ASC Lecture III

- Put up the outline again for review

- Say more about why Grassmannians
  Can't be $U(N) \times (\mathbb{R}^*)^{\otimes N}$

- Put up tables of $\left\{ \begin{matrix} \text{real} \\ \text{complex} \end{matrix} \right.$ Clifford $\}$ structure

- Review where we've been.
  Start @ p. 47

⑩ Real & Complex Clifford Algebras

Clifford algebras can be defined for any Vector space w/ nondegenerate quadratic form. But here we will be concerned with

$$k = \mathbb{R}, \mathbb{C}$$

and $\quad Q = \pm \mathbb{1}_n \quad$ (in suitable basis).

Once again the distinction between $\mathbb{R}, \mathbb{C}$ is important.

① $\mathbb{C}l_n$: Super algebra / $\mathbb{C}$ with <u>odd</u> generators e_i, $i = 1, \cdots, n$

$$e_i e_j + e_j e_i = 2\delta_{ij} \mathbb{1} \quad (\text{or } -2\delta_{ij}\mathbb{1})$$

② $Cl_n \quad n \in \mathbb{Z}$: Superalgebra / $\mathbb{R}$ with odd generators e_i, $i = 1, \cdots, |n|$ and

$$\begin{aligned} e_i e_j + e_j e_i &= +2\delta_{ij}\mathbb{1} \qquad n > 0 \\ &= -2\delta_{ij}\mathbb{1} \qquad n < 0 \end{aligned}$$

Exercise: $\mathrm{sdim}_{\mathbb{R}} Cl_n = \left(2^{|n|-1} \mid 2^{|n|-1} \right)$

$\mathrm{sdim}_{\mathbb{C}} \mathbb{C}l_n = \left(2^{n-1} \mid 2^{n-1} \right)$

Now we can write:

$$\mathbb{C}l_n = \mathbb{C}l_1 \hat{\otimes} \cdots \hat{\otimes} \mathbb{C}l_1$$

$$Cl_n = \underbrace{Cl_\epsilon \hat{\otimes} \cdots \hat{\otimes} Cl_\epsilon}_{|n| \text{ times}} \qquad \epsilon = \mathrm{sgn}(n)$$

2 Warning! The tensor product of matrix reps of $\mathcal{A}_1, \mathcal{A}_2$ is NOT a rep. of $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ because of the Koszul sign rule. That's why the rep. theory of Clifford is not completely trivial.

Now let's look at the structure of the Clifford algebras:

$$\mathbb{C}l_0 := \mathbb{C}$$

$$\mathbb{C}l_1 = \mathbb{C} \oplus \mathbb{C}e \qquad e^2 = 1$$

Note: $\mathbb{C}l_1$ is NOT a mtx superalgebra - just by dimensions

So in the world of Morita equivalence $\mathbb{C}l_1$ is an irreducible object - "atomic" - can't make it equivalent to something simpler.

~~Hence~~ Contrast that with $\mathbb{C}l_1$ as an ungraded algebra:

$$P_{\pm} = \frac{1}{2}(1 \pm e) \quad \text{projectors}$$

$$\mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C}$$

Go on to $\mathbb{C}l_2$: Has faithful irrep $\mathcal{M} = \mathbb{C}^{1|1}$

$$e_1 \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \longrightarrow \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

both odd: <u>Note!</u> Cannot take $e_2 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ X

$$\mathbb{C}l_2 \cong \mathrm{End}(\mathbb{C}^{1|1})$$

$\therefore$ Morita equivalent to $\mathbb{C}l_0 = \mathbb{C}$

But now $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \hat{\otimes} \mathbb{C}l_2$

So Morita classes have mod-two periodicity:

	Graded	Ungraded
$\mathbb{C}l_{2k}$	$\mathrm{End}\left(\mathbb{C}^{2^{k-1}\vert 2^{k-1}}\right)$	$\mathbb{C}(2^k)$
$\mathbb{C}l_{2k+1}$	$\mathbb{C}l_{2k}\hat{\otimes}\mathbb{C}l_1$	$\mathbb{C}(2^k)\oplus\mathbb{C}(2^k)$

Morita: $[\mathbb{C}l_0]$, $[\mathbb{C}l_1]$ form group $\cong \mathbb{Z}_2$

At this point we are very close to one way of defining K-theory.

We have essentially computed the complex K-theory of a point.

See my notes.

But we are going to focus on something else.

END ASC LECTURE II

Let's look at the analogous structure of the real Clifford algebras

Similarly, we have the structure of real Clifford algebras:

	Graded	Ungraded
Cl_4	$\mathbb{H} \otimes \mathrm{End}(\mathbb{R}^{1\|1})$ ⑰	$\mathbb{H}(2)$ ⑱
Cl_3	$\mathbb{H} \hat{\otimes} Cl_{-1}$ ⑬	$\mathbb{C}(2)$ ⑭
Cl_2	$\mathbb{C}[\varepsilon_+]$ ⑦	$\mathbb{R}(2)$ ⑧
Cl_1	$\mathbb{R}[\varepsilon_+]$ ⑤	$\mathbb{R} \oplus \mathbb{R}$ ⑥
Cl_0	$\mathbb{R}$ ①	$\mathbb{R}$ ②
Cl_{-1}	$\mathbb{R}[\varepsilon_-]$ ③	$\mathbb{C}$ ④
Cl_{-2}	$\mathbb{C}[\varepsilon_-]$ ⑨	$\mathbb{H}$ ⑩
Cl_{-3}	$\mathbb{H} \hat{\otimes} Cl_{+1}$ ⑪	$\mathbb{H} \oplus \mathbb{H}$ ⑫
Cl_{-4}	$\mathbb{H} \otimes \mathrm{End}(\mathbb{R}^{1\|1})$ ⑮	$\mathbb{H}(2)$ ⑯

Comments: ①, ② clear

③, ④ Cl_{-1} NOT Morita equiv. to a matrix algebra.
"atomic" as a graded algebra.
But ungraded $\varepsilon_-^2 = -1$ rep. faithfully as $\sqrt{-1}$

$x + y\varepsilon_- \longrightarrow x + \sqrt{-1}\, y \qquad Cl_{-1} \cong \mathbb{C}$

⑤, ⑥ Similarly for Cl_{+1}. But note $P = \frac{1}{2}(1 \pm \varepsilon_+)$ gives ungraded algebra $\mathbb{R} \oplus \mathbb{R}$

NOT MOD TWO PERIODIC!

⑦,⑧

Cl_{+2} Why not a matrix superalgebra?

$$e_1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_2 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ?$$

NO e_2 is not odd. So again Cl_{+2} is "atomic" although as an ungraded rep it is a matrix algebra.

⑨,⑩

Cl_{-2} : likewise "irreducible"

Ungraded $e_1 \to i$, $e_2 \to j$ gives $\mathbb{H}$.

⑪ ⑫ ⑬ ⑭

$$Cl_{\mp 3} \cong \mathbb{H} \hat{\otimes} Cl_{\pm 1}$$

$$e_1 \longrightarrow \underline{i} \otimes e$$
$$e_2 \longrightarrow j \otimes e$$
$$e_3 \longrightarrow \underline{k} \otimes e$$

From ungraded $Cl_{\pm 1}$ get the ungraded results.

⑮ ⑯ ⑰ ⑱

Cl_{-4} :

$$e_1 \longrightarrow \begin{pmatrix} 0 & i \\ \underline{i} & 0 \end{pmatrix}$$

$$e_2 \longrightarrow \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \qquad Cl_{-4} \cong \mathbb{H} \otimes End(\mathbb{R}^{1|1})$$

$$e_3 \to \begin{pmatrix} 0 & \underline{k} \\ \underline{k} & 0 \end{pmatrix}$$

$$e_4 \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and similarly for Cl_{+4} so Morita equivalent to each other and to $\mathbb{H}$. $Cl_{-4} = Cl_{+4}$

Now we start to see mod 8 periodicity:

- $$\begin{aligned} Cl_5 &= Cl_1 \hat{\otimes} Cl_4 \\ &= Cl_1 \hat{\otimes} \mathbb{H} \otimes \mathrm{End}(\mathbb{R}^{1|1}) \\ &= Cl_{-3} \hat{\otimes} \mathrm{End}(\mathbb{R}^{1|1}) \end{aligned}$$

$\underline{5 = -3 \mod 8}$

- $$\begin{aligned} Cl_6 &= Cl_3 \hat{\otimes} Cl_3 \\ &= Cl_{-1} \hat{\otimes} Cl_{-1} \hat{\otimes} \mathbb{H} \otimes \mathbb{H} \\ &= Cl_{-2} \hat{\otimes} \mathrm{End}(\mathbb{R}^{4|0}) \end{aligned}$$

Exercise: a.) $Cl_7 = Cl_{-1} \otimes \mathrm{End}(\mathbb{R}^{4|4})$

b.) $Cl_{\pm 8} = \mathrm{End}(\mathbb{R}^{8|8})$

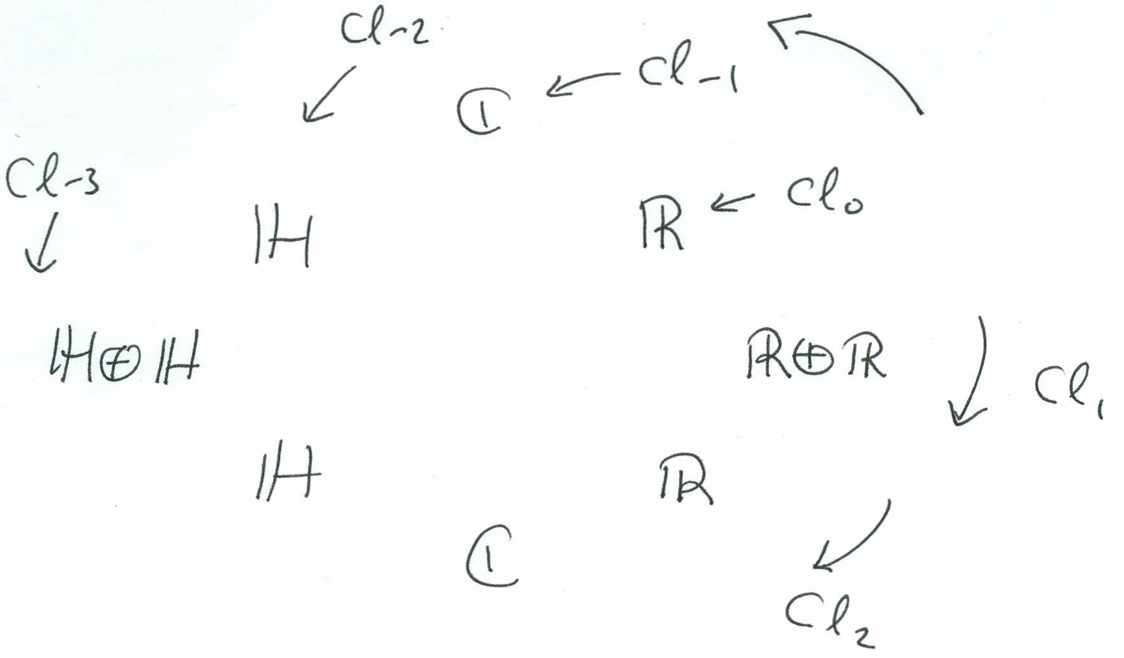
<u>Conclusion</u>: There are 8 Morita equivalence classes of real Clifford algebras $[D_\alpha]$

$$[Cl_n] = [D_\alpha] \qquad \alpha = n \mod 8$$

$$[D_\alpha] \cdot [D_\beta] = [D_{\alpha+\beta}]$$

Morita classes form group $\mathbb{Z}/8\mathbb{Z}$

In terms of <u>ungraded</u> algebras we have the Bott clock, or Bott genetic code



This gives the Wedderburn type and then just fit dimensions.

e.g. Cl_{65} $\quad 65 = 1 \bmod 8$ so

$$Cl_{65} = \mathbb{R}(2^{32}) \oplus \mathbb{R}(2^{32})$$

⑪ Generalized Dyson Problem & 10-Fold Way

Now let us return to symmetry in Q.M. Recall we argued that on general grounds there must be a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading:

$$G \xrightarrow{(\phi, \chi)} \mathbb{Z}_2 \times \mathbb{Z}_2$$

And if G is compatible with dynamics then

$$\boxed{\rho(g) H = \chi(g) H \rho(g), \quad \forall g \in G}$$

This is easy. What is tricky is just how one should interpret this equation.

- For a gapped Hamiltonian $\Pi = \mathrm{sgn}(H)$ is defined and defines a $\mathbb{Z}_2$-grading on H.S. Then $\rho(g)$ should be odd for $\chi(g) = -1$. (Fidkowski-Kitaev; Freed-Moore; G. Thiang...

- $\mathcal{H}$ has an a priori $\mathbb{Z}_2$-grading, $\rho(g)$ is odd when $\chi(g) = -1$, then H is in the super-commutant.

2nd is viewpoint taken here.

Def: A (G,ϕ,χ)-rep is a $\mathbb{C}$-$\underline{\underline{s}}$vs V

$$\rho(g) = \begin{cases} u & \phi(g)=+1 \\ a & \phi(g)=-1 \end{cases} \quad \begin{array}{c} | \\ \varepsilon \\ | \end{array} \quad = \begin{cases} \text{even} & \chi(g)=+1 \\ \text{odd} & \chi(g)=-1 \end{cases}$$

So: If H is compatible it is in the super-commutant and if $\chi(g)=-1$ it is an odd element of the s-commutant.

Gen. Dyson Problem: Given a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded group and a (G,ϕ,χ) rep $\mathcal{H}$, what is the ensemble of compatible Hamiltonians?

Can answer this just like before

Def: A superalgebra $\mathcal{A}$ is a superdivision algebra if every nonzero homogeneous element is invertible.

Example: $\mathbb{C}\ell_1 = \mathbb{C} \oplus \mathbb{C}e$

But note: $1+e$ is NOT invertible, so $\mathbb{C}\ell_1$ is not a division algebra (in accord w/ Frobenius).

Thm (Super-Schur)

1.) A homogeneous intertwiner between two (G,ϕ,χ) irreps either vanishes or is a (graded) isomorphism.

2.) If V is a (G,ϕ,χ)-irrep then

$$\mathrm{Hom}^G_{\mathbb{C}}(V,V) \text{ is a superdivision algebra}$$

Pf: Same as usual, paying close attention to $\mathbb{Z}_2$-grading.

So $\{(V_\lambda,\rho_\lambda)\}$ complete list of distinct irreps (no graded isom.'s)

$$\mathcal{H} = \oplus\, S_\lambda \otimes V_\lambda$$

~~$\mathrm{Hom}^G_{\mathbb{C}}(\mathcal{H})$~~ $\mathrm{End}^G_{\mathbb{C}}(\mathcal{H}) =$ Supercommutant of $\mathcal{A}$ gen. by $\rho(g), I$

$$\cong \bigoplus_\lambda \mathrm{Mat}_{s_\lambda}(D^s_\lambda)$$

D^s_λ = superdivision algebra associated to (V_λ,ρ_λ)

Thm (Corallary of Wall's Thm): There are 10 superdivision algebras / $\mathbb{R}$

- $\mathbb{R}, \mathbb{C}, \mathbb{H}$
- $\mathbb{C}l_1$, $Cl_{\pm 1}$, $Cl_{\pm 2}$, $Cl_{\pm 3}$

What about $Cl_{\pm 4}$? $\cong \mathbb{H} \otimes End(\mathbb{R}^{1|1})$

Contains $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ homog., nonzero, noninvertible.

All Morita equiv. classes of Clifford / $\mathbb{R}, \mathbb{C}$ are labelled uniquely by one of the 10 real superdivision algebras.

Now, can put a $*$-structure on D^s_λ

So we have the generalized 10-foldway

$$\mathcal{E} = \bigoplus_{\lambda} \mathcal{E}_\lambda$$

- $\chi = 1$ $\mathcal{E}_\lambda$ one of 3 Dyson classes
- $\chi \neq 1$ $\mathcal{E}_\lambda$ = Odd elements of $Mat_{S_\lambda}(D^s_\lambda)$

N.B. This is a very unorthodox presentation. No free fermions!! Now let's connect to literature

⑫ The 10 CT-Groups & Superdivision Algebras

Now we would like to start making contact with the literature on topological phases.

Our group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ - graded so:

$$\begin{tikzcd} 1 \arrow[r] & G_0 \arrow[r] \arrow[d, phantom, "\shortparallel"] & G \arrow[r, "(\phi,\chi)"] & U \arrow[r] \arrow[d, phantom, "\cap"] & 1 \\ & \ker\phi \cap \ker\chi & & M_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \end{tikzcd}$$

WLOG we can choose generators of $M_{2,2}$ so that:

$$\boxed{\begin{array}{lll} \bar{\phi}(\bar{T}) = -1 & & \bar{\phi}(\bar{C}) = -1 \\ \bar{\chi}(\bar{T}) = +1 & [redacted] & \bar{\chi}(\bar{C}) = -1 \end{array}}$$

The literature makes the simplifying assumption that

$$G = G_0 \times \tilde{U}$$

$\tilde{U}$ = ϕ-twisted extension of $U \subset M_{2,2}$

This need not be the case!
But we will assume it for the sake of argument.

We can "factor out" the G_0:

$$\mathcal{H} \cong \bigoplus_{G_0^\vee} \underset{\underset{\tilde{U}}{\circlearrowleft}}{S_\lambda} \otimes \underset{\underset{G_0}{\circlearrowleft}}{V_\lambda}$$

So our questions are:

- What are the possible $\tilde{U}$'s ?
- What are the (ϕ,χ)-reps of these $\tilde{U}$'s ?

There are 10 possible $\tilde{U}$'s and we call them the 10 "CT-groups."

5 possibilities for $\bar{U}$!

$$\begin{tikzcd}
 & M_{2,2} = \langle \bar{C}, \bar{T} \rangle & \\
\langle \bar{T} \rangle \arrow[ur] & \langle \bar{C}\bar{T} \rangle \arrow[u] & \langle \bar{C} \rangle \arrow[ul] \\
 & \{1\} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Denote lifts into $\tilde{U}$ by C, T.

Then $C^2 = \pm 1$, $T^2 = \pm 1$ indpt of lift

<u>WLOG</u>: Can choose $CT = TC$

Now we have the table [DO NOT ERASE!]

$U \subset M_{2,2}$	$\dot{T}^2$	C^2	[Clifford]	[illegible]	
$\{1\}$			$[\mathbb{C}l_0]$		
⊛ $\langle : \bar{C}\bar{T} \rangle$			$[\mathbb{C}l_1]$		
$\langle \bar{T} \rangle$	+1		$[Cl_0]$		
$\langle \bar{T} \rangle$	−1		$[\mathbb{H}] = [Cl_{\pm 4}]$		
$\langle \bar{C} \rangle$		+1	$[Cl_{+2}]$		
$\langle \bar{C} \rangle$		−1	$[Cl_{-2}]$		
$M_{2,2}$	+1	+1	$[Cl_{+1}]$		
$M_{2,2}$	+1	−1	$[Cl_{-1}]$		
$M_{2,2}$	−1	+1	$[Cl_{+3}]$		
$M_{2,2}$	−1	−1	$[Cl_{-3}]$		

⊛ WLOG S lifting $\bar{C}\bar{T}$ has $S^2 = 1$ because $\bar{C}\bar{T}$ ~~has~~ $\phi = +1$.

So, there are 10 CT-groups.

And there are 10 superdivision algebras, or, equivalently 10 Morita classes of complex and real Clifford algebras.

Are these related?

Not necessarily! I also (fortunately) have 10 fingers!

Which of the 10! associations should we make?

Thm [Freed & Moore; Fidkowski & Kitaev]

$\exists$ 1-1 correspondence between

$$\underset{\tilde{U}}{\text{CT-Groups}} \Longleftrightarrow \text{Morita classes of } \mathbb{R}, \mathbb{C}, \text{Clifford } [\mathcal{D}]$$

Such that

$$\text{Cat}\left\{ \begin{matrix} (\phi,\chi)\text{-reps} \\ \text{of } \tilde{U} \end{matrix} \right\} \cong \text{Cat}\left\{ \begin{matrix} \text{Graded reps of} \\ \text{any } \mathcal{A} \text{ in Morita} \\ \text{class } [\mathcal{D}] \end{matrix} \right\}$$

Pf: $U = 1$: A (G, ϕ, χ)-rep is just a $\mathbb{Z}_2$-graded $\mathbb{C}$-v.s. $V = \mathbb{C}^{n_+|n_-}$

This is a graded $\mathbb{C}\ell_0 = \mathbb{C}$ - module.

2. $\mho = \langle \overline{CT} \rangle \Rightarrow S \in \tilde{\mho}$

- $S^2 = 1$, $\{S\}$ $\mathbb{C}$- linear + odd.

$\therefore$ $\{S\} = \rho(e)$ e gen's $\mathbb{C}l_1$.

3. $\mho = \langle \overline{T} \rangle$ we worked this out in lecture 1: T is even

$T^2 = 1$ V_+ graded $\mathbb{R}$-v.s. = Cl_0 module

$T^2 = -1$ $V_{\mathbb{R}}$ is an $\mathbb{H}$-module

4. $\mho = \langle \overline{C} \rangle$ $C \in \tilde{\mho}$

C : $\mathbb{C}$-antilinear, odd, squares to ± 1

$\rho(e_1) := C$ $\rho(e_2) = iC$, odd, anticommute,

act linearly on $V_{\mathbb{R}}$

$\rho(e_1)^2 = \rho(e_2)^2 = C^2$

5. $\mho = M_{2,2}$; $V = (\tilde{U}, \phi, \chi)$ - module

5A. $T^2 = +1$ $\quad V_+ = \mathrm{Fix}(T)$ still $\mathbb{Z}_2$-graded (T even)

C acts on V_+, is odd, gen's $Cl_{\pm 1}$ according to sign of C^2

5B: $T^2 = -1$: $\rho(e_1) = C$ $\quad \rho(e_2) = iC$ $\quad \rho(e_3) = iCT$

Satisfy relations of $Cl_{\pm 3}$ according to sign of C^2 : Makes $V_{\mathbb{R}}$ a graded $Cl_{\pm 3}$ - module.

Now one has to define a map going the other way and verify that the two compositions are equivalent to 1.

⑬ Free Fermion Dyson Problem & 10 Symm. Spaces

Now we want to classify free fermion Hamiltonians.

13A// Fermions

Data :

- Mode space of Majorana operators

$\mathcal{M} \cong \mathbb{R}^N$, w/ Eucl. metric $\Longrightarrow$ Cliff$(\mathcal{M})$

$$c_i c_j + c_j c_i = 2\delta_{ij}\mathbb{1}$$

c_i – generate real Clifford algebra

- $*$ algebra of operators

$$\mathcal{A} = \text{Cliff}(\mathcal{M}) \otimes \mathbb{C}$$

$*$: $\uparrow$ transpose $\otimes$ cplx conj.

- Fock space repn : Choose a cplx str I on $\mathcal{M}$

$$\mathcal{M}\otimes\mathbb{C} := V \cong \underset{\frac{1}{2}(1 - I\otimes i)V}{\underset{\parallel}{W}} \oplus \underset{\frac{1}{2}(1 + I\otimes i)V}{\underset{\parallel}{\overline{W}}}$$

$\mathcal{H}_F = \Lambda^* W$ W acts by exterior mult
$\overline{W}$ acts by contraction

To connect with standard treatments

Given I choose basis so that

$$I\, c_{2j-1} = -c_{2j}$$
$$I\, c_{2j} = c_{2j-1}$$

$$a_j = \frac{1}{2}(c_{2j-1} - i c_{2j}) \in \overline{W}$$
$$\overline{a_j} = \frac{1}{2}(c_{2j-1} + i c_{2j}) \in W$$

$$\mathcal{H}_F = \underset{\overset{\|}{\mathbb{C}}}{\Lambda^0 W} \oplus \Lambda^1 W \oplus \cdots$$

vac. line : annihilated by a_j.

<u>Free</u> Fermions :

$H \in \mathcal{A}$ is quadratic :

$$H = \frac{i}{4} \sum_{i,j} A_{ij} c_i c_j \,, \quad \left\{ \begin{array}{l} A^{tr} = -A \\ \underline{\text{real}} \end{array} \right.$$

Def: (G, ϕ) acts on FF's if

$$\exists \quad S : G \longrightarrow O(\mathcal{M})$$
$$S_g : e_j \longrightarrow \sum (S_g)_{j'j}\, e_{j'}$$

And it is extended to $\mathrm{Aut}_{\mathbb{R}}(\mathcal{A})$ using ϕ.

Exercise: Show that if (G,ϕ) acts on FF then it is compatible with H if $\exists$ homom. $\tau : G \to \mathbb{Z}_2$ s.t.

$$\boxed{S_g A S_g^{tr} = \tau(g) A \ , \ \forall g \in G}$$

FFDP: Given (G,ϕ) acting on FF, What is the ensemble of compatible FF Hamiltonians?

13B// Lie Algebras

Now observe that $so(N) = \left\{ \begin{array}{c} N \times N \text{ real} \\ \text{antisymmetric} \\ \text{matrices} \end{array} \right\}$ is a Lie algebra $[X_1, X_2] \in so(N)$ if X_i are

~~We are int~~

So the ensemble of all FF Ham's is $so(N)$. We are interested in the subspace

$$so(N) \supset \mathcal{A} := \left\{ A \mid S_g A S_g^{tr} = \tau(g) A \ , \forall g \in G \right\}$$

$\mathcal{A}$ is, in general, not a Lie algebra:

$$S_g [A_1, A_2] S_g^{tr} = [A_1, A_2]$$

By the same token:

$$k = \{ X \in so(N) \mid S_g X S_g^{tr} = X \}$$

is a Lie subalgebra.

Form the direct sum:

$$\mathfrak{g} = k \oplus \mathfrak{p}$$

This is a Lie algebra:

$$[k_1 \oplus p_1, k_2 \oplus p_2] := \left([k_1, k_2] + [p_1, p_2] \right) \oplus \left([k_1, p_2] + [p_1, k_2] \right)$$

Note it has a Lie algebra automorphism

$$\theta = \begin{cases} +1 & \text{on } k \\ -1 & \text{on } \mathfrak{p} \end{cases}$$

13C// Cartan Symmetric Spaces

Cartan classified pairs $(\mathfrak{g}, \theta)$ where $\mathfrak{g}$ = semisimple L.A and θ is a L.A. involution: $\theta^2 = 1$. This means there is a decomposition

$$\mathfrak{g} = \underset{\theta=+1}{k} \oplus \underset{\theta=-1}{\mathfrak{p}}$$

I will say something about the lassification momentarily, but first let s look at this structure at the e group level

$\mathfrak{g}, \mathfrak{k}$ generate groups G, K

θ is diffl of $\Theta : G \to G$, $\Theta^2 = 1$

$K = \mathrm{Fix}(\Theta)$

$T_K(G/K) \cong \mathfrak{p}$

G/K is a special kind of homogeneous space known as a "symmetric space"

Very beautifully, there is an actual *subspace* of G isomorphic to G/K:

$$G/K \xrightarrow{\approx} \mathcal{O} := \mathrm{Fix}^-(\Theta) := \{ g \mid \Theta(g) = g^{-1} \}$$

"antifixed points"

$$gK \longmapsto \Theta(g)\, g^{-1}$$

$$T_1 \mathcal{O} \cong \mathfrak{p}.$$

In our case we see that the ensemble of FF Hamiltonians compatible with (G, ϕ) is going to be the tangent space at 1 of a Cartan-embedded symmetric space.

One can show (easy):

- $\mathfrak{g}, k$ are compact
- $\mathfrak{g}, k$ are classical: A, B, C, D

This is the main statement of AZ, HHZ:

<u>Thm</u>: Ensembles of FF H's compatible with a symmetry (G, ϕ) are 4 identified with tangent space @ 1 of Cartan embedded compact, classical, symmetric space.

13D// Cartan's Classification

Cartan classified (G, Θ), with G [illegible] simple, and found 10 infinite sequences and ~~some~~ a finite set of exceptional cases.

The exceptional cases involve exceptional Lie groups. We just want the irreducible compact classical symmetric spaces.

$\mathbb{K}$ ~~$\mathbb{D}$~~ $= \mathbb{R}, \mathbb{C}, \mathbb{H}$. (Use $\mathbb{K}$ NOT $\mathbb{D}$ since $\mathbb{D}$ looks like a dimension)

1, 2, 3

$$G = U(N, \mathbb{K}) \times U(N, \mathbb{K})$$

$$\Theta(g_1, g_2) = (g_2, g_1)$$

$$K = U(N, \mathbb{K})_{diag}$$

$$G/K \cong U(N, \mathbb{K})$$

4, 5, 6

$$G = U(N, \mathbb{K}) \qquad g_0 = \begin{pmatrix} \mathbb{1}_{n_+} & \\ & -\mathbb{1}_{n_-} \end{pmatrix}$$

$$n_+ + n_- = N$$

$$\Theta(g) = g_0 g g_0^{-1}, \quad K = U(n_+, \mathbb{K}) \times U(n_-, \mathbb{K})$$

G/K: Grassmannians

7: $G = U(N, \mathbb{C})$, $\Theta(g) = g^*$, $K = O(N)$

8. $G = O(2N)$, $\Theta(g) = I_0 g I_0^{-1}$ $K \cong U(N)$

9. $G = USp(2N)$, $\Theta(g) = -i g i$, $K \cong U(N)$

10. $G = U(2N)$, $\Theta(g) = I_0 g^* I_0^{-1}$, $K \cong USp(2N)$

⑭ Matching Symmetric Spaces To Superdivision Algebras

Once again we can ask: Is there some natural 1–1 correspondence between the 10 symmetric spaces and the 10 superdivision algebras, or equivalently the 10 Morita classes of $\mathbb{R}, \mathbb{C}$ Clifford algebras

One way to do this is the following table:

~~Now we claim that~~

Once again, we have 10 items, but ~~[illegible]~~

Can we give a ♮ assignment to the 10 CT groups or the 10 Morita classes of C.A.'s?

We claim there is [Fill in as you go along]

UC $M_{2,2}$	T^2	C^2	[Clifford]	Symm Space	Cartan's Label.
$\{1\}$			$[\mathbb{C}l_0]$	$U/U\times U$	AIII
$\langle \bar{C}\bar{T}\rangle$			$[\mathbb{C}l_1]$	$U\times U/U$	A
$\langle \bar{T}\rangle$	$+1$		$[Cl_0]$	$O/O\times O$	BDI
$\langle \bar{T}\rangle$	-1		$[\mathbb{H}]=[Cl_4]$	$Sp/Sp\times Sp$	CII
$\langle \bar{C}\rangle$		$+1$	$[Cl_{+2}]$	Sp/U	CI
$\langle \bar{C}\rangle$		-1	$[Cl_{-2}]$	O/U	DIII
$M_{2,2}$	$+1$	$+1$	$[Cl_{+1}]$	U/O	AI
$M_{2,2}$	$+1$	-1	$[Cl_{-1}]$	$O\times O/O$	D
$M_{2,2}$	-1	$+1$	$[Cl_{+3}]$	$Sp\times Sp/Sp$	C
$M_{2,2}$	-1	-1	$[Cl_{-3}]$	U/Sp	AII

We derive the table as fallows:

We use a parallel construction for both $\mathbb{R}, \mathbb{C}$ Clifford algebras.

The construction is basically due to J. Milnor, in his book on Morse Theory and was applied to topological insulators in a paper of M. Stone et. al., although my treatment will differ in several important details.

Consider $\mathbb{C}l_{2d}$ with generators J_i in the Dirac rep$^{\underline{n}}$ $\mathcal{S} \cong \mathbb{C}^{2^d}$

We take: $J_i^{\dagger} = -J_i$ & $J_i^2 = -1$

Set $2^d = 2r$. Think of d as large so r is a large power of 2.

Now we define a nested sequence of groups:

$$\underset{U(2r)}{\underset{\parallel}{G_0}} \supset G_1 \supset G_2 \supset \cdots$$

For $k \geq 1$:

$$G_k := \{ g \in G_0 \mid g J_s = J_s g, \; s = 1, \dots k \}$$

Claim: This sequence of groups is isomorphic to

$$U(2r), \; U(r) \times U(r), \; U(r), \; U(\tfrac{r}{2}) \times U(\tfrac{r}{2}), \; \dots$$

We can see this somewhat directly:

WLOG $J_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \Rightarrow G_1 = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}$

$$\sim \begin{pmatrix} A+B & \\ & A-B \end{pmatrix}$$

So $G_1 \cong U(r) \times U(r)$

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow G_2 = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \right\} \cong U(r)$$

Moreover: For $k > 2$: $\begin{cases} J_k = j_k \otimes \sigma^3 \quad j_k \text{ generate } \mathbb{C}l_{2d-2} \\ \text{just acts within blocks} \end{cases}$

So the pattern repeats.

Now we have a sequence of homogeneous spaces $O_k \cong G_{k-1}/G_k$, $k \geq 1$

In fact they are Cartan symmetric spaces.

One nice way to see it is:

Step 1: $\Theta_1(g) = J_1 g J_1^{-1}$ Involution on G_0

$$G_1 = \mathrm{Fix}(\Theta_1)$$

$$O_1 = \{g \in G_0 \mid \Theta_1(g) = g^{-1}\} \cong G_0/G_1$$
$$= \{g \in G_0 \mid (J_1 g)^2 = -1\}$$

It will be convenient to consider the translated copy:

$$\tilde{O}_1 := J_1 O_1 = \{g \in G_0 \mid g^2 = -1\}$$

Step 2: $\Theta_2(g) = J_2 g J_2^{-1}$ Involution of G_1

$$G_2 = \mathrm{Fix}(\Theta_2)$$

$$O_2 = \mathrm{Fix}^-(\Theta_2) \subset G_1 \qquad O_2 \cong G_1/G_2$$

Can show: $J_2 O_2 = \tilde{O}_2 = \{g \in G_0 \mid g^2 = -1 \text{ \& } \{g, J_1\} = 0\}$

… and so forth ….

$$\Theta_k(g) = J_k\, g\, J_k^{-1} \qquad \text{Involution on } G_{k-1}$$

$$G_k = \mathrm{Fix}(\Theta_k)$$

$$\mathcal{O}_k = \mathrm{Fix}^-(\Theta_k) \cong G_{k-1}/G_k$$

$$J_k\mathcal{O}_k = \tilde{\mathcal{O}}_k = \{ g\in G_0 \mid g^2=-1 \ \& \ \{g, J_s\}=0,\ s=1,\cdots k-1 \}$$

One can show:

$$\tilde{\mathcal{H}}_k := T_{J_k}\tilde{\mathcal{O}}_k = J_k\cdot T_1\mathcal{O}_k$$

$$= \{ A\in u(2r) \mid \{A, J_s\}=0,\ s=1,\cdots,k \}$$

But now we recognize this as a symmetry-constrained set of FF Hamiltonians:

$$u(2r) \hookrightarrow o(4r)$$

as the commutant of $I_0 = \begin{pmatrix} 0 & 1\!\!1_{2r} \\ -1\!\!1_{2r} & 0 \end{pmatrix}$

$$a+ib \longrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

View $u(2r)$ as the commutant of $I_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $o(4r)$. So embed A, J_s according to

$$X + iY \longrightarrow \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$$

$$\tilde{\mathfrak{P}}_k = \left\{ A \in o(4r) \;\middle|\; \begin{array}{l} I_0 A I_0^{-1} = A \\ J_s A J_s^{-1} = -A \quad s = 1, \dots, k \end{array} \right\}$$

$$G = \langle I_0, J_1, \dots, J_k \rangle$$

generate a group $\tau(I_0) = +1$, $\tau(J_s) = -1$

This makes ~~an ungraded~~ $V = \mathcal{M} \otimes \mathbb{C}$ an ungraded repⁿ of $\mathbb{C}\ell_k$ (In fact W, $\overline{W}$ are seperately reps.)

Subtlety: Ungraded reps $\mathbb{C}\ell_k \Longleftrightarrow$ Graded Reps of $\mathbb{C}\ell_{k+1}$

" " $C\ell_{-k} \Longleftrightarrow$ Graded Reps of $C\ell_{-(k+1)}$

Now the above discussion was designed so that everything goes over nicely to the real case:

Cl_{-8d} : Real irrep $\mathbb{R}^{2^{4d}} = \mathbb{R}^{16r}$

$J_i^{tr} = -J_i$ and $J_i^2 = -1$

$G_0 \supset G_1 \supset \cdots$

$G_0 = O(16r)$ G_k = Commutant of first k Clifford gen's.

What is the series of groups now?

Recall Weyl's thm + Wedderburn type!

As ungraded algebras Cl_{-k} has type

k	Cl_{-k}	~~Commutant~~
0	$\mathbb{R}$	
1	$\mathbb{C}$	
2	$\mathbb{H}$	
3	$\mathbb{H} \oplus \mathbb{H}$	
4	$\mathbb{H}$	
5	$\mathbb{C}$	
6	$\mathbb{R}$	
7	$\mathbb{R} \oplus \mathbb{R}$	
8	$\mathbb{R}$	

So commutant has same type so the groups are :

$O(16r)$
$U(8r)$
$Sp(4r)$
$Sp(2r) \times Sp(2r)$
$Sp(2r)$
$U(2r)$
$O(2r)$
$O(r) \times O(r)$
$O(r)$

and $O_k = G_k / G_{k-1}$ as before

Now

$$\tilde{\mu}_k = T_{J_k}\tilde{O}_k = J_k T_1 O_k$$

$$= \{ A \in o(16r) \mid \{A, J_s\} = 0 \quad s = 1, \dots k \}$$

Making V an ungraded rep of Cl_{-k}

$\Longleftrightarrow$ graded rep of $Cl_{-(k+1)}$

so $O_1 = O/U \Longleftrightarrow Cl_{-1}^{\text{ungrad}} \Longleftrightarrow Cl_{-2}^{\text{graded}}$

This gives the table.

~~≥≥ Warning!! My table differs from Fid-Kitaev, Ludwig et.al. by a shift of $Cl_k \to Cl_{1-k}$. Perhaps because we are asking slightly different questions and/or using graded vs. ungraded algebras.~~

22 Warning!!

My table differs from That of Ludwig et.al. and Fidkowski-Kitaev by a substitution

$$Cl_K \to Cl_{1-K}$$

That is: If I associate symmetric space $\mathcal{S}_K$ with Cl_K they will associate it with Cl_{1-K}.

This is probably do to using graded vs. ungraded C.A.'s and different choices of operator grading - but this needs to be checked.

(15) Bott Periodicity & Morse Theory

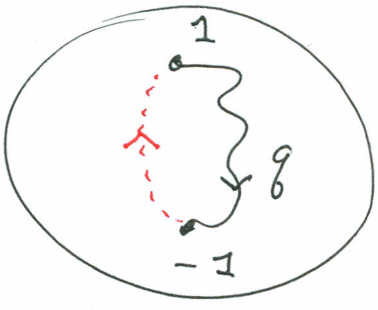
Finally, I want to give a taste of how these symmetric spaces (rather, their large r limits) arise in topology.

The key idea is to study Morse theory on a space of paths. Lets start with

$$\mathcal{P} = \left\{ g : [0,1] \longrightarrow G_0 \;\middle|\; \begin{matrix} g(0) = \mathbb{1} \\ g(1) = -\mathbb{1} \end{matrix} \right\}$$

$G_0 = U(2r)$ or $O(16r)$. We can work in parallel.

Fact: $\mathcal{P}$ has the same homotopy type as the based loop space $\Omega_* G_0$



So choosing a standard path from 1 to −1 to every open path assign a closed path.

Can show: Homotopy equivalence.

Now let us view these paths as trajectories of a particle moving in G_0. It has a natural action

$$S = - \int_0^1 dt \, \mathrm{Tr}(g^{-1}\dot{g})^2$$

View this as a ~~real positive~~ real-valued function $S : \mathcal{P} \longrightarrow \mathbb{R}$.

View it as a <u>Morse</u> <u>function</u>

<u>Fact</u>: The space of minimal action trajectories (MAT) is a "good approximation" to $\mathcal{P}$, and hence to $\Omega_* G_0$ as $r \to \infty$.

$$\pi_j(\mathrm{MAT}) = \pi_j(\mathcal{P}) = \pi_j(\Omega_* G_0) \qquad r >> j$$

Find <u>M</u>inimal <u>A</u>ction <u>T</u>rajectories

EOM: $\quad g(t) = \exp(\pi t A) \qquad A \in \mathfrak{g}_0$

BC: $\quad A \sim i \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{2r} \end{pmatrix} \qquad a_i \in 2\mathbb{Z}+1$

for $\mathfrak{g}_0 = \mathfrak{u}(2r)$

Note there is a whole manifold of M.A.T.'s because A need only be conjugate to this diag. matrix.

Similarly for $\mathfrak{g}_0 = o(16r)$.

Minimal Action: $S \sim \sum a_i^2 \Rightarrow a_i = \pm 1$

$\Rightarrow A^2 = -1$

Lemma: $A^+ = -A$ & $A^2 = -1 \Rightarrow A \in U(2r)$

Conversely $A^+ A = 1$ & $A^2 = -1 \Rightarrow A \in u(2r)$

Similarly for $O(16r)$ and $o(16r)$.

So we can identify the manifold of MAT's with a subset of the Lie group

$$MAT(P) = \{ g \in G_0 \mid g^2 = -1 \} \cong \tilde{Q}_1$$

~~[illegible]~~

~~[illegible]~~ So $\Omega_* G_0 \underset{h.e.}{\sim} G_0/G_1 \quad r \to \infty$

Now, do it again

$$\mathcal{P}_1 := \left\{ q : [0,1] \to \tilde{\mathcal{O}}_1 \;\middle|\; \begin{matrix} q(0) = J_1 \\ q(1) = -J_1 \end{matrix} \right\}$$

$$MAT(\mathcal{P}_1) = \left\{ q(t) = J_1 e^{\pi t A} \;\middle|\; \begin{matrix} A^2 = -1 \\ \{A, J_1\} = 0 \end{matrix} \right\}$$

Then $\tilde{A} = J_1 A$ also satisfies $\tilde{A}^2 = -1$

So we apply our lemma and

$$\cong \tilde{\mathcal{O}}_2 = \{ g \in G_0 \mid g^2 = -1 \ \& \ \{g, J_1\} = 0 \}$$

Morse theory $\Rightarrow$ $\Omega_* \tilde{\mathcal{O}}_1 \underset{h.e.}{\sim} \tilde{\mathcal{O}}_2 \quad r \to \infty$

And so on:

$$\mathcal{P}_k := \left\{ q : [0,1] \to \tilde{\mathcal{O}}_k \;\middle|\; \begin{matrix} q(0) = J_k \\ q(1) = -J_k \end{matrix} \right\}$$

$$MAT(\mathcal{P}_k) \cong \tilde{\mathcal{O}}_{k+1}$$

$$\Omega_* \tilde{\mathcal{O}}_k \underset{h.e.}{\sim} \tilde{\mathcal{O}}_{k+1} \quad r \to \infty$$

Now consider the complex case

$$\tilde{O}_1 \approx \frac{U(2r)}{U(r) \times U(r)}$$

$$\tilde{O}_2 \approx \frac{U(r) \times U(r)}{U(r)} \approx U(r)$$

So $\Omega^2_* U(2r) \sim U(r) \quad r \to \infty$

$\Omega^2_* U \sim U$ <u>Bott periodicity</u>!

Similarly: Mod 8 periodicity for real case.

<u>Def</u>: A sequence of spaces with basepoint $\{\mathcal{E}_q\}_{q \in \mathbb{Z}}$ is a <u>loop spectrum</u> if there are homotopy equivalences

$$\mathcal{E}_q \xrightarrow{\approx} \Omega_* \mathcal{E}_{q+1}$$

Given a loop spectrum one defines a cohomology theory by

$$E^q(X) := [X, \mathcal{E}_q]$$

- $\mathcal{E}_q = K(q, \mathbb{Z})$ $\quad q \geq 0 \quad \Longrightarrow H^q$
- $\mathcal{E}_q = \dots, \frac{U \times U}{U}, \frac{U}{U \times U}, \dots \quad \Longrightarrow K^q$

Final Remarks:

1.) We have indicated one path from FF to K-theory. This is the beginning of one interpretation of Kitaev's periodic table, but really we have only done one *Column* $d=0$:

We did not include d-dim'l locality. Kitaev has done this and has arguments that the homotopy groups just get shifted by d.

2.) We have also not included interactions. Again, Kitaev has defined

spectrum of spaces of interacting
ice Hamiltonians with invertible,
short-range-entangled ground states.
t differs from the K-theory spectrum.
At the same time, Freed+ Hopkins
ve been classifying invertible topological
field theories and seem to be converging
to the same spectrum.