Three Pedestrian Overpasses Between Number Theory And Physics

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Moonshine Phenomena, Supersymmetry, and Quantum Codes

1.A: Some Background

RCFT Approach To FLM

The original RCFT explanation of Monstrous Moonshine begins with 24 free *chiral* bosons with target space the Leech torus := $\mathbb{R}^{24}/\Lambda$

$\Lambda \subset \mathbb{R}^{24}$ is the Leech lattice,

D25-brane

Moreover, target space torus has a very special ``B-field''

$$S = \int d^2\sigma \, (\, G_{\mu\nu} \, \partial_i x^{\mu} \, \partial^i x^{\nu} + B_{\mu\nu}\epsilon^{ij}\partial_i x^{\mu}\partial_j x^{\nu})$$

$\mathbb{Z}_2$ −Orbifold

Now gauge the global symmetry:

$\vec{x} \rightarrow -\vec{x}$ for $\vec{x} \in \mathbb{R}^{24}/\Lambda$

$$\mathcal{H}_\Lambda = \mathcal{H}_\Lambda^+ \bigoplus \mathcal{H}_\Lambda^-$$

Nontrivial Gauge Bundle on S^1
Twist Fields

Identify order two points in the torus $\mathbb{R}^{24}/\Lambda$

$$T_2(\Lambda) \coloneqq \Lambda/2\Lambda$$

Orbifold breaks translation symmetry on Leech torus down to $T_2(\Lambda)$

B −field defines a symplectic form on $T_2(\Lambda)$

$$B(\lambda_1, \lambda_2) = (-1)^{\lambda_1 \cdot \lambda_2}$$

Noncommutative Translations - 2/2

Unbroken translation symmetry realized on Hilbert space via a nontrivial central extension

$$0 \to \mathbb{Z}_2 \to \mathcal{H}\big(T_2(\Lambda)\big) \to T_2(\Lambda) \to 0$$

$$T(\lambda_1)T(\lambda_2) = \epsilon(\lambda_1, \lambda_2)T(\lambda_1 + \lambda_2)$$

$$\frac{\epsilon(\lambda_1, \lambda_2)}{\epsilon(\lambda_2, \lambda_1)} = (-1)^{\lambda_1 \cdot \lambda_2}$$

Early example of noncommutative geometry on D-branes induced by a B-field

Let $\mathcal{S}$ be the unique irreducible representation of the Heisenberg group $\mathcal{H}(T_2(\Lambda))$:

Construct it using γ −matrices.

$\mathcal{S}$: ``Spinor representation "

$$\mathcal{H}_T = \mathcal{F} \otimes \mathcal{S} = \mathcal{H}_T^+ \oplus \mathcal{H}_T^-$$

FLM Module

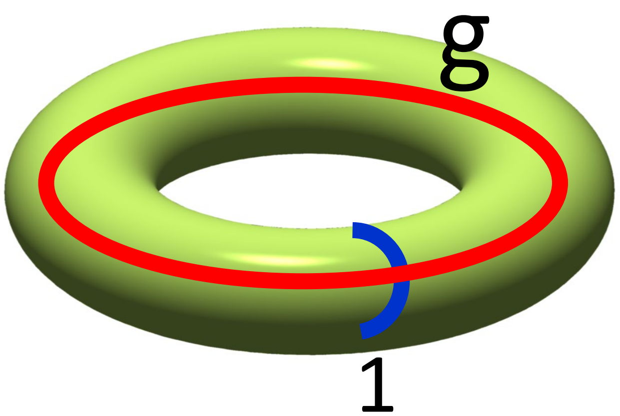
$$\mathcal{H}_{FLM} = \mathcal{H}_{\Lambda}^{+} \oplus \mathcal{H}_{T}^{+}$$

FLM & Borcherds:

The automorphism group of the VOA $\mathcal{H}_{FLM}$ is the Monster Group

Payoff: Conceptual Explanation of Modularity

$$Th_g(q) = Tr_{\mathcal{H}_{FLM}}\, g q^{L_0 - \frac{c}{24}} =$$



⇒ Modularity

This is the gold standard for the conceptual explanation of Moonshine-modularity

A truly satisfying conceptual explanation of genus zero properties remains elusive.

Important progress: Duncan & Frenkel 2009; Paquette, Persson, Volpato 2017

1.B: Statement Of The Problem

1988:

Beauty and the Beast: Superconformal Symmetry in a Monster Module

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Abstract. Frenkel, Lepowsky, and Meurman have constructed a representation of the largest sporadic simple finite group, the Fischer–Griess monster, as the automorphism group of the operator product algebra of a conformal field theory with central charge $c = 24$. In string terminology, their construction corresponds to compactification on a $\mathbf{Z}_2$ asymmetric orbifold constructed from the torus $\mathbf{R}^{24}/\Lambda$, where Λ is the Leech lattice. In this note we point out that their construction naturally embodies as well a larger algebraic structure, namely a super-Virasoro algebra with central charge $\hat{c} = 16$, with the supersymmetry generator constructed in terms of bosonic twist fields.

(Super-) Conformal Symmetry:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \qquad n, m \in \mathbb{Z}$$

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

Superconformal symmetry ⇒ supercurrent:

$$T(z)\, T_F(w) \sim \frac{\frac{3}{2}T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w} + \cdots$$

$$T_F(z) = \sum_r G_r z^{-r-\frac{3}{2}}$$

$$T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

There are no dimension 3/2 fields in $\mathcal{H}_{FLM}$

Associated to a nonanomalous $\mathbb{Z}_2$ is a ``spin lift'' - a ``2d spin conformal field theory'' [Lin & Shao: systematic study]

$$\mathcal{H}_{B\&B} = \mathcal{H}_{\Lambda} \oplus \mathcal{H}_{T}$$

has fields with conformal dimension in $\mathbb{Z} + \frac{1}{2}$

What is the actual supercurrent?

Not known.

Not easy.

Today I will fill in this gap.

It is very recent work with R. Singh

1.C: Solution Of The Problem

In one of our (several) attempts to explain Umbral Moonshine, Jeff Harvey and I discovered a curious relation between supercurrents in certain superconformal 2d field theories and quantum error correcting codes.

Moonshine, Superconformal Symmetry, and Quantum Error Correction

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Jeffrey A. Harvey,[1] Gregory W. Moore[2]

Work with Jeff focused on a K3 sigma model and Conway Moonshine

We showed that the superconformal current could be constructed using a special spinor determined by a code.

Jeff and I speculated the same pattern would appear in the construction of the superconformal generator in $\mathcal{H}_{B\&B}$

This turns out to be correct

With a student,
Ranveer Singh,
we have indeed realized
the supercurrent in this way

For every spinor $\Psi \in \mathcal{S}$ we have a dimension 3/2 primary field $V_\Psi \in \mathcal{H}_T$

$$V_\Psi(z_1) V_\Psi(z_2) \sim$$

$$\sim \frac{\bar{\Psi}\Psi}{z_{12}^3} + \frac{1}{8}\frac{\bar{\Psi}\Psi}{z_{12}} T(z_2) + \frac{1}{z_{12}} \sum_{\lambda:\lambda^2=4} \kappa_\lambda(\Psi) e^{i\lambda\cdot x(z_2)} \cdots$$

For any Ψ such that $\kappa_\lambda(\Psi) = 0$ for all $\lambda \in \Lambda: \quad \lambda^2 = 4$

$\Rightarrow V_\Psi$ is a supercurrent

We need to compute $\kappa_\lambda(\Psi)$

We need to know about
the OPE of *bosonic* twist fields .....

.... challenging .....

THE CONFORMAL FIELD THEORY OF ORBIFOLDS

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Conformal Field Theories, Representations and Lattice Constructions

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$$\sim \frac{\bar{\Psi}\Psi}{z_{12}^3} + \frac{1}{8}\frac{\bar{\Psi}\Psi}{z_{12}} T(z_2) + \frac{1}{z_{12}} \sum_{\lambda:\lambda^2=4} \kappa_\lambda(\Psi) e^{i\,\lambda\cdot x(z_2)} \ldots$$

$$\kappa_\lambda(\Psi) \sim \langle \Psi, T(\lambda)\Psi \rangle$$

$$T(\lambda) \in \mathcal{H}\big(T_2(\Lambda)\big)$$

A Strategy To Find A Suitable Ψ

For any Abelian subgroup $\hat{\mathcal{L}} \subset \mathcal{H}(T_2(\Lambda))$

$$P = \sum_{[\lambda] \in \hat{\mathcal{L}}} T(\lambda)$$

is proportional to a projection operator

$\hat{\mathcal{L}}$ maximal $\Rightarrow$ P is rank one

So we seek maximal subgroups $\hat{\mathcal{L}}$ such that V_Ψ is a supercurrent for $\Psi \in Im(P)$

Method to find a suitable $\hat{\mathcal{L}} \subset \mathcal{H}\left(T_2(\Lambda)\right)$:

Find a lattice $\Lambda_{sc} \subset \Lambda$ such that

$$\lambda_1, \lambda_2 \in \Lambda_{sc} \Rightarrow \lambda_1 \cdot \lambda_2 = 0 \; mod \; 2$$

$$2\Lambda \underset{2^{12}}{\subset} \Lambda_{sc} \underset{2^{12}}{\subset} \Lambda$$

$$\lambda \in \Lambda_{sc} \Rightarrow \lambda^2 = 0 \; mod \; 4$$

Nonzero $\lambda \in \Lambda_{sc} \Rightarrow \lambda^2 > 4$

Choose an isomorphism $T_2(\Lambda) \cong \mathbb{F}_2^{24}$

$$\hat{\mathcal{L}} \to \mathcal{L} \to \mathcal{C} \subset \mathbb{F}_2^{24}$$

$$\lambda^2 = 4 \quad \Rightarrow \langle \Psi, T(\lambda)\Psi \rangle = 0$$

because of the error correcting properties of $\mathcal{C}$

Existence of Λ_{sc} $\Rightarrow$ V_Ψ is a superconformal current in $\mathcal{H}_{B\&B}$ for $\Psi \in Im\, P$

Example of a sublattice Λ_{sc}

Dong, Li, Mason, Norton:
There is an isometric embedding of $\sqrt{2}L$ into the Leech lattice for every Niemeier lattice L

$$\Lambda_{sc} \cong \sqrt{2}\Lambda$$

Are there others?
Does $\mathcal{H}_{B\&B}$ have $\mathcal{N} > 1$ supersymmetry ?

Embarrassment Of Riches

Dong, Li, Mason, Norton: There are

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embeddings $\sqrt{2}\Lambda \hookrightarrow \Lambda$

For each embedding ι we get a self-dual doubly even code $\mathcal{C}_{12}^{\iota} \subset \mathbb{F}_2^{24}$

Inequivalent codes give different supercurrents

Theorem (Pless and Sloane): There are 9 inequivalent self-dual doubly even dimension 12 codes in $\mathbb{F}_2^{24}$

So, our construction can yield up to 9 distinct supercurrents

We are trying to find $\mathcal{N}$

Using the quaternary Golay code we can show that $\mathcal{N} \geq 2$

Time Reversal In Chern-Simons-Witten Theory

When does 3d Chern-Simons-Witten theory have a time reversal symmetry?

General theory based on compact group G and a ``level'' $k \in H^4(BG;\mathbb{Z})$

Which (G, k) give T-reversal invariant theories?

Related: When does Reshetikhin-Turaev-Witten topological field theory factor through the unoriented bordism category?

Some nontrivial examples of T-invariant CSW theories appeared in several recent papers

[Seiberg & Witten 2016; Hsin & Seiberg 2016; Cordova, Hsin & Seiberg]

$$G = PSU(N) \qquad\qquad k = N$$

But there is no systematic understanding.

With my student Roman Geiko we have recently carried out a systematic study for

Spin Chern-Simons Theory with torus gauge group $G \cong U(1)^r$

$$S = \frac{1}{4\pi} \int K_{IJ}\, A_I\, d\, A_J$$

K_{IJ} : $r \times r$ nondegenerate, integral symmetric matrix: determines integral lattice L

Classical T-reversal:

$\exists \; U \in GL(r, \mathbb{Z})$ such that

$$UKU^{tr} = -K$$

(Note: $\sigma(L) = 0$)

But there can be quantum T-reversal symmetries not visible classically.

Rank 2 examples studied by Seiberg & Witten; Delmastro & Gomis

The quantum theory does not depend on all the details of L

What *does* it depend on?

Finite Abelian group $\mathcal{D}(L) := L^{\vee}/L$

a.k.a ``group of anyons''   a.k.a. ``group of 1-form symmetries''

Quadratic Refinement (spin of anyons) :

$$q_W(x) = \frac{1}{2}(\tilde{x}, \tilde{x} - W) + \frac{1}{8}(W, W) \ \ mod\, \mathbb{Z}$$

$$\frac{1}{\sqrt{|\mathcal{D}(L)|}} \sum_{x \in \mathcal{D}(L)} e^{2\pi i\, q_W(x)} = e^{2\pi i \frac{\sigma(L)}{8}}$$

Theorem
[Belov & Moore; Freed,Lurie,HopkinsTeleman]

The quantum theory only depends on the equivalence class of the triple $(\mathcal{D}, q, \bar{\sigma})$

$$q: \mathcal{D} \to \mathbb{R}/\mathbb{Z} \qquad \bar{\sigma} \in \mathbb{Z}/24\mathbb{Z}$$

$$\frac{1}{\sqrt{|\mathcal{D}|}} \sum_{x \in \mathcal{D}} e^{2\pi i \, q(x)} = e^{2\pi i \frac{\bar{\sigma}}{8}}$$

Conversely, every such triple arises from some torus CSW theory

Equivalence of triples

$$(\mathcal{D}, q, \bar{\sigma}) \cong (\mathcal{D}', q', \bar{\sigma})$$

$\exists$ isomorphism $f: \mathcal{D} \to \mathcal{D}'$

$\exists \ \Delta' \in \mathcal{D}'$

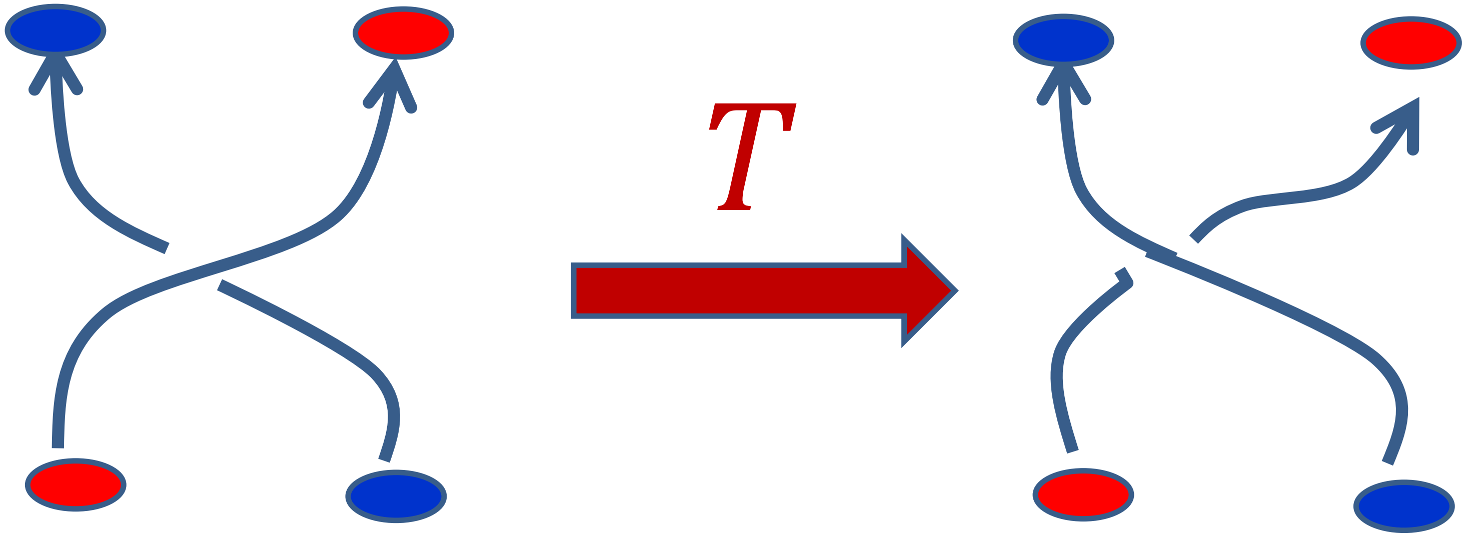
$$q(x) = q'(f(x) + \Delta')$$

T-Reversal Criterion

$$[(\mathcal{D}, q, \bar{\sigma})] = [(\mathcal{D}, -q, -\bar{\sigma})]$$

q: Determines the spin of anyons

b: Determines the braiding of anyons



Simpler Problem: The Witt Group (1936)

$$b(x,y) = q(x+y) - q(x) - q(y) + q(0)$$

Throw away $q, \bar{\sigma}$ and just keep b.

Classify $[(\mathcal{D}, b)]$

$$[(\mathcal{D}_1, b_1)] + [(\mathcal{D}_2, b_2)] := [(\mathcal{D}_1 \oplus \mathcal{D}_2, b_1 \oplus b_2)]$$

Abelian monoid $\mathcal{DB}$

$$\mathcal{DB} = \bigoplus_p \ \mathcal{DB}_p$$

Odd p: $\mathcal{DB}_p$ is generated by forms on $\mathbb{Z}/p^r\mathbb{Z}$

X_{p^r}: $b(1,1) = p^{-r}$ $\qquad$ Y_{p^r}: $b(1,1) = \theta p^{-r}$

θ: Quadratic nonresidue modulo p^r

$p = 2$ Many generating forms:

$$A_{2^r}, B_{2^r}, C_{2^r}, \dots \ , F_{2^r}$$

Submonoid $\mathcal{Spl}$ Split forms:

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$$

$$\mathcal{D}_1 = \mathcal{D}_1^{\perp}$$

$$\mathcal{Witt} \ := \ \mathcal{DB}/\mathcal{Spl}$$

Abelian group whose structure is known.

Wall, Miranda, Kawauchi & Kojima

determine relations on the generators

$$\mathcal{Witt} \cong \bigoplus_p \mathcal{Witt}_p$$

p odd: $\mathcal{Witt}_p \cong \bigoplus_{k \geq 1} \mathcal{W}_p^k$

$\mathcal{W}_p^k \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \qquad \left(-\frac{1}{p}\right) = (-1)^{\frac{p-1}{2}} = 1$

$\mathcal{W}_p^k \cong \mathbb{Z}_4 \qquad \left(-\frac{1}{p}\right) = (-1)^{\frac{p-1}{2}} = -1$

$$\mathcal{S}p\ell \subset \ \mathcal{DB}^T := \{ [\mathcal{D}, b] = [\mathcal{D}, -b] \} \subset \mathcal{DB}$$

Roman computed generators for the (infinite) Abelian subgroup

$$\mathcal{DB}^T / \mathcal{S}p\ell$$

and then refined it to T −invariant triples

Theorem: A T-invariant triple $[(\mathcal{D}, q, \bar{\sigma})]$ must be a direct sum of

$\mathscr{D}$	b	$\hat{q}$	$\sigma \bmod 8$
$\mathbb{Z}/p^r,\ p \equiv 1 \bmod 4$	X_{p^r} Y_{p^r}	ux^2/p^r vx^2/p^r	$r(p^2-1)/2$ $r(p^2-1)/2+4r$
$\mathbb{Z}/p^r,\ p \equiv 3 \bmod 4$	X_{p^r}	ux^2/p^r	$r(p^2-1)/2$
$\mathbb{Z}/2$	A_2	$x^2/4-1/8$	0
$(\mathbb{Z}/2)^2$	E_2	$xy/2$	0
$(\mathbb{Z}/4)^4$	$4A_{2^2}$	$(x_1^2+x_2^2+5x_3^2+5x_4^2)/8$	4
$\mathbb{Z}/2^r \times \mathbb{Z}/2^r,\ r \geqslant 1$	E_{2^r}	$xy/2^r + \alpha(x/2+y/2)$	0
$\mathbb{Z}/2^m \times \mathbb{Z}/2^m,\ m \geqslant 2$	F_{2^m}	$(x^2+xy+y^2)/2^m$	$4(m+1)$
$(\mathbb{Z}/2^m)^4,\ m \geqslant 2$	$4A_{2^m}$	$\sum_{i=1}^4 x_i^2/2^{m+1}$	4
$(\mathbb{Z}/2^m)^2,\ m \geqslant 2$	$A_{2^m}+B_{2^m}$	$x^2/2^{m+1}+3y^2/2^{m+1}$	$4(m+1)$
$(\mathbb{Z}/2^n)^2,\ r \geqslant 3$	$A_{2^n}+D_{2^n}$	$x^2/2^{n+1}+7y^2/2^{n+1}$	0
$(\mathbb{Z}/2^r)^4,\ r \geqslant 3$	$3A_{2^n}+C_{2^n}$	$\sum_{i=1}^3 x_i/2^{n+1}+5y^2/2^{n+1}$	$4n$

Table 3. T-invariant quartets. Here, $\left(\frac{-1}{p}\right)=1$, $\left(\frac{2u}{p}\right)=1$, $\left(\frac{2v}{p}\right)=-1$, $r \geqslant 1$, $m \geqslant 2$, $n \geqslant 3$, $\alpha \in \{0,1\}$. Note, we can add $1/2$ to $\hat{q}$ and 4 to σ in any line to obtain another quartet.

Example: $L \cong A_4$ and $L \cong D_4$ can be primitively embedded into E_8 (Nikulin)

These are positive definite, and cannot be T-invariant classically

Nevertheless, they are quantum T-invariant

Conjecture for the general (non-spin) case:

$$(G,k) \to CSW(G,k) \to \quad MTC(G,k)$$

Witten

Moore & Seiberg

$$WZW(G,k)$$

Definition [Lee & Tachikawa; Kong & Zhang]: The time reversal of an MTC $\mathcal{C}$ with braiding $B_{x,y}: x \otimes y \to y \otimes x$ and ribbon structure $\theta_x: x \to x$ is the MTC $\mathcal{C}^{rev}$ with

$$B_{x,y}^{rev} := B_{y,x}^{-1} \qquad \theta_x^{rev} := \theta_x^{-1}$$

A CSW theory is time reversal invariant if there is an equivalence of MTC's

$$MTC(G,k)^{rev} \cong MTC(G,k)$$

There is a mathematical notion of a Witt group of (nondegenerate) braided fusion categories.

[Davydov, Müger, Nikshych, Ostrik 2010]

$\mathcal{C}_1 \sim \mathcal{C}_2$ if there exist fusion categories $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

$$\mathcal{C}_1 \otimes Z(\mathcal{D}_1) \cong \mathcal{C}_2 \otimes Z(\mathcal{D}_2)$$

CONJECTURE

A (bosonic) $CSW(G,k)$ is T-invariant

iff

$[MTC(G,k)]$ is order 2 in $\mathcal{W}itt$

Condition On Higher Gauss Sums

Higher Gauss sums $\sum_x d_x^2 \theta_x^n$ studied in

[Ng, Schopieray, Wang 2018;
Kaidi, Komargodski,Ohmori,Seifnashri, Shao 2021]

are all real.

The examples of Seiberg et. al. satisfy this condition.

Topological Interfaces

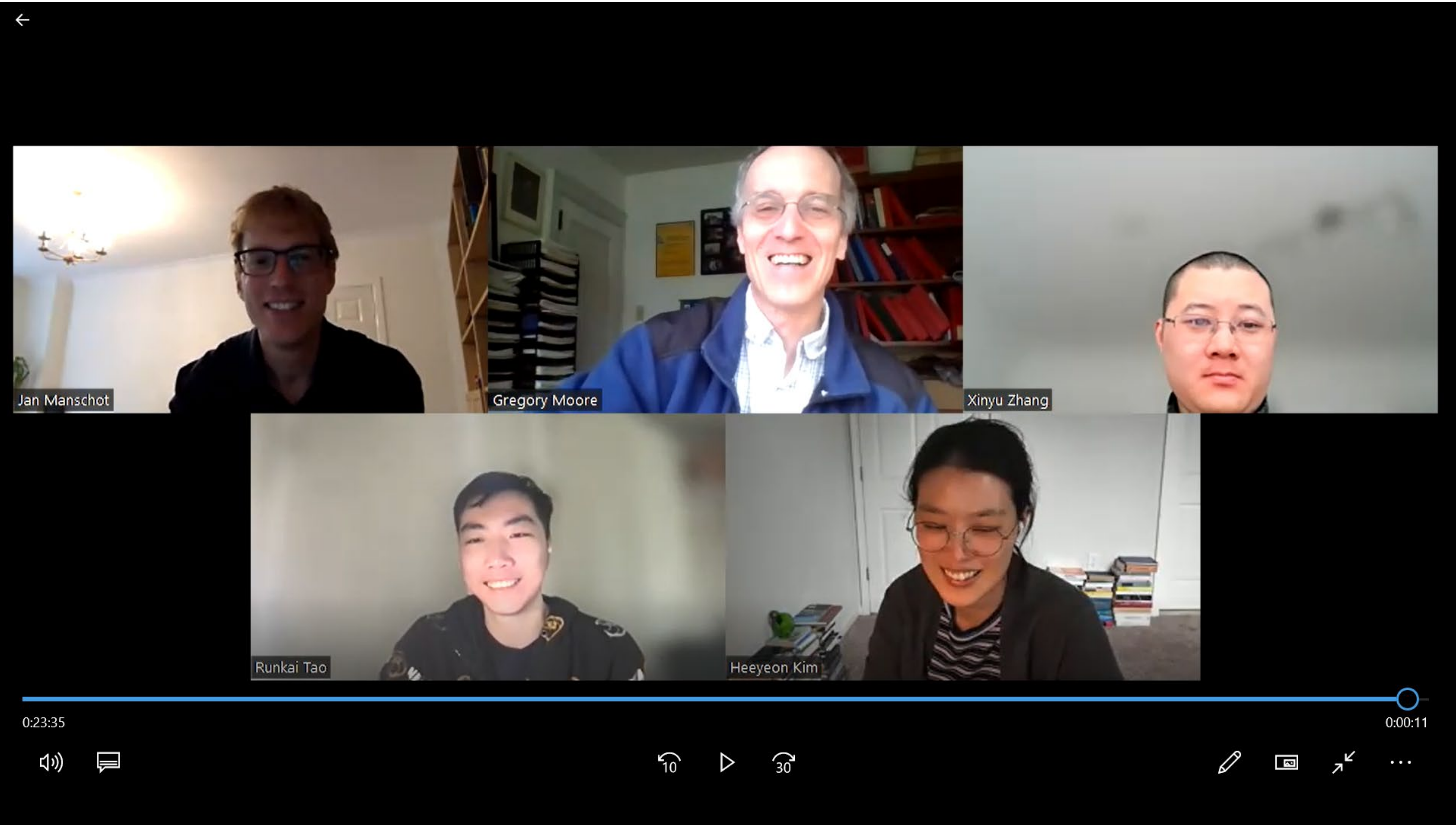
It is always true that $\mathcal{C} \otimes \mathcal{C}^{rev} \cong Z(\mathcal{D})$ and therefore there is a topological gapped boundary condition for $\mathcal{C} \otimes \mathcal{C}^{rev}$ [Freed & Teleman]

Conjecture is equivalent to existence of a topological interface between $CS(G,k)$ and its time-reversal

(Related to work of Kapustin & Saulina.)

U-Plane For 5d SYM And Four-Manifold Invariants

``K-Theoretic Donaldson Invariants''



Five Dimensions

Partial Topological Twist of 5d SYM on $X \times S^1$

Reduces to SQM on the moduli space of instantons:

(Requires that $\mathcal{M}$ be Spin-c)

$$\mathcal{R} := R\,\Lambda$$

$$Z[\mathcal{R}] = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} \hat{A}(T\,\mathcal{M}_k)$$

[Nekrasov (1996); Losev, Nekrasov, Shatashvili (1997); ....]

+ important generalization ...

Chern-Simons Observables

$U(1)_{inst}$ symmetry with current $J = Tr(f \wedge f\ )$

Couple to background gauge field A: $n := \left[\frac{F(A)}{2\pi}\right] \in H^2(X, \mathbb{Z})$

$$\mathcal{O}(n) = \int_{\Sigma(n)\times S^1} Tr\left(a\, da + \frac{2}{3} a^3\right) + \cdots$$

$= \int_{X\times S^1} F(A) \wedge Tr\left(\, a\, da + \frac{2}{3} a^3\right)$ + susy completion

$$Z(\mathcal{R}, n) := \langle\, e^{\mathcal{O}(n)} \rangle$$

Five Dimensions

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} ch\big(L(n)\big)\, \hat{A}(\mathcal{M}_k)$$

Using both the Coulomb branch integral (a.k.a. the U-plane integral) and, independently, localization techniques, we make contact with the work of mathematicians

2006:

K-THEORETIC DONALDSON INVARIANTS VIA INSTANTON COUNTING

LOTHAR GÖTTSCHE, HIRAKU NAKAJIMA, AND KŌTA YOSHIOKA

To Friedrich Hirzebruch on the occasion of his eightieth birthday

ABSTRACT. In this paper we study the holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank 2 semistable sheaves on an algebraic surface X, which can be viewed as K-theoretic versions of the Donaldson invariants. In particular if X is a smooth projective toric surface, we determine these invariants and their wall-crossing in terms of the K-theoretic version of the Nekrasov partition function (called 5-dimensional supersymmetric Yang-Mills theory compactified on a circle in the physics literature). Using the results of [43] we give an explicit generating function for the wall-crossing of these invariants in terms of elliptic functions and modular forms.

2019:

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

ABSTRACT. We conjecture a Verlinde type formula for the moduli space of Higgs sheaves on a surface with a holomorphic 2-form. The conjecture specializes to a Verlinde formula for the moduli space of sheaves. Our formula interpolates between K-theoretic Donaldson invariants studied by the first named author and Nakajima-Yoshioka and K-theoretic Vafa-Witten invariants introduced by Thomas and also studied by the first and second named authors. We verify our conjectures in many examples (e.g. on K3 surfaces).

$$b_2^+(X) = 1$$

Derived a wall-crossing formula

Differs from GNY.

Agrees with GNY.

(Suitably interpreted.)

This raises some puzzles…

$$Z^J(\mathcal{R}, n) = \Phi^J(\mathcal{R}, n) + Z^J_{SW}(\mathcal{R}, n)$$

$J \in H^2(X, \mathbb{R})$: $J = * J$ & $J^2 = 1$ & $J \in Positive\ LC$

Z^J_{SW} : Contribution of SW invariants

$\Phi^J(\mathcal{R}, n)$: 4d Coulomb branch integral

One can deduce Z^J_{SW} from Φ^J

For 5d SYM gauge group of rank 1:
Coulomb branch = $\mathbb{C}$

Measure is singular at 4 special points and ∞

SW special Kahler geometry is subtle

a: cylinder valued

$$\mathcal{F} \sim R^{-2} Li_3(e^{-2Ra}) + \cdots$$

$$+ Instanton\ corrections$$

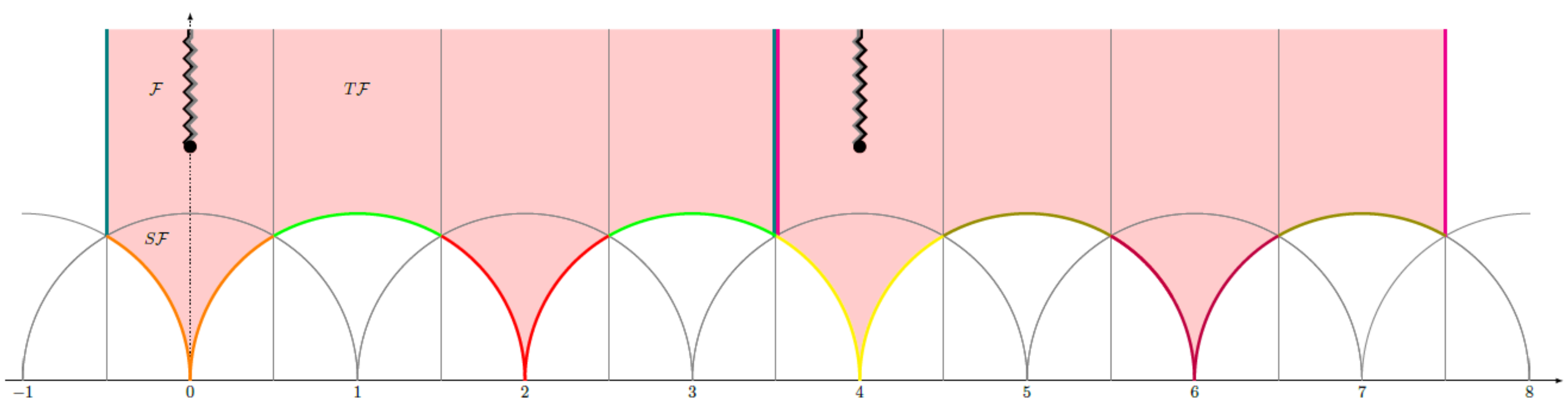
[Nekrasov, 1996]

Modular Parametrization Of U —plane

$$\left(\frac{U}{R}\right)^2 + \tilde{u}(\tau)^2 = 8 + 4(\mathcal{R}^2 + \mathcal{R}^{-2})$$

$$\tilde{u}(\tau) = 2\left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} + \frac{\vartheta_3(\tau)}{\vartheta_2(\tau)}\right)$$

Hauptmodul for $\Gamma^0(8)$



$$\Phi^J(\mathcal{R},n) = \int_{\mathcal{F}} d\tau d\bar{\tau}\, \nu(\tau)\ C(\tau)^{n^2}\ \Psi^J\left(\tau, \frac{v(\tau)}{2} n\right)$$

$$\nu(\tau) = \frac{\vartheta_4^{13-b_2}}{\eta^9}\ \frac{1}{\sqrt{1-2\,\mathcal{R}^2 u(\tau)+\mathcal{R}^4}}$$

$$4u(\tau) = \tilde{u}(\tau)^2 - 8$$

$$C(\tau) = \frac{\vartheta_4\left(\tau, \frac{v(\tau)}{2}\right)}{\vartheta_4(\tau)}$$

$$\frac{\vartheta_1\left(\tau, \frac{v(\tau)}{2}\right)}{\vartheta_4\left(\tau, \frac{v(\tau)}{2}\right)} = -\mathcal{R}$$

$$\Psi^J(\tau, z) = \sum_{k \in H^2(X,\mathbb{Z})} \left(\frac{\partial}{\partial \bar{\tau}} E_k^J \right) q^{-\frac{k^2}{2}} e^{-2\pi i \, k \cdot z} \, (-1)^{k \cdot K}$$

$$E_k^J = Erf\left(\sqrt{Im \tau} \, \left(k + \frac{Im \, z}{Im \, \tau} \right) \cdot J \right)$$

$$\nu(\tau) = \frac{\vartheta_4^{13-b_2}}{\eta^9} \, \frac{1}{\sqrt{1 - 2\,\mathcal{R}^2 u(\tau) + \mathcal{R}^4}}$$

$$C(\tau) = \frac{\vartheta_4\left(\tau, \frac{v(\tau)}{2}\right)}{\vartheta_4(\tau)}$$

$$\Phi^J(\mathcal{R}, n) = \int_{\mathcal{F}} d\tau d\bar{\tau} \, \nu(\tau) \, C(\tau)^{n^2} \, \Psi^J\left(\tau, \frac{v(\tau)}{2} n\right)$$

Wall-Crossing Formula @ ∞

$$\Phi^{J} - \Phi^{J'} = \left[\nu \, C^{n^2} \Theta^{J,J'} \right]_{q^0}$$

$$\sum_k \left[sgn\left\{ \left(k + \frac{n}{2} \frac{Im\, v(\tau)}{Im\, \tau} \right) \cdot J \right\} - \{ J \to J' \} \right] q^{-\frac{k^2}{2}} e^{-2\pi i \, k \cdot n \frac{v(\tau)}{2}} (-1)^{k \cdot K}$$

$\nu, C, \Theta^{J,J'}$ are functions of τ and of $\mathcal{R}$

Subtle order of limits: $\mathcal{R} \to 0$ vs. $\Im\tau \to \infty$

A. First expand in $\mathcal{R}$ around $\mathcal{R} = 0$ then take the constant q^0 term at each order in $\mathcal{R}$

This agrees with GNY

B. First expand in q and extract the constant q^0 term

1. Results differ from GNY

2. Terms involving negative powers of $\mathcal{R}$

$$Z(\mathcal{R}, n) = \sum_{k=0}^{\infty} \mathcal{R}^{d_k/2} \int_{\mathcal{M}_k} e^{c_1(L(n))} \hat{A}(\mathcal{M}_k)$$

Did we make a technical mistake?

Probably not:

Using toric localization and the 5d instanton partition function we derived exactly the same formula for wall-crossing @ ∞

Moreover, using the wall-crossing behavior of $\Phi^J(\mathcal{R}, n)$ at the strong coupling cusps allows one to ***derive*** Z_{SW}^J $\Rightarrow$ partition function for $b_2^+ > 1$

$$G(\mathcal{R}, n) = \frac{2^{2\chi+3\sigma-\chi_h}}{(1-\mathcal{R}^2)^{\frac{1}{2}n^2+\chi_h}} \sum_c SW(c) \left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{c\cdot\frac{n}{2}}$$

$$Z(\mathcal{R}, n) = \sum_{\xi\in\mu_4} \xi^{-\chi_h} G(\xi\mathcal{R}, n)$$

Agrees with, and generalizes, GKW Conjecture 1.1

Explicit evaluation of our U-plane integral Φ^J for special values of J involves some interesting technical considerations in the theory of Jacobi-Maass forms

The Special Period Point

For any manifold with $b_2^+ = 1$

$\exists$ special J_0 such that $\Psi_\nu^{J_0}$ factorizes:

$$\Psi^{J_0}(\tau, z) = f(\tau, z)\ \Theta_{L\_}(\tau, z)$$

$$f(\tau, z) = \sum_{k \in \mathbb{Z}} \partial_{\bar{\tau}} E_k^J\ q^{-\frac{1}{4}k^2}\ e^{-2\pi i\, k \cdot z}$$

Measure As A Total Derivative

$$\Omega = d\,\Lambda \qquad \Lambda = d\tau\,\mathcal{H}\ \hat{G}$$

Where we can write $\hat{G}$ explicitly so that Λ is:

1. Well-defined
2. Nonsingular away from $\tau \in \{cusps\}$
3. Good q_i expansion near cusps

Harmonic Jacobi-Maass Forms

These conditions determine $\widehat{G}$ uniquely.

Modular completion of an Appel-Lerche sum

$$F(\tau, z) \sim \frac{e^{-2\pi i\, z}}{\vartheta_4(2\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n^2-\frac{1}{4}}}{1+e^{4\pi i\, z} q^{2n-1}}$$

$$z = n_0 \frac{v(\tau)}{2} \qquad n_0 := n \cdot J$$

The modular completion is not unique because we can add a meromorphic modular function to $F\left(\tau, \frac{n_0 v(\tau)}{2}\right)$

We need to choose the one with no unwanted poles in the fundamental domain.

This is technically challenging for general values of n_0

All this should generalize to (anomaly-free) 6d SYM theories on $X \times \mathbb{E}$

$$\hat{A}(\mathcal{M}_k) \to Ell(\mathcal{M}_k, q)$$

So far, we did not use any K-theory in describing the ``K-theoretic Donaldson invariants''

It would be very desirable to do so, because the 6d version, analogously formulated could be quite interesting:

Conjecture:

Integrals in elliptic cohomology of distinguished classes defined by the susy sigma model with target space $\mathcal{M}_k$ define smooth invariants of four-manifolds

Summary

The FLM/Beauty & Beast formulation of the Monster group has underlying extended superconformal symmetry with $\mathcal{N} \geq 2$

Complete classification of T-invariant quantum spin CSW theories for torus gauge group. The general case is stated, conjecturally, in terms of a Witt group

Twisted 5d SYM computes $\hat{A}$-genera of instanton moduli spaces, but the physical path integral leads to puzzling discrepancies with the mathematical results of GNY/GKW and the general predictions of UV localization.


NOT
That's all Folks!