ASC Lecture II

- Put up the outline again for review.
- Answer question about 1diml Berry phase.
- Review briefly lecture 1 esp. 4A.

The 4 boxed spaces are examples of <u>Cartan symmetric spaces</u>, and their tangent spaces at the origin will ~~be~~ define 4 of the 10 ensembles of free fermion Hamiltonians in the AZ scheme.

4 Representation Theory For $\mathbb{Z}_2$-Graded Groups

Now recall what I said about $\mathbb{Z}_2$-graded group in QM.

4A Let (G,ϕ) be a $\mathbb{Z}_2$-graded group

<u>Def:</u> 1.) A <u>(G,ϕ)-rep</u>, denoted (V,ρ) is a $\mathbb{C}$-v.s. V with

$$\rho : G \longrightarrow \mathrm{End}(V_{\mathbb{R}})$$

$$\rho(g) = \begin{cases} \mathbb{C}\text{-linear} & \phi(g) = +1 \\ \mathbb{C}\text{-antilinear} & \phi(g) = -1 \end{cases}$$

2.) An <u>intertwiner</u> $\mathcal{O} : (V_1, \rho_1) \xrightarrow{\mathcal{O}} (V_2, \rho_2)$ is a $\mathbb{C}$-linear map:

$$\begin{tikzcd} (V_1,\rho_1) \arrow[r, "\mathcal{O}"] \arrow[d, "\rho_1(g)"] & (V_2,\rho_2) \arrow[d, "\rho_2(g)"] \\ (V_1,\rho_1) \arrow[r, "\mathcal{O}"] & (V_2,\rho_2) \end{tikzcd}$$

$\mathrm{Hom}^G_{\mathbb{C}}(V_1, V_2)$ = ~~real~~ $\mathbb{R}$-v.s. of intertwiners

(Note: If $\phi \neq 1$ then $\mathcal{O}$ intertwiner $\Rightarrow$ $i\mathcal{O}$ <u>not</u> an intertwiner)

<u>Expl</u> $(M_2^{\pm}, \phi)$ – reps

M_2^+ : $\rho(T)$ $\mathbb{C}$-antilin. $\rho(T)^2 = 1$

$\therefore$ an (M_2^{+}, ϕ) rep <u>is</u> a complex vector-space w/ real structure.

M_2^- $\rho(T)$ $\mathbb{C}$-antilinear $\rho(T)^2 = -1$

$\therefore$ an (M_2^-, ϕ) rep <u>is</u> a quaternionic structure ($\therefore \Rightarrow V \cong \mathbb{C}^{2n}$ has <u>even</u> dimension $\Rightarrow$ Kramers' theorem)

4B/// Most of the rep. theory of (G, ϕ) is just like ordinary rep theory: Unitary, reducible etc. One big difference is in Schur's lemma. To state it nicely we need:

<u>Def</u>: A division algebra $\mathcal{A}$ is such that $\forall a \neq 0$, a^{-1} exists.

~~<u>Thm</u> (Schur) 1. $\mathrm{Hom}_{\mathbb{C}}^{G}(V_1, V_2) = 0$ for irreps $V_1 \neq V_2$~~

~~2. $Z(V, \rho) := \mathrm{Hom}_{\mathbb{C}}$~~

Thm (Schur)

1. An intertwiner between two irreps is either zero, or an isomorphism

2. If (V,ρ) is a (G,ϕ) irrep then

$$\mathrm{End}^G_{\mathbb{C}}(V) = \mathrm{Hom}^G_{\mathbb{C}}(V,V) = \text{Real division algebra}$$

Pf: . . .

So: What are the real division algebras?

Thm (Frobenius): $\exists$ precisely 3 associative real division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}$

That's the "three" in the "threefold way"

END ASC LECTURE I

Expl 1 $(G,\phi) = M_2^+$

Irrep $V = \mathbb{C}$, $\rho(T) = \mathcal{C}$.

$$\mathrm{Hom}^G_{\mathbb{C}}(V,V) = \{ \mathcal{O} : z \to rz \mid r \in \mathbb{R} \} \cong \mathbb{R}$$

Expl 2: $(G,\phi) = M_2^-$; $V = \mathbb{C}^2$

$$\rho(e^{i\theta})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{i\theta} z_1 \\ e^{i\theta} z_2 \end{pmatrix} \quad \& \quad \rho(T)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix}$$

How shall we compute the vector space of intertwiners?

Note that the operators $\rho(e^{i\theta})$ and $\rho(T)$ generate an <u>algebra</u> of operators, and the intertwiners must be $\mathbb{C}$-linear operators commuting with this algebra.

In fact, quite generally, for a (G,ϕ)-rep V, let

$$\mathcal{A} = \text{algebra of operators gen. by } \rho(g), I \in \mathrm{End}(V_{\mathbb{R}})$$

$$\mathrm{End}^G_{\mathbb{C}}(V) = \mathcal{A}' = \left\{ \mathcal{O} : V_{\mathbb{R}} \to V_{\mathbb{R}} \;\middle|\; \begin{array}{l} \mathcal{O}a = a\mathcal{O} \\ \forall a \in \mathcal{A} \end{array} \right\}$$

$$= \text{"commutant of } \mathcal{A} \text{ in } \mathrm{End}(V_{\mathbb{R}})\text{"}$$

This is itself an algebra, and $\mathcal{A}'' = \mathcal{A}$.

Now let's get back to our example. We use a trick:

$$\mathbb{C}^2 \xrightarrow{\approx} \mathbb{H}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow z_1 + z_2 \underline{j}$$

$$= x_1 + y_1 \underline{i} + x_2 \underline{j} + y_2 \underline{k}$$

$$\rho(e^{i\theta}) = c(\theta) + s(\theta) L(\underline{i}) \quad \forall \theta$$

$$\rho(T) = L(\underline{j})$$

Together these generate the full algebra

$$\{L(\underline{q})\} \cong \mathbb{H}$$

Now recall that $\{R(\underline{q})\} \cong \mathbb{H}^{opp}$ is in the commutant and since

$$\mathbb{H} \otimes \mathbb{H}^{opp} \cong \mathrm{End}(\mathbb{R}^4)$$

this is the full commutant.

So, this gives an example of the third case of Schur's lemma.

4C// Complete Reducibility

Thm: Suppose G is compact. Any (G,ϕ)-rep is completely reducible:

- $\{(V_\lambda, \rho_\lambda)\}$ complete list of distinct irreps

$$\underset{\rho(g)}{\overset{\circlearrowleft}{V}} \cong \bigoplus_\lambda \underset{Id}{\overset{\circlearrowleft}{S_\lambda}} \otimes \underset{\rho_\lambda(g)}{\overset{\circlearrowleft}{V_\lambda}}$$

S_λ = <u>real</u> vector space of degeneracies

If we choose a suitable basis this is just block diagonalization

$$\rho(g) = \begin{array}{c} \lambda \longrightarrow \\ \downarrow \begin{bmatrix} \ddots & & \\ & \mathbb{1}_{S_\lambda} \otimes \rho_\lambda(g) & \\ & & \ddots \end{bmatrix} \end{array}$$

$$\mathbb{1}_{S_\lambda} \otimes \rho_\lambda(g) = \left.\begin{bmatrix} \rho_\lambda(g) & & & \\ & \rho_\lambda(g) & & \\ & & \ddots & \\ & & & \rho_\lambda(g) \end{bmatrix}\right\} s_\lambda \text{ blocks}$$

So the operators commuting with $\rho(g)$ act on the blocks, with entries in $\mathrm{End}_{\mathbb{C}}^{G}(V_\lambda)$:

$$\mathrm{End}_{\mathbb{C}}^{G}(V) \cong \bigoplus_{\lambda} \mathrm{End}(S_\lambda) \otimes \underbrace{\mathrm{End}_{\mathbb{C}}^{G}(V_\lambda)}_{= D_\lambda \to \mathbb{R}, \mathbb{C}, \mathbb{H} \text{ depending on } V_\lambda}$$

$$\cong \bigoplus_{\lambda} \mathrm{Mat}_{s_\lambda}(D_\lambda)$$

<u>Def</u>: An algebra consisting of a direct sum of matrix algebras over $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is called a semisimple algebra and the list of D's which occur is the <u>Wedderburn</u> <u>type</u>.

Put differently:

$\mathcal{A}$ = algebra gen. by $\rho(g)$, $I \in \mathrm{End}(V_{\mathbb{R}})$

$$\mathrm{End}_{\mathbb{C}}^{G}(V) = \text{Commutant of } \mathcal{A} \text{ in } \mathrm{End}(V_{\mathbb{R}}) = \mathcal{A}'$$

Now introduce some notation:
If K is any algebra let

$$K(m) := \mathrm{Mat}_m(K)$$

$$mK := \left\{ \begin{pmatrix} k & & \\ & \ddots & \\ & & k \end{pmatrix} \middle| k \in K \right\} \subset K(m)$$
$$\cong K$$

Thm (Weyl). G cpt

$$d_\lambda = \dim_{\mathbb{C}} V_\lambda \qquad \tau_\lambda = \begin{cases} 2 d_\lambda & D_\lambda = \mathbb{R} \\ d_\lambda & = \mathbb{C} \\ \frac{1}{2} d_\lambda & = \mathbb{H} \end{cases}$$

$$\mathcal{A} \cong \bigoplus_\lambda s_\lambda \, D_\lambda(\tau_\lambda)$$

$$\mathcal{A}' \cong \bigoplus_\lambda \tau_\lambda \, D_\lambda^{opp}(s_\lambda)$$

Remarks: (0.) N.B. The Wedderburn type of $\mathcal{A}, \mathcal{A}'$ is the same - we'll make use in (15)

(1) Complete reducible not nec. true for general G

(2) Nevertheless it does apply to many examples of noncpt groups, notably crystallographic groups.

For example if G = lattice in Euclidean Space then we have Bloch's thm:

$$\mathcal{H} \cong \int_{\text{Brill.}}^{\oplus} dk \, \mathcal{H}_k$$

(Thats "Bloch" with an "h"!)

(5) Quantum Aut's Compatible With Dynamics

We saw that a group preserving quantum probabilities is $\mathbb{Z}_2$-graded

In physical situations there is also a <u>second grading</u>:

$$\tau : G \to \mathbb{Z}_2$$

telling whether the symmetry preserves or reverses the orientation of time.

Suppose (for simplicity) that our system is time-translation invariant. Then unitary evalution is generated by a self-adjoint Hamiltonian:

$$U(t) = \exp\left(-\frac{it}{\hbar} H\right)$$

<u>Def:</u> A (G,ϕ) rep $\mathcal{H}$ is compatible with dynamics if

$$\boxed{\rho(g)\, U(t)\, \rho(g)^{-1} = U(\tau(g)t)} \quad \forall g \in G$$

<u>Ex:</u> Show this is equivalent to

$$\boxed{\rho(g)\, H \rho(g)^{-1} = \chi(g) H} \quad \forall g \in G.$$

where $\chi(g) = \phi(g)\tau(g)$

Note: $\phi \cdot \tau \cdot \chi = 1$

Which two of the three are given a priori depends on how we formulate the problem.

~~Remarks:~~

~~1) If $\chi(g) = -1$ for some g~~

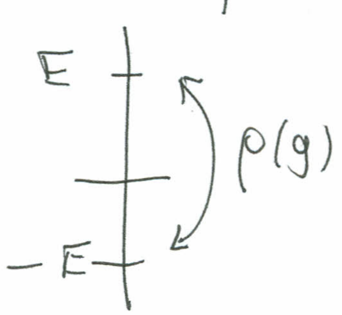
General principles of Q.M. $\Rightarrow$ a "symmetry group" always comes with a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading

$$(\phi, \chi) : G \longrightarrow M_{2,2} := \mathbb{Z}_2 \times \mathbb{Z}_2$$

Remarks:

1.) If $\chi(g) = -1$ for some $g \in G$

Then Spec(H) is symmetric around $E = 0$



In general, there is no distinguished zero of energy, but here there is.

2.) So, if H is bounded from below, but not above then $\chi = 1$.

In fact all QM textbooks say $\chi=1$, i.e. they all say $\phi = \tau$

3.) Nevertheless there are physically interesting examples with $\chi \neq 1$, that is, with antilinear "symmetries" that preserve time orientation -

~~I will give a list later in lecture II, but Charlie Kane already mentioned "chiral symmetries" in polyacetylene.~~

For the moment ~~Henceforth in this lecture~~ we take $\chi=1$.

Then ~~In the next~~ we will allow $\chi \neq 1$ and that will lead to the 10-fold way.

4.) $1 \to \underset{G_0}{\overset{\parallel}{\ker\tau}} \to G \xrightarrow{\tau} \mathbb{Z}_2 \to 1$

In general this is NOT split, i.e.

$$G \neq G_0 \times \mathbb{Z}_2 \quad \text{NOR} \quad G_0 \rtimes \mathbb{Z}_2$$

"nonsymmorphic magnetic crystallographic groups"

$\exists$ 230 3D crystallographic groups but 1651 magnetic crystall. groups!

5.) (For experts) in the right spot?

When discussing band structure with magnetic crystallographic symmetry you need the double-cover.

Should the magnetic point group be a subgp of $Pin^+(3)$ or $Pin^-(3)$?

The repn theory is different so there is a physical difference. I claim it is in $Pin^+(C)$ and I can explain afterwards.

Of course a crystal breaks Lorentz invariance : $O(1,3) \longrightarrow \mathbb{Z}_2 \times O(3)$

Nevertheless we know that the rel. corrections to band structure such as spin orbit and Darwin terms can be obtained from the Dirac Hamiltonian

$$\left(c\alpha\cdot p + \beta mc^2 + V(\vec{r}) \right)\psi = \left(mc^2 + \varepsilon \right)\psi$$

But now we are constrained in our choice of Dirac rep. If we take

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{matrix} \leftarrow \text{large} \\ \leftarrow \text{small} \end{matrix}$$

and we get Schröd + Rel. corrections for ψ_1

So let's recall the P,T invce of the Dirac eq.

$$\left(i\gamma^\mu \partial_\mu - mc - \gamma^0 V \right) \psi = 0$$

Time reversal: $(T\psi)(x) = \xi C \psi^*(Tx)$

$$C(\gamma^i)^* C^{-1} = -\gamma^i$$
$$C(\gamma^0)^* C^{-1} = \gamma^0$$

$T^2 = -1$ for any phase.

But $(P_{\hat n}\psi)(x) = \xi_{\hat n} (\hat n \cdot \vec\gamma) \gamma_5 \psi\, P(x)$

$$\xi_{\hat n}^2 = \pm 1 \ ?$$

Equation doesn't tell you, but ~~we should~~ if we turn off V we should be representing Pin(1−, 3+) and not some other signature - by rel. invariance.

Now if we do have V note that in this basis

$$C = \begin{pmatrix} \sigma^2 & \\ & -\sigma^2 \end{pmatrix} \qquad \hat n \cdot \vec\gamma\, \gamma_5 = -i \begin{pmatrix} \hat n \cdot \vec\sigma & \\ & -\hat n \cdot \vec\sigma \end{pmatrix}$$

ARE BLOCK DIAGONAL.

Now we have

$$
\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, no head] & \tilde{O}_m = \mathrm{Pin}^+(3)\times\mathbb{Z}_2 \arrow[r] \arrow[d] & O(3)\times\mathbb{Z}_2 \arrow[d] & \\
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathrm{Pin}(1-,3+) \arrow[r] & O(1,3) \arrow[r] & 1
\end{tikzcd}
$$

If we pull back the 4D ϕ-irrep of $\mathrm{Pin}(1-,3+)$ it becomes reducible, a sum of 2 2D irreps for the large and small components.

Moreover the correct cover of the magnetic point group is the pullback

$$
\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & \hat{P}(C) \arrow[r] \arrow[d] & P(C) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \tilde{O}_m = \mathrm{Pin}^+(3)\times\mathbb{Z}_2 \arrow[r] & O(3)\times\mathbb{Z}_2 \arrow[r] & 1
\end{tikzcd}
$$

Finally the correct double cover space group for band structure is:

$$
\begin{tikzcd}
1 \arrow[r] & L(C) \arrow[r] \arrow[d, equal] & \hat{G}(C) \arrow[r] & \hat{P}(C) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & L(C) \arrow[r] & G(C) \arrow[r] & P(C) \arrow[r] & 1
\end{tikzcd}
$$

⑥ Dyson's 3-Fold Way

Now finally we return to the Dyson problem.

Usually we start with H and ask for the symmetries. ~~Instead~~ Dyson turned it around:

$$\boxed{\begin{array}{l}\underline{\text{DP}}: \text{ Given a } (G,\phi)\text{-rep } \mathcal{H}, \\ \text{what is the ensemble of compatible} \\ \text{Hamiltonians (with } \chi = 1\text{). ?}\end{array}}$$

$$\mathcal{H} = \bigoplus_{\lambda} S_\lambda \otimes V_\lambda$$

$$Z(\mathcal{H},\rho) \cong \bigoplus_{\lambda} \mathrm{Mat}_{s_\lambda}(D_\lambda)$$

$$\cup$$

$$\mathcal{E} = \bigoplus_{\lambda} \mathrm{Herm}_{s_\lambda}(D_\lambda) \quad \begin{cases} \text{real symm } D_\lambda = \mathbb{R} \\ \text{Herm. } D_\lambda = \mathbb{C} \\ \text{Quat. Herm } D_\lambda = \mathbb{H} \end{cases}$$

So we look at $\mathrm{Herm}_N(D)$

Each ensemble has a $\circlearrowleft U(N,D)$ invt measure with statistically indpt matrix elements: $D = \mathbb{C}$:

$$d\mu = \frac{1}{Z} \prod_i dH_{ii} \prod_{i<j} d^\beta H_{ij} \, e^{-N \operatorname{Tr} H^2}$$

This becomes a prob. distribution on the eigenvalues:

$$d\mu(\lambda_1, \dots \lambda_N) = \frac{1}{Z} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N\sum \lambda_i^2} \prod d\lambda_i$$

$$\beta = \begin{cases} 1 & \mathbb{R} \\ 2 & \mathbb{C} \\ 4 & \mathbb{H} \end{cases} = \dim_{\mathbb{R}} D$$

~~Now, I'd like to put this into some context:~~

~~As mentioned RMT originated in nuclear physics.~~

(M) Ensembles of Hamiltonians With A Gap

We can restrict our ensembles above by requiring the existence of a "gap" in the spectrum.

Often this means ~~assuming~~ a lower bound on the energy gap between the ~~lowest and first~~ ground state and the first excited state.

But more generally we could choose an energy level E_0 and require all the Hamiltonians in our ensemble to sit inside

$$\mathcal{E}_N(E_0) = \{ H \in \mathrm{Herm}_N(\mathbb{C}) \mid (H - E_0)^{-1} \text{ exists} \}$$

Now we can describe the homotopy type of this ensemble easily.

If we only care about ~~the~~ the topology of the ensemble then we can use the technique of spectral flattening.

Set $E_0 = 0$.

Then ~~we~~ for $H \in \mathcal{E}_N(E_0)$ $\mathrm{sgn}\, H$ makes sence and we can deform $H \longrightarrow \mathrm{sgn}\, H$

Note:

1. $\operatorname{sgn} H = \Pi(H)$ satisfies $\Pi(H)^2 = 1$

2. If $\wp : H_1 \rightsquigarrow H_2$ is a cont. path in $\mathcal{E}_N(0)$

then $\bar{\wp} : \Pi(H_1) \rightsquigarrow \Pi(H_2)$ is a cont. path in $\mathcal{E}_N(0)$

We can deformation retract $\mathcal{E}_N(0)$ to the spectrally flattened Hamiltonians.

But now, if $\Pi^2 = 1$ then

$$P_\pm = \frac{1}{2}(1 \pm \Pi) = \text{pair of orthogonal projectors}$$

If P_+ has rank k then it is in

$$\begin{aligned} Gr_{k,N}(\mathbb{C}) &= \{\text{Space of rank } k \text{ projectors}\} \\ &= \{\text{Space of } k\text{-dim'l subspaces of } \mathbb{C}^N\} \\ &\cong U(N,\mathbb{C}) / U(k) \times U(N-k) \end{aligned}$$

$$\mathcal{E}_N(E_0) \underset{h.e.}{\sim} \coprod_{k=0}^{N} Gr_{k,N}(\mathbb{C})$$

As with the space of pure states these spaces have interesting large k, N limits and define important "classifying spaces" in topology.

One can do exactly the same thing with $\mathbb{R}, \mathbb{H}$ - Hermitian Hamiltonians and we just replace

$$U(N,\mathbb{C}) \longrightarrow \begin{array}{l} U(N,\mathbb{R}) = O(N) \\ U(N,\mathbb{H}) = Sp(N) \cong USp(2N) \end{array}$$

This gives us 3 more series of Cartan symmetric spaces.

⑧ Some Physical Systems With $\chi \neq 1$

~~To motivate~~

Let us now return to the general symmetry constraints. We saw that any group of symmetries in Q.M. is $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded, but that usually it is assumed that $\chi = 1$.

To motivate the consideration of $\chi \neq 1$ let us return to the history of RMT. First entered condensed matter physics in the 1980's in the subject of mesoscopic physics – physics on scales ~~where~~ larger than atomic but quantum interference is still important.

(Bigger than a virus and smaller than a bacterium.)

Good example: Universal Conductance Fluctuations

Conductance of electrons in a disordered mesoscopic sample, gold ring or Si MOSFET exhibits reproducible fluctuations (say as a function of B) of order e^2/h independent of sample size, disorder, and other things.

Explained by RMT (Beenakker review)

- Verbaarschot: statistics of ev's of Dirac op.

$$\not{D} = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}$$

in a random gauge field.

- Altland + Zirnbauer: e^- transport in disordered hybrid systems of norm/sc metal

- Katz + Sarnak: Statistics of zeroes of L-functions in number theory - these are generalizations of the Riemann ζ-function.

All clearly showed the need for ensembles of Hamiltonians distinct from the 3 Dyson classes.

This culminated in the "10-fold way" Classification of AZ, HHZ - describing the symmetry classes of free fermion Hamiltonians.

Around 2008 A. Ludwig et. al. and A. Kitaev applied the same mathematics (and some new elements, like adding a gap and using K-theory) to the classification of phases of matter in the context of gapped free fermion systems.

So the primary examples with $\chi \neq 1$ ~~here~~ are ensembles of free fermion Hamiltonians

$$H = \frac{i}{4} \sum A_{ij} e_i e_j$$

e_i - Majorana

We'll come back to this.

But, are there other examples of physical systems with $\chi \neq 1$?

Answer is "Yes" But the question is how natural these examples are:

1. Lattice systems on bipartite lattices can easily have such symmetries:

$$\Lambda = \Lambda_0 \cup \Lambda_1$$

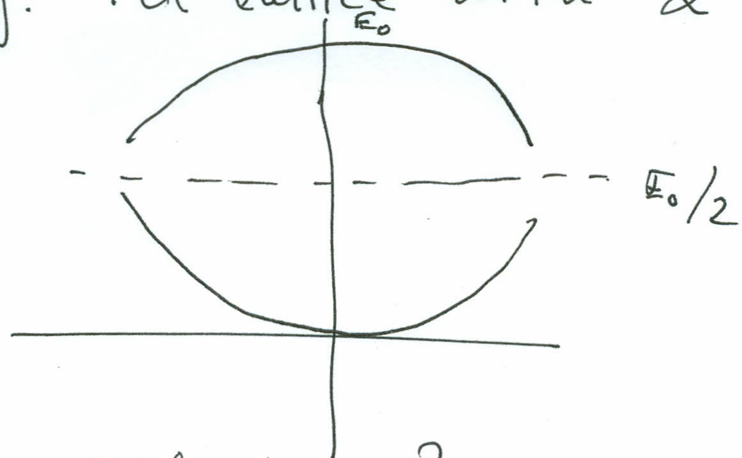
$$H = \sum_{\substack{\langle ij \rangle \\ i \in \Lambda_0\, j \in \Lambda_1}} J \sigma_i^z \sigma_j^z + \sum_{i \in \Lambda_1} h \sigma_i^y$$

$$g = \prod_{i \in \Lambda_1} \sigma_i^x$$

OR SSH MODEL OF POLYACTELYNE (Su Schrieffer-Heeger)

2. Photonic bandstructure (DeNittis + Lein, ?)

3. One can engineer phonon band structures e.g. 1d lattice with 2 atoms/unit cell



4. Cold atoms?

5. Hilbert-Polya Hamiltonian whose spectrum gives nontrivial zeroes of $\zeta(s)$ (if it exists)

So now we want to give the analog of Dyson's 3-fold way, but allowing for $\chi \neq 1$.

This will turn out to be a 10-fold way, but to explain that I need to do some more math.

9 $\mathbb{Z}_2$-Graded (Super-) Linear Algebra

<u>Def</u>: a.) A $\mathbb{Z}_2$-Graded (Super-) Vector Space/k

$$V = \underbrace{V^0}_{\text{even}} \oplus \underbrace{V^1}_{\text{odd}}$$

b.) Homogeneous vectors are vectors entirely in V^0, V^1: $\deg v = |v| = 0, 1 \bmod 2$

<u>Expl1</u>: If V has an operator Γ with $\Gamma^2 = 1$ then use ± 1 eigenspaces.
For example $\Gamma = (-1)^F$ in theories w/ fermions

<u>Expl2</u>: If V is a real v.s. w/ cplx str. I
Then $\mathrm{End}(V)$ has a $\mathbb{Z}_2$-Grading

$$\Gamma : A \longrightarrow I A I^{-1}$$

<u>Expl3</u>: $k = \mathbb{R}, \mathbb{C}$ $\quad k^{n_+|n_-} := \underbrace{k^{n_+}}_{\text{even}} \oplus \underbrace{k^{n_-}}_{\text{odd}}$

$(n_+|n_-)$ is called the superdimension

Now if V, W are s.v.s.'s

Then $Hom(V,W)$ is a svs:

$$Hom(V,W)^0 = Hom(V^0, W^0) \oplus Hom(V^1, W^1)$$
$$Hom(V,W)^1 = Hom(V^0, W^1) \oplus Hom(V^1, W^0)$$

If we choose bases of homogen. vectors we have matrices:

$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ even op's $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ odd op's

We can also speak of superalgebras:

<u>Def:</u> A <u>superalgebra</u> $\mathcal{A}$ is a svs/k with a k-bilinear even product

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$
$$\deg(a a') = \deg(a) + \deg(a')$$

- graded commute $a a' = (-1)^{|a||a'|} a' a$

(Convention: If I write a formula with "deg" in it – it is understood that the vectors are homogeneous.)

- Super center $Z_s(\mathcal{A}) = \{ a \mid ab = (-1)^{|a||b|} ba \ \forall\, b \in \mathcal{A} \}$

(might or might not be the same as the ungraded center)

<u>Graded Tensor Product</u> $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$

- SVS
- $(a_1 \hat{\otimes} a_2)(b_1 \hat{\otimes} b_2) = (-1)^{|a_2||b_1|} a_1 b_1 \hat{\otimes} a_2 b_2$

"Koszul sign rule" - can have dramatic consequences.

<u>Example:</u> <u>Matrix superalgebras</u> :

V - s.v.s $\quad \mathrm{End}(V) = \mathrm{Hom}(V, V)$

is a superalgebra.

Not all superalgebras are matrix superalgebras - as we'll see.

Finally, we need the notion of "Morita equivalence" of two superalgebras.

~~[illegible]~~

The proper math definition is that $\mathcal{A}_1, \mathcal{A}_2$ are Mortia equivalent if there is an equivalence of their category of representations, but we won't go into that.

All we need is the following:

<u>Def:</u> $\mathcal{A}_1 \underset{\text{Morita}}{\sim} \mathcal{A}_2$ if $\exists$ svs V

$$\mathcal{A}_1 \cong \mathcal{A}_2 \hat{\otimes} \operatorname{End} V \quad \left(\begin{matrix}\text{or vice}\\ \text{versa}\end{matrix}\right)$$

Denote Morita equiv. class by $[\mathcal{A}]$, and note there is a product:

$$[\mathcal{A}_1] \cdot [\mathcal{A}_2] := [\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2]$$

it might or might not have an inverse.