ PROVIDE EXAMPLES OF SUPER-DIVISION ALGEBRAS.

EXAMPLE: Cl_{+2} = 4 REAL DIMENSIONAL SUPERALGEBRA GEN. BY ODD e_1, e_2 WITH

$$e_1e_2 = -e_2e_1 \qquad e_1^2 = e_2^2 = 1$$

$$Cl^0_{+2} = \mathbb{R} \oplus \mathbb{R}e_1e_2$$

$$Cl^1_{+2} = \mathbb{R}e_1 \oplus \mathbb{R}e_2$$

NOTE THAT:

$$(x + y\,e_1e_2)(x - y\,e_1e_2) = x^2 + y^2$$

$$(xe_1 + ye_2)^2 = x^2 + y^2$$

FROM WHICH YOU CAN CHECK THE SUPER-DIVISION PROPERTY

ALSO NOTE THAT

1.) $(1+e_1)(1-e_1) = 0$

SO NON-HOMOGENEOUS ELEMENTS MIGHT NOT BE INVERTIBLE: A SUPER-DIVISION ALGEBRA NEED NOT BE A DIVISION ALGEBRA!

2.) Cl_{+2} IS NOT A SUPER-MATRIX ALGEBRA!

AS AN UNGRADED ALGEBRA WE CAN REPRESENT

$$\underbrace{\rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{ODD}} \qquad \underbrace{\rho(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{EVEN !?!}}$$

SO AS AN UNGRADED ALGEBRA

$$Cl_{+2} \approx \mathbb{R}(2) = 2\text{x}2 \text{ REAL MATRICES}$$

THEOREM (WALL 1963; DELIGNE NOTES ON SPINORS)

THERE ARE TEN SUPER-DIVISION ALGEBRAS OVER $\mathbb{R}$!

WE WILL RELATE THEM TO THE 10 CT GROUPS BELOW

WE SEE 8 OF THEM LISTING THE LOW-DIM'L CLIFFORD ALGEBRAS / $\mathbb{R}$:

OF COURSE

$$Cl_0 = \mathbb{R} := \underbrace{D_0^s}_{\text{NOTATION FOR SUPER-DIVISION ALGEBRA}}$$

THE OTHERS ARE RELATED TO C.A.'S:

CLIFFORD ALGEBRA	AS UNGRADED ALGEBRA	AS SUPERALGEBRA
Cl_{-1}	$\mathbb{C}$	D^s_{-1}
Cl_{+1}	$\mathbb{R} \oplus \mathbb{R}$	D^s_{+1}
Cl_{-2}	$\mathbb{H}$	D^s_{-2}
Cl_{+2}	$\mathbb{R}(2)$	D^s_{+2}
Cl_{-3}	$\mathbb{H} \oplus \mathbb{H}$	D^s_{-3}
Cl_{+3}	$\mathbb{C}(2)$	D^s_{+3}
$Cl_{\pm 4}$	$\mathbb{H}(2)$	$\mathrm{End}(\mathbb{R}^{1\|1}) \hat{\otimes} D^s_{\pm 4}$
Cl_{-5}	$\mathbb{C}(4)$	$\mathrm{End}(\mathbb{R}^{1\|1}) \hat{\otimes} D^s_{+3}$
Cl_{+5}	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathrm{End}(\mathbb{R}^{1\|1}) \hat{\otimes} D^s_{-3}$
Cl_{-6}	$\mathbb{R}(8)$	$\mathrm{End}(\mathbb{R}^{2\|2}) \hat{\otimes} D^s_{+2}$
Cl_{+6}	$\mathbb{H}(4)$	$\mathrm{End}(\mathbb{R}^{2\|2}) \hat{\otimes} D^s_{-2}$
Cl_{-7}	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathrm{End}(\mathbb{R}^{3\|3}) \hat{\otimes} D^s_{+1}$
Cl_{+7}	$\mathbb{C}(8)$	$\mathrm{End}(\mathbb{R}^{3\|3}) \hat{\otimes} D^s_{-1}$

REMARKS:

1. $\hat{\otimes}$ HERE IS THE GRADED TENSOR PRODUCT OF SUPERALGEBRAS

2.

$$Cl_n \hat{\otimes} Cl_m \cong Cl_{n+m}$$

IF $\underbrace{n \geq 0 \text{ and } m \geq 0}$ OR $\underbrace{n \leq 0 \text{ and } m \leq 0}$

3. $D^s_{\pm 4} = \mathbb{H}$. NOTE THAT

$$Cl_{+4} \cong Cl_{-4}.$$

4. WHAT HAPPENS AT $n = \pm 8$?

FOR THE FIRST TIME WE DO GET A MATRIX SUPERALGEBRA

$$Cl_{\pm 8} \cong \mathrm{End}\left(\mathbb{R}^{8|8}\right)$$

MORITA EQUIVALENCE

TWO (SUPER)ALGEBRAS R_1, R_2 ARE MORITA EQUIVALENT IF THERE IS A (SUPER) VECTOR SPACE V WITH

$$R_1 \cong R_2 \hat{\otimes} \mathrm{End}(V)$$

(OR THE OTHER WAY AROUND)

THE POINT OF MORITA EQUIVALENCE IS THAT $\mathrm{End}(V)$ HAS A UNIQUE REPRESENTATION SO THAT R_1 AND R_2 HAVE "THE SAME" REPRESENTATION THEORY.

(TECHNICALLY $Rep(R_1)$ & $Rep(R_2)$ ARE EQUIVALENT CATEGORIES.)

WITH THE NOTION OF MORITA EQUIVALENCE WE HAVE

$$Cl_n \hat{\otimes} Cl_m \approx Cl_{n+m}$$

FOR ALL $n, m \in \mathbb{Z}$ (EITHER SIGN!).

SINCE Cl_{8n} IS A SUPER-MATRIX ALGEBRA THE MORITA CLASSES FORM AN ABELIAN GROUP $\approx \mathbb{Z}/8\mathbb{Z}$.

THIS IS OUR FIRST ENCOUNTER WITH BOTT PERIODICITY.

THE SUBSCRIPT α ON THE SUPER DIVISION ALGEBRA D^s_α $\alpha = 0, \pm 1, \pm 2, \pm 3, \pm 4$ IS THE MORITA CLASS IN $\mathbb{Z}/8\mathbb{Z}$.

CLIFFORD ALGEBRAS / $\mathbb{C}$

WHEN WE CHANGE THE GROUND FIELD TO $\mathbb{C}$ THE STRUCTURE SIMPLIFIES:

1. $\mathbb{C}\ell_n = \mathbb{C}\ell_{-n}$ SINCE WE CAN CHANGE GENERATORS $e_j \to \sqrt{-i}\, e_j$

2. $\mathbb{C}\ell_0$, $\mathbb{C}\ell_1$ ARE SUPERDIVISION ALGEBRAS / $\mathbb{C}$ (AND THEREFORE ALSO / $\mathbb{R}$)

$\mathbb{C}\ell_0 \cong \mathbb{C} := D_0^{\mathbb{C}}$ IS PURELY EVEN

$\mathbb{C}\ell_1 = \mathbb{C} \oplus \mathbb{C}\cdot e := D_1^{\mathbb{C}}$

BUT $\mathbb{C}l_2$ IS A MATRIX SUPERALGEBRA $\mathbb{C}l_2 \cong \mathrm{End}(\mathbb{C}^{1|1})$ BECAUSE WE CAN TAKE

$$\rho(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho(e_2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

NOW $\mathbb{C}l_{n+2} = \mathrm{End}(\mathbb{C}^{1|1}) \hat{\otimes} \mathbb{C}l_n$

$$\mathbb{C}l_{2n} \cong \mathrm{End}\left(\mathbb{C}^{2^{n-1}|2^{n-1}}\right)$$

$$\mathbb{C}l_{2n+1} \cong \mathrm{End}\left(\mathbb{C}^{2^{n-1}|2^{n-1}}\right) \hat{\otimes} \mathbb{C}l_1$$

SO MORITA EQUIVALENCE CLASSES / $\mathbb{C}$ FORM THE GROUP $\mathbb{Z}/2\mathbb{Z}$.

THIS IS BOTT PERIODICITY OVER THE COMPLEX #'S.

IN CONCLUSION, WE HAVE 10 SUPERDIVISION ALGEBRAS

$$\mathbb{C}l_0,\ \mathbb{C}l_1,\ Cl_0,\ Cl_{\pm 1},\ Cl_{\pm 2},\ Cl_{\pm 3}$$

AND $\mathbb{H}$.

NOW WE RETURN TO THE 10 CT GROUPS.

13. $\mathbb{Z}_2$-GRADED COREPS OF THE CT GROUPS

RETURN TO $M_{2,2} = \langle \bar{T}, \bar{C} \rangle$

$\bar{T}^2 = \bar{C}^2 = 1 \quad \bar{T}\bar{C} = \bar{C}\bar{T}.$

WE GAVE IT A $\mathbb{Z}_2$-GRADING

$$\phi(\bar{T}) = \phi(\bar{C}) = -1.$$

NOW LET US DEFINE A SECOND $\mathbb{Z}_2$-GRADING:

$$c : M_{2,2} \longrightarrow \mathbb{Z}_2$$

$$c(\bar{T}) = +1 \; , \; c(\bar{C}) = -1$$

WE DEFINE A <u>CT-MODULE</u> TO BE A $\mathbb{Z}_2$-GRADED COREP (WRT ϕ) V OF THE $\mathbb{Z}_2$-GRADED (WRT c) CT GROUP $\underline{\mathfrak{m}}^{c}$.

THEOREM: THERE IS A 1-1 CORRESPONDENCE BETWEEN CT GROUPS $\underrightarrow{\text{III}^{\tau}}$ AND MORITA CLASSES OF CLIFFORD ALGEBRAS OVER $\mathbb{R}, \mathbb{C}$ (EQUIVALENTLY, OF SUPER-DIVISION ALGEBRAS OVER $\mathbb{R}$), SUCH THAT THERE IS AN EQUIVALENCE OF REPRESENTATION THEORIES.

IDEA OF PROOF: GIVEN A CT MODULE V OF $\underline{\text{III}^{\tau}}$ WE HAVE POTENTIAL CLIFFORD GENERATORS:

$$C, \; iC, \; iCT \quad \underline{\underline{\text{OR}}} \quad CT$$

FOR EXAMPLE, IF $\underline{\mathbb{M}} = \{1, \bar{C}\}$

THEN

$$\rho(e_1) = C \qquad \rho(e_2) = iC$$

ARE <u>ODD</u> OPERATORS ON V

AND

$$\rho(e_1)\rho(e_2) + \rho(e_2)\rho(e_1) = 0$$

BECAUSE C IS ANTI-LINEAR

NOW THE TWO ϕ-TWISTED EXTS:

$$C^2 = (iC)^2 = +1 \ \underline{\text{OR}} \ -1$$

SHOW THAT V IS A CLIFFORD MODULE FOR

$$Cl_{+2} \ \underline{\text{OR}} \ Cl_{-2}$$

WHAT IS THE ROLE OF T?

IT IS EVEN AND ANTILINEAR.

THE ROLE OF T IS TO IMPOSE A REAL OR QUATERNIONIC STRUCTURE.

REMARK: THERE IS A MORE DIRECT WAY TO CONSTRUCT A SUPER-DIVISION ALGEBRA OUT OF A CT GROUP:

$$D = \bigoplus_{g \in |||^\tau} L_g$$

WHERE $L_g \cong \mathbb{C}$. THE CONSTRUCTION IS BEST EXPLAINED AFTER WE HAVE DISCUSSED CENTRAL EXTENSIONS OF GROUPOIDS.

IN SUMMARY THE CORRESPONDENCE IS:

SUBGROUP $\mathbb{H} \subset M_{2,2}$	T^2	C^2	CLIFFORD (MORITA CLASS)
$\{1\}$			$\mathbb{C}l_0$
$\{1, \bar{S}\}$			$\mathbb{C}l_1$
$\{1, \bar{T}\}$	$+1$		Cl_0
$M_{2,2}$	$+1$	-1	Cl_{-1}
$\{1, \bar{C}\}$		-1	Cl_{-2}
$M_{2,2}$	-1	-1	Cl_{-3}
$\{1, \bar{T}\}$	-1		Cl_{-4}
$M_{2,2}$	-1	$+1$	Cl_{-5}
$\{1, \bar{C}\}$		$+1$	Cl_{-6}
$M_{2,2}$	$+1$	$+1$	Cl_{-7}

14. FINITE-DIMENSIONAL FREE FERMIONS WITH SYMMETRY

14 A. FINITE DIMENSIONAL FERMIONIC SYSTEMS (FDFS)

WE DEFINE A FDFS TO CONSIST OF

- A REAL VECTOR SPACE $\mathcal{M}$ (THE MODE SPACE) OF EVEN DIMENSION WITH A POSITIVE DEFINITE BILINEAR FORM b.
- AN IRREDUCIBLE $*$- MODULE $\mathcal{H}_F$ FOR THE $*$-ALGEBRA

$$\mathcal{A} = \mathrm{Cliff}(\mathcal{M}, b) \otimes \mathbb{C}$$

WLOG WE MAY TAKE $\mathcal{M} = \mathbb{R}^{2N}$ WITH THE EUCLIDEAN ALGEBRA. THEN $Cliff(\mathcal{M}, b)$ IS JUST $C\ell_{2N}$:

CHOOSE AN ON BASIS c_i FOR $\mathcal{M}$

THEN

$$c_i c_j + c_j c_i = 2\delta_{i,j}$$

$i,j = 1, \ldots, 2N$

N.B. THIS CLIFFORD ALGEBRA IS PHYSICALLY & CONCEPTUALLY <u>COMPLETELY DIFFERENT</u> FROM THE ONE ASSOCIATED TO THE CT GROUPS $\underline{\text{III}}^{\tau}$. HERE THE c_i REPRESENT PHYSICAL FERMIONIC MODES.

NOW WE FORM

$$\mathcal{A} = \mathrm{Cliff}(\mathcal{M}, b) \otimes \mathbb{C}$$

IT IS A CLIFFORD ALGEBRA $/\mathbb{C}$

BUT ALSO CONTAINS A $*$-STRUCTURE:

$$a \longmapsto a^* \qquad (ab)^* = b^* a^*$$

SO THAT $c \in \mathcal{M} \longrightarrow c^* = c$

[N.B. SINCE WE HAVE A SUPERALGEBRA WE MIGHT WISH TO DEFINE A $*$-STRUCTURE SO THAT

$$(ab)^* = (-1)^{|a| \cdot |b|} b^* a^*$$

FOR HOMOGENEOUS ELEMENTS, a, b.

DEFINING

$$a^{\tilde{*}} = \begin{cases} a^* & |a| = 0 \\ i a^* & |a| = 1 \end{cases}$$

CONVERTS BACK TO THE USUAL CONVENTION]

SINCE $\mathbb{C}\ell_{2N}$ IS A MATRIX SUPERALGEBRA WE CAN TAKE

$$\mathcal{H}_F \cong \mathbb{C}^{2^{N-1}|2^{N-1}}$$

THIS IS A FERMIONIC FOCK SPACE.

THERE IS A UNIQUE SUCH MODULE <u>UP TO ISOMORPHISM</u>, HERE, UP TO A CHOICE OF VACUUM FOR THE $\{c_i\}$.

THE ISOMORPHISM CAN BE PHYSICALLY SIGNIFICANT - SEE COMMENTS ON BOGOLIUBOV TMN'S BELOW.

THE $*$ MODULE MEANS THAT IF WE PUT THE USUAL HERMITIAN CONJUGATION $*$-STRUCTURE ON $\mathrm{End}(\mathcal{H}_F)$ THEN

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathrm{End}(\mathcal{H}_F) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & \longmapsto & \hat{a} \end{array}$$

SATISFIES

$$\widehat{a^*} = (\hat{a})^*$$

14B. FDFS WITH SYMMETRY

LET (G,ϕ) BE A $\mathbb{Z}_2$-GRADED GROUP. THEN (G,ϕ) ACTS AS A SYMMETRY ON THE FDFS IF

- THERE IS A HOMOMORPHISM

$$\alpha : G \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{M}, b) = O(\mathcal{M}, b) \cong O(2N)$$

- THERE IS A HOMOMORPHISM

$$\rho : G \longrightarrow \mathrm{Aut}_{qtm}(\mathcal{H}_F)$$

MAKING $\mathcal{H}_F$ A COREP. OF (G,ϕ)

- THESE ARE COMPATIBLE:

$$\rho(g)\,\hat{a}\,\rho(g)^{-1} = \widehat{\alpha(g)\cdot a}$$

IN TERMS OF A BASIS WE ARE SAYING THAT

$$\rho(g)\,\hat{c}_j\,\rho(g)^{-1} = \sum_{m=1}^{2N} S(g)_{mj}\,\hat{c}_m$$

where $g \mapsto S(g) \in O(2N)$ IS A REPRESENTATION OF G BY ORTHOGONAL MATRICES.

REMARKS

1. $S(g) \in O(2N) \iff$ CCR's PRESERVED

2. THE DATA OF THE COREP ϕ ENTERS IN THE EXTENSION OF α TO $\mathcal{A}$.

14C. FREE FERMION DYNAMICS

PHYSICAL OBSERVABLES, SUCH AS THE HAMILTONIAN, ARE SELF-CONJUGATE ELEMENTS OF THE $*$-ALGEBRA OF OPERATORS $\mathcal{A}$.

<u>DEF:</u> <u>FREE</u> FERMION DYNAMICS IS DEFINED BY ANY HAMILTONIAN QUADRATIC IN $\mathcal{M}$:

$$H = \frac{\sqrt{-1}}{4} \sum_{j,k} A_{jk} c_j c_k + h$$

WHERE $h \in \mathbb{R}$ IS A CONSTANT.

NOTE THAT OFF-DIAGONAL ELEMENTS OF A_{jk} ARE, WLOG, ANTISYMMETRIC, AND DIAGONAL ELEMENTS GIVE A CONSTANT

SINCE H IS HERMITIAN WE CAN - WLOG - TAKE A_{jk} TO BE A REAL, ANTI-SYMMETRIC MATRIX.

WE CAN THEREFORE IDENTIFY THE ENSEMBLE OF FREE FERMION HAMILTONIANS WITH

$$so(2N) \oplus \mathbb{R}$$

WHERE $so(2N)$ IS THE SET OF $2N \times 2N$ REAL ANTISYMMETRIC MATRICES.

OF COURSE, $so(2N)$ IS ALSO A LIE ALGEBRA. MORE ON THAT LATER.

14D. SYMMETRIES OF FREE FERMION DYNAMICS

NOW SUPPOSE WE HAVE A FDFS WITH SYMMETRY, AND WE HAVE A FREE HAMILTONIAN.

SO WE HAVE THE DATA

$$(\mathcal{M}, b, \mathcal{H}_F, G, \phi, \alpha, \rho, H)$$

WE SAY THAT (G, ϕ) ACTS AS A SYMMETRY OF THE FREE FERMION DYNAMICS IF THERE EXISTS A HOMOMORPHISM

$$t : G \longrightarrow \{\pm 1\}$$

SO THAT

$$\boxed{\rho(g)\,U(\tau)\,\rho(g)^{-1} = U(\tau)^{t(g)} = U(t(g)\tau)}$$

THIS IS EQUIVALENT TO THE EXISTENCE OF A HOMOMORPHISM

$$c : G \longrightarrow \{\pm 1\}$$

WITH

$$\boxed{\rho(g)\,\hat{H}\,\rho(g)^{-1} = c(g)\,\hat{H}} \quad (*)$$

WHERE, AS USUAL,

$$\phi \cdot c \cdot t = 1 .$$

IN TERMS OF THE MATRICES $S = S(g)$ AND A_{jk} DEFINED ABOVE

$$(*) \Longleftrightarrow \boxed{S A S^{tr} = t(g) A}$$

(EXERCISE)

14E. NAMBU-DIRAC SPACE

AN UNUSUAL FEATURE OF FREE FERMIONS IS THAT, IN ADDITION TO $\mathcal{H}_F$ AND $\mathcal{H}_{DN}$ THERE IS ANOTHER HILBERT SPACE AND HAMILTONIAN ASSOCIATED TO THE SYSTEM.

HISTORICALLY, THE SUBJECT BEGAN WITH THE DIRAC-NAMBU STRUCTURE IN THE "FIRST-QUANTIZATION" OF THE DIRAC EQUATION. IT WAS (AND CONTINUES TO BE) A SOURCE OF CONFUSION.

(THIS SPECIAL STRUCTURE IS ALSO IMPORTANT IN THE APPLICATION TO K-THEORY.)

ON $\mathcal{M}$ LET US <u>CHOOSE</u> A COMPLEX STRUCTURE: $I : \mathcal{M} \to \mathcal{M}$ WITH $I^2 = -1$.

THIS CHOICE ALLOWS US TO INTRODUCE "CREATION" AND "ANNIHILATION" OPERATORS

$$\mathcal{M} \otimes \mathbb{C} \cong V \oplus \overline{V}$$

$$V = P_- \mathcal{M} = \{ v \in \mathcal{M} \otimes \mathbb{C} \mid Iv = iv \}$$

$$\overline{V} = P_+ \mathcal{M} = \{ v \in \mathcal{M} \otimes \mathbb{C} \mid Iv = -iv \}$$

$$P_{\pm} = \frac{1}{2} \left(1 \pm I \otimes i \right) : \text{PROJECTION OPERATORS}$$

V IS THE SPACE OF "CREATION OPERATORS a^+"

$\overline{V}$ IS THE SPACE OF "ANNIHILATION OPERATORS a"

HAVING CHOSEN I, WE NOW GET A CONCRETE $*$-MODULE

$$\mathcal{H}_F \cong \Lambda^* V = \bigoplus_{j=0}^{N} \Lambda^j V$$

$\mathcal{H}_F$ IS JUST FERMIONIC FOCK SPACE, WITH

$$\Lambda^0 V \cong \mathbb{C}$$

CORRESPONDING TO THE VACUUM LINE:

$$\hat{a}|\Omega\rangle = 0 \quad \forall\ a \in \overline{V}$$

FOR EXAMPLE, IF I IS THE CANONICAL

$$\left.\begin{aligned} I_0(c_{2j-1}) &= -c_{2j} \\ I_0(c_{2j}) &= c_{2j-1} \end{aligned}\right\} \quad j=1,\dots,N$$

THEN WE DEFINE

$$a_j^+ := P_- c_{2j-1} = \tfrac{1}{2}(1 - I\otimes i)c_{2j-1}$$

$$a_j^+ = \tfrac{1}{2}(c_{2j-1} + i\, c_{2j})$$

$$a_j := P_+ c_{2j-1} = \tfrac{1}{2}(1 + I\otimes i)c_{2j-1}$$

$$a_j = \tfrac{1}{2}(c_{2j-1} - i\, c_{2j})$$

AND WE HAVE THE FAMILIAR EQS:

$$\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0$$

$$\{a_j, a_k^\dagger\} = \delta_{jk} \qquad j,k = 1, \dots, N$$

$$a_j |\Omega\rangle = 0 \qquad j = 1, \dots N$$

$$\Lambda^\ell V \cong \mathrm{Span}\{ a_{j_1}^\dagger \cdots a_{j_\ell}^\dagger |\Omega\rangle \}$$

BUT BEAR IN MIND THAT THIS STRUCTURE DEPENDS ON A <u>CHOICE</u> OF COMPLEX STR. I ON $\mathcal{M}$.

$$I \in O(2N)/U(N)$$

NOW V IS ITSELF A HILBERT SPACE, WITH HERMITIAN FORM:

$$h(v_1, v_2) := b(\overline{v_1}, v_2)$$

THEREFORE

$$\mathcal{H}_{DN} := V \oplus \overline{V} \cong M \otimes \mathbb{C}$$

WITH THE DIAGONAL HERMITIAN FORM IS A <u>HILBERT SPACE</u>.

WE WILL CALL IT THE DIRAC-NAMBU HILBERT SPACE.

MOREOVER, THE HAMILTONIAN ELEMENT $H \in \Lambda^2 V$ DEFINES AN HERMITIAN OPERATOR H_{DN} ON $\mathcal{H}_{DN}$:

$$c \in \mathcal{M} \longmapsto [H, c] \in \mathcal{A}$$

$$c_i \longmapsto \frac{\sqrt{-1}}{2} \sum A_{ji} c_j$$

AND EXTEND THIS $\mathbb{C}$-LINEARLY.

15. THE FREE FERMION DYSON PROBLEM & THE ALTLAND-ZIRNBAUER CLASSIFICATION

15A. STATEMENT OF THE PROBLEM

SUPPOSE WE ARE GIVEN A FDFS WITH SYMMETRY, SO WE HAVE DATA

$$\left(\mathcal{M}, b, \mathcal{H}_F, G, \phi, \alpha, \rho\right)$$

NOW WE PROVIDE ONE MORE PIECE OF DATA, A HOMOMORPHISM

$$t : G \to \{\pm 1\}$$

OR, EQUIVALENTLY,

$$c : G \longrightarrow \{\pm 1\}$$

THE "FREE FERMION DYSON PROBLEM" IS TO FIND THE ENSEMBLE $\mathcal{E}_{FF}$ OF <u>FREE</u> HAMILTONIANS $\hat{H} \in \mathrm{End}(\mathcal{H}_F)$ COMPATIBLE WITH THE SYMMETRY:

$$\rho(g)\hat{H} = c(g)\hat{H}\rho(g)$$

EQUIVALENTLY, WRITING

$$\hat{H} = \frac{\sqrt{-1}}{4}\sum_{j,k} A_{jk}\,\hat{c}_j\,\hat{c}_k$$

WE WANT

$$\boxed{\mathrm{Ad}_{\alpha(g)} A := S(g) A S(g)^{tr} = t(g) A} \quad (*)$$

N.B. WE HAVE CHANGED THE DYSON PROBLEM FOR $(G, \phi, \mathcal{H}_F)$ IN <u>TWO</u> <u>WAYS</u>

1. WE RESTRICT TO <u>FREE</u> HAMILTONIANS

2. WE ALLOW "SYMMETRIES" TO ANTICOMMUTE WITH H, BY ALLOWING $c : G \to \{\pm 1\}$ TO BE NONTRIVIAL.

<u>REMARK</u>: OF COURSE, EQUATION (*) DOES NOT INVOLVE c, ϕ, ρ. SO IF WE INSIST ON KEEPING $c = 1$ WE CAN MODIFY ϕ ACCORDINGLY.

15B. FREE HAMILTONIANS & THE ORTHOGONAL LIE ALGEBRA

DEF. $so(M)$ = LIE ALGEBRA OF $M \times M$ REAL ANTISYMMETRIC MATRICES

EXERCISE: LET $T_{ij} := e_{ij} - e_{ji}$ FOR $i \neq j$, $i,j = 1, \cdots, M$. HERE e_{ij} = Matrix with 1 in (i,j) place and zero otherwise

THEN T_{ij} FORM A BASIS FOR $so(M)$ WITH STRUCTURE CONSTANTS

$$[T_{ij}, T_{kl}] = \delta_{j,k} T_{il} \pm 3 \text{ TERMS}$$

REMARK: THE QUADRATIC ELEMENTS IN THE CLIFFORD ALGEBRA

$$\mathrm{Cliff}(\mathcal{M}, b) \approx Cl_{2N}$$

FORM A LIE ALGEBRA ISOMORPHIC TO $so(2N)$:

DEFINE $\tau_{ij} := \frac{1}{2} c_i c_j \quad i \neq j$

SO $\tau_{ij} = -\tau_{ji}$ AND

$$[\tau_{ij}, \tau_{k\ell}] = \delta_{jk} \tau_{i\ell} \pm 3 \text{ TERMS}$$

- $\mathcal{H}_{DN}$ IS THE VECTOR REP. $\mathbb{R}^{2N} \otimes \mathbb{C}$
- $\mathcal{H}_F$ IS THE (NONCHIRAL) SPIN REP.

15C. CARTAN DECOMPOSITIONS

LET US RETURN TO THE FREE FERMION DYSON PROBLEM, WHICH WE HAVE REDUCED TO FINDING THE SET OF $A \in so(2N)$ SUCH THAT:

$$Ad_{\alpha(g)} A := S(g)\, A\, S(g)^{tr} = t(g) A$$

DEFINE:

$$k := \{ A \mid Ad_{\alpha(g)} A = A \}$$

$$p := \{ A \mid Ad_{\alpha(g)} A = t(g) A \}$$

NOTE THAT k IS A LIE ALGEBRA: $A_1, A_2 \in k \Longrightarrow [A_1, A_2] \in k$

BUT $\mathfrak{p}$ IS <u>NOT</u> A LIE ALGEBRA IN GENERAL:

SUPPOSE A_1 & $A_2 \in \mathfrak{p}$, i.e.

$$Ad_{\alpha(g)} A_i = t(g) A_i \qquad i = 1, 2$$

THEN WE COMPUTE:

$$Ad_{\alpha(g)}\left([A_1, A_2]\right) = [A_1, A_2]$$

SO $[A_1, A_2] \in k$, NOT $\mathfrak{p}$!

SIMILARLY:

$$[k, \mathfrak{p}] \subset \mathfrak{p}$$

NEVERTHELESS, WE CAN PUT A LIE ALGEBRA STRUCTURE ON

$$\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$$

$$[k_1 \oplus p_1, k_2 \oplus p_2] := \left([k_1, k_2] + [p_1, p_2]\right) \oplus \left([k_1, p_2] + [p_1, k_2]\right)$$

IN LIE ALGEBRA THEORY A CARTAN INVOLUTION IS A LIE ALGEBRA AUTOMORPHISM Θ WHICH SQUARES TO $+1$.

$$\mathfrak{g} = \underbrace{\mathfrak{k}}_{\Theta = +1} \oplus \underbrace{\mathfrak{p}}_{\Theta = -1}$$

WHEN $\mathfrak{g}, \mathfrak{k}$ ARE SIMPLE LIE ALGEBRAS THEY EXPONENTIATE TO GROUPS $\mathcal{G}, \mathcal{K}$ AND $\mathfrak{p}$ IS IDENTIFIED WITH THE TANGENT SPACE OF $\mathcal{G}/\mathcal{K}$ AT THE COSET $\mathcal{K}$

$$T_{\mathcal{K}}(\mathcal{G}/\mathcal{K}) \cong \mathfrak{p}$$

THEN $\mathcal{G}/\mathcal{K}$ ARE CALLED <u>CARTAN</u> <u>SYMMETRIC</u> <u>SPACES</u>

THUS THE ENSEMBLE $\mathcal{E}_{FI}$ MAY BE IDENTIFIED WITH THE TANGENT SPACE TO $\mathcal{G}/\mathcal{K}$

IT IS ALSO NOT HARD TO SHOW:

1. $\mathfrak{g}$, $\mathfrak{k}$ ARE <u>COMPACT</u> LIE ALGEBRAS. THEY EXPONENTIATE TO COMPACT LIE GROUPS.

2. $\mathfrak{g}$, $\mathfrak{k}$ ARE CLASSICAL LIE ALGEBRAS, i.e. OF ABCD TYPE.

THESE STATEMENTS FOLLOW FROM THE EMBEDDING

$$\mathfrak{g} \longrightarrow o(2N) \oplus o(2N)$$

$$k \oplus p \longmapsto (k+p) \oplus (k-p)$$

15D CARTAN'S LIST OF THE SYMMETRIC SPACES

CARTAN CLASSIFIED THE SYMMETRIC SPACES WITH $\mathfrak{g}$, $\mathfrak{k}$ SIMPLE LIE ALGEBRAS.

THERE ARE COMPACT AND NONCOMPACT FORMS, AND SOME INVOLVE EXCEPTIONAL LIE ALGEBRAS,

WHEN WE RESTRICT TO COMPACT FORMS WITH $\mathfrak{k}, \mathfrak{p}$ OF CLASSICAL TYPE: A_N, B_N, C_N, D_N (WHICH HAVE LARGE N LIMITS) WE GET

TEN CASES

A	$U(r)/U(r-s)\times U(s)$
AIII	$U(r)\times U(r)/U(r)$
BDI	$O(r)/O(r-s)\times O(s)$
D	$O(r)\times O(r)/O(r)$
DIII	$O(r)/U(r)$
AII	$U(r)/Sp(r/2)$
CII	$Sp(r)/Sp(r-s)\times Sp(s)$
C	$Sp(r)\times Sp(r)/Sp(r)$
CI	$Sp(r)/U(r)$
AI	$U(2r)/O(r)$

IS THERE A RELATION WITH THE 10 AZ GROUPS AND THE 10 SUPER-DIVISION ALGEBRAS?

YES! WE HAVE ALREADY EXPLAINED THE RELATION OF CT-MODULES AND CLIFFORD MODULES: IF OUR HILBERT SPACE IS A CT-MODULE THEN IT IS A CLIFFORD MODULE. BUT THE TWISTINGS OF KR THEORY ON $pt//\mathbb{Z}_2$ AND $S^0//\mathbb{Z}_2$ ARE GIVEN (UP TO ISOMORPHISM) BY THE MORITA CLASSES OF CLIFFORD ALGEBRAS.

THE TEN TWISTINGS GIVE THE GROUPS

$$K^{-j}(pt) \quad j = 0, 1$$

$$KO^{-j}(pt) \quad j = 0, 1, 2, 3, 4, 5, 6, 7$$

IN K-THEORY WE USE A SEQUENCE "OF CLASSIFYING SPACES" $\mathcal{S}_j$ SO THAT

$$h^{-j}(X) = [X, \mathcal{S}_j]$$

FOR THE COHOMOLOGY THEORIES $h = K, KO$ THE $\mathcal{S}_j$ ARE THE LARGE N LIMIT OF THE TEN CARTAN SYMMETRIC SPACES.

FOR THE MOMENT WE JUST GIVE THE SOLUTION TO THE FREE-FERMION DYSON PROBLEM:

WE DECOMPOSE $\mathcal{M}$ INTO ISOTYPICAL COMPONENTS FOR G

$$\mathcal{M} = \bigoplus_{\lambda} N_{\lambda} \otimes_{\mathbb{R}} V_{\lambda}$$

V_{λ} = IRREPS OF G OVER $\mathbb{R}$.

$$\mathcal{E}_{FF} = \prod_{\lambda} \mathcal{P}_{\lambda}$$

WHERE $\mathcal{P}_{\lambda}$ IS THE TANGENT SPACE TO ONE OF THE 10 CLASSES OF SYMMETRIC SPACES.

THIS IS A VERSION OF THE "ALTLAND-ZIRNBAUER CLASSIFICATION."

SEE:

- Heinzner, Huckleberry, Zirnbauer "Symmetry Classes of Disordered Fermions" arXiv:math-ph/0411040
- Zirnbauer, "Properties of random matrix theory," Oxford Handbook of Random Matrix Theory
- A. Ludwig, et. al. arXiv:0912.2157

FOR ALTERNATIVE, SIGHTLY DIFFERENT (AND INCOMPATIBLE !??!) ACCOUNTS...

REMARK: A VERY SIMILAR DISCUSSION ALSO APPLIES TO FREE BOSONS:

$(\mathcal{M}, b)$: Real symplectic vector space

$$\mathcal{A} = \mathrm{Heis}(\mathcal{M}, b) = \left(T(\mathcal{M}) \otimes \mathbb{C}\right) / \mathcal{J}$$

$T(\mathcal{M})$ = Tensor algebra

$\mathcal{J}$ = Ideal generated by

$$v v' - v' v - \sqrt{-1}\, b(v, v') \cdot \mathbb{1}$$

$\forall\, v, v' \in \mathcal{M}$

To get a Fock space choose a complex structure I on $\mathcal{M}$ so that $b(Iv, Iv') = b(v,v')$

Then

$$\mathcal{M} \otimes \mathbb{C} \cong V \oplus \bar{V}$$

$$\mathcal{H}_{Fock} \cong \mathrm{Sym}^* V$$

A symmetry is defined by

$$\alpha : G \longrightarrow Sp(\mathcal{M}, b; \mathbb{R})$$

$$\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H}_F)$$

With

$$\rho(g)\, \hat{a}\, \rho(g)^{-1} = \widehat{\alpha(g)\cdot a}$$

as before.

Now, up to a c-number, a free boson Hamiltonian is of the form

$$H = H^{ij} v_i v_j$$

$\{v_i\}$ basis for $\mathcal{M}$.

Choosing a symplectic basis so that

$$b(v_i, v_j) = J_{ij} = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)$$

We have:

$$(HJ)^{ij} \in sp(\mathcal{M}, b; \mathbb{R})$$

$$= \{ A \in M_{2N}(\mathbb{R}) \mid A^{tr} J + JA = 0 \}$$

Now

$$\rho(g)\, \hat{V}_i\, \rho(g)^{-1} = \sum_m S_{mi}(g)\, \hat{V}_m$$

So

$$\rho(g)\, \hat{H}\, \rho(g)^{-1} = c(g)\, \hat{H}$$

$$\Longleftrightarrow$$

$$\left(S(g) H S(g)^{tr} \right)^{ij} = c(g)\, H^{ij}$$

$$\Longleftrightarrow$$

$$S(g) A S(g)^{-1} = c(g) A$$

$$A^{ij} = (HJ)^{ij}$$

$\therefore$

$$\mathfrak{p} := \{ A \mid \mathrm{Ad}_{\alpha(g)}(A) = c(g) A \}$$

For bosons $c(g) = 1$ so $\mathfrak{p}$ is already a Lie subalgebra of $sp(\mathcal{M}, b; \mathbb{R})$.

It need not be compact since the Killing form on $sp(\mathcal{M}, b; \mathbb{R})$ is not definite.

It will be a classical Lie algebra, being the commutant of a matrix group

16. REALIZING ALL THE CLASSES USING BOTT'S SEQUENCE OF GROUPS

WE NOW GIVE EXAMPLES TO SHOW THAT ALL 10 CASES CAN ACTUALLY BE REALIZED.

LET US BEGIN WITH A CLIFFORD ALGEBRA Cl_{-8d} FOR SOME SUFFICIENTLY LARGE POSITIVE INTEGER d.

THIS IS A MATRIX ALGEBRA, SO WE CAN REPRESENT e_i BY $J_i \in \mathrm{End}(\mathbb{R}^{2N})$ (WITH $2N = 2^{4d} := 16r$)

OF COURSE, WE HAVE

$$J_i^2 = -1 \quad i = 1, \cdots, d$$

$$J_i J_j + J_j J_i = 0 \quad i \neq j, \; i,j = 1, \cdots, d$$

BUT WLOG WE CAN ALSO TAKE

$$J_i \in O(2N), \text{ so } J_i^{tr} = -J_i.$$

<u>DEF.</u> THE <u>SUBGROUP</u> OF $C\ell_n$ GENERATED BY e_i, $i = 1, \cdots, n$ IS $Pin^{\pm}(n)$ WHERE $\pm = \mathrm{sign}(n)$.

REMARK: $1 \to \mathbb{Z}_2 \to Pin^{\pm}(n) \longrightarrow O(n) \to 1$

THE e_i ARE LIFTS OF REFLECTIONS IN COORDINATE PLANES.

THUS RETURNING TO OUR ORTHOGONAL CLIFFORD OPERATORS J_i, $i=1,\cdots,8d$ WE HAVE A COLLECTION OF HOMOMORPHISMS

$$\alpha_n : Pin^-(n) \hookrightarrow O(2N)$$

For all $n=1,\cdots,8d$.

NOW DEFINE A SEQUENCE OF GROUPS

$$G_0 = O(2N) = O(16r)$$

$$G_i = \{ g \in G_0 \mid g J_s = J_s g \,,\ s=1,\cdots i \}$$

$i=1,2,\cdots$

WHAT IS G_1 ?

$J_1^2 = -1$ IS A COMPLEX STRUCTURE ON $\mathbb{R}^{2N}$.

$\Rightarrow \quad G_1 \approx U(N) = U(8r)$

WHAT ABOUT G_2 ?

$J_1^2 = -1$ & $J_2^2 = -1$ & $\{J_1, J_2\} = 0$

DEFINE A <u>QUATERNIONC STR.</u> ON $\mathbb{R}^{2N}$

$\Rightarrow \quad G_2 \approx Sp(4r)$,

THE GROUP OF $4r \times 4r$ UNITARY MATRICES OVER $\mathbb{H}$.

WHAT ABOUT G_3 ?

NOW $P_{\pm} = \frac{1}{2}(1 \pm J_1 J_2 J_3)$

ARE PROJECTION OPERATORS WHICH COMMUTE WITH THE $\mathcal{H}$- STR. GENERATED BY J_1, J_2:

$$G_3 \simeq Sp(2r) \times Sp(2r)$$

CONTINUING, WE GET THE SEQUENCE OF GROUPS G_i :

$$O(16r) \supset U(8r) \supset Sp(4r) \supset Sp(2r) \times Sp(2r) \supset$$

$$\supset Sp(2r) \supset U(2r) \supset O(2r) \supset O(r) \times O(r) \supset$$

$\boxed{\supset O(r)}$ ← PERIODIC !

NOW CONSIDER:

$$\mathcal{O}_i = \left\{ h \in O(16r) \;\middle|\; \begin{array}{l} h^2 = -1 \quad \} \\ \{h, J_s\} = 0 \quad s = 1, \cdots i \end{array} \right\}$$

NOTE THAT $J_{i+1} \in \mathcal{O}_i$.

THE TANGENT SPACE

$$T_{J_{i+1}} \mathcal{O}_i = \left\{ A \in so(16r) \;\middle|\; \{A, J_s\} = 0, \; s = 1, \cdots i{+}1 \right\}$$

TAKING $G = Pin^-(i+1)$ AND

$\alpha = \alpha_{i+1} : G \hookrightarrow O(16r)$ AND

$t(e_s) = -1$ WE RECOGNIZE

$$\mathfrak{p} = T_{J_{i+1}} \mathcal{O}_i$$

$$\mathfrak{k} = \mathfrak{g}_{i+1} = \text{LIE ALGEBRA OF } G_{i+1}$$

ON THE OTHER HAND,

$$\mathcal{O}_i \approx G_i / G_{i+1}$$

IS A CLASSICAL SYMMETRIC SPACE.

INDEED, ONE CAN SHOW THAT EVERY $h \in \mathcal{O}_i$ IS OF THE FORM

$$h = g J_{i+1} g^{-1} \quad , \quad g \in O(2N)$$

THIS SHOWS THE 8 CASES BDI $\rightarrow$ AI ABOVE ARE REALIZED.

REMARK: THE LIE ALGEBRAS OF THE GROUPS G_i ARE

$$\mathfrak{g}_i = \{ A \in \mathfrak{o}(16r) \mid [A, J_s] = 0,\ s = 1, \cdots, i \}$$

ONE CAN SHOW THAT $\theta = \mathrm{Ad}(J_{i+1})$ IS A CARTAN INVOLUTION OF $\mathfrak{g}_i$ AND

$$\mathfrak{g}_i = \underbrace{\mathfrak{g}_{i+1}}_{\theta = +1} \oplus \underbrace{\mathfrak{p}_i}_{\theta = -1}$$

THEN THE ISOMORPHISM

$$\mathfrak{p}_i \longrightarrow T_{J_{i+1}} \Theta_i$$

$$a \longmapsto J_{i+1} a$$

FINALLY FOR A COMPLEX ANALOG TO CAPTURE THE TWO REMAINING CASES WE TAKE

- $G = U(1)$ AND $t = 1$ (NECESSARILY)

LET $R(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

AND DEFINE $\alpha : G \to O(2N)$:

$$\alpha : e^{i\theta} \longmapsto \begin{pmatrix} R(\theta) & & \\ & \ddots & \\ & & R(\theta) \end{pmatrix}$$

IT IS EASY TO SHOW

$$\mathfrak{k} \cong \mathfrak{u}(N) \text{ \& } \mathfrak{p} \cong \mathfrak{u}(N)$$

SO

$$\mathcal{G}/\mathcal{K} = U(N) \times U(N) / U(N)$$

INDEED, INTRODUCING THE STANDARD COMPLEX STRUCTURE

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

(WHICH GENERATES $\mathrm{Im}\,\alpha \subset O(2N)$)

$$a_i \longrightarrow e^{i\theta} a_i, \quad a_i^{+} \rightarrow e^{-i\theta} a_i^{+}$$

SO THE MOST GENERAL $\hat{H}$ IS

$$\hat{H} = \sum_{i,j=1}^{N} h_{ij}\, a_i^{+} a_j$$

WITH h_{ij} HERMITIAN.

FOR THE LAST CASE TAKE

$$G = Pin^c(1) = \{ \lambda S \mid S^2 = 1,\ \lambda S = S\lambda \}$$

$$t(S) = -1$$

$$\alpha(S) = \left(\begin{array}{c|c} \mathbb{1}_{2n} & \\ \hline & \mathbb{1}_{2N-2n} \end{array} \right)$$

$\alpha(e^{i\theta}) =$ AS BEFORE.

NOW $\quad k \cong u(n) \oplus u(N-n)$

$$k \oplus p \cong u(N)$$

SO

$$G/K \cong U(N) / U(n) \times U(N-n)$$

17. EXAMPLE OF A FREE FERMION OSCILLATOR

TAKE $\mathcal{M} = \mathbb{R}^2$ WITH STANDARD METRIC.

WE HAVE

$$A = \epsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon \in \mathbb{R}$$

WRT ON BASIS $\{c_1, c_2\}$

THERE ARE TWO COMPLEX STRUCTURES

$$I_\pm : \begin{array}{l} c_1 \longrightarrow \mp c_2 \\ c_2 \longrightarrow \pm c_1 \end{array}$$

Then

$$a^\dagger = \frac{1}{2}\left(1 - I \otimes i\right) c_1 = \begin{cases} \frac{1}{2}(c_1 + i c_2) & I_+ \\ \frac{1}{2}(c_1 - i c_2) & I_- \end{cases}$$

$$a = \frac{1}{2}\left(1 + I \otimes i\right) c_1 = \begin{cases} \frac{1}{2}(c_1 - i c_2) & I_+ \\ \frac{1}{2}(c_1 + i c_2) & I_- \end{cases}$$

These satisfy

$$a^2 = (a^\dagger)^2 = 0 \qquad \{a, a^\dagger\} = 1$$

NOW $\quad H = \frac{i\epsilon}{4}(c_2 c_1 - c_1 c_2)$

FOR $I = I_+$:

$$H = \frac{\epsilon}{2}(a^\dagger a - a a^\dagger) = \epsilon\left(a^\dagger a - \frac{1}{2}\right)$$

Now we can consider examples of symmetries:

$G = O(2) \quad \alpha = \mathrm{Id} \quad \phi = \det$

If $P: c_1 \to c_1, \quad c_2 \to -c_2$

$P a P^{-1} = a \quad P a^{+} P^{-1} = a^{+}$

So $\quad P H P^{-1} = H \Rightarrow c(g) = 1.$

$\Rightarrow t(g) = \det(g).$

If instead we take

$G = O(2), \ \alpha = \mathrm{Id}, \ \phi(g) = 1$

then

$PaP^{-1} = a^{+}$ $\quad$ $Pa^{+}P^{-1} = a$

So $\quad PHP^{-1} = -H \Longrightarrow$

$$c(g) = \det(g)$$

Once again, $t(g) = \det(g)$, as expected.

If instead we take

$$G = M_2 = \{1, \bar{C}\}$$

$$\phi(\bar{C}) = -1$$

but $\alpha : G \longrightarrow O(2)$ is the trivial homomorphism then

$$\bar{C} a \bar{C}^{-1} = a^{+} \quad \bar{C} a^{+} \bar{C}^{-1} = a$$

so

$$\bar{C} H \bar{C}^{-1} = -H$$

and hence $t(\bar{C}) = +1$.

Thus, depending on the symmetry group already for a single fermionic H.O. We can realize all 3 nontrivial possibilities:

$\phi(g)$	-1	$+1$	-1
$c(g)$	-1	-1	$+1$
$t(g)$	$+1$	-1	-1

for some group element $g \in G$.

18. CONCLUSIONS