Comments On Global Anomalies In Six-Dimensional Supergravity

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Work in progress with SAMUEL MONNIER

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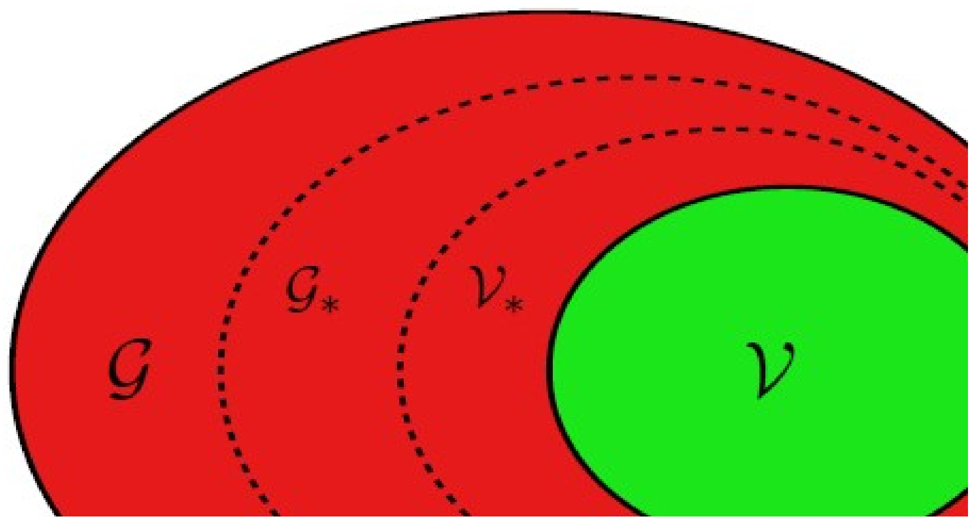
1. **Introduction & Summary Of Results**
2. Six-dimensional Sugra & Green-Schwarz Mech.
3. Quantization Of Anomaly Coefficients.
4. F-Theory Check
5. Geometrical Anomaly Cancellation, η-Invariants & Wu-Chern-Simons
6. Technical Tools
7. Conclusion & Discussion

Motivation

Relation of consistent theories of quantum gravity to string theory.

Recall the Taylor-Venn diagram:



State of art summarized in
Brennan, Carta, and Vafa 1711.00864

Brief Summary Of Results

Focus on 6d sugra

(More) systematic study of global anomalies

Result 1: Quantization of anomaly coefficients

Result 2: a-coefficient is a characteristic vector in lattice of string charges.

Result 3: Mathematically precise formulation of the 6d Green-Schwarz anomaly cancellation

Result 4: Check in F-theory: Requires knowing the global form of the (identity component of) the gauge group.

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(Pre-) Data For 6d Supergravity

(1,0) sugra multiplet + vector multiplets + hypermultiplets + tensor multiplets

VM: Choose a compact (reductive) Lie group G.

HM: Choose a quaternionic representation $\mathcal{R}$ of G

TM: Choose an integral lattice Λ of signature (1,T)

Pre-data: $(G, \mathcal{R}, \Lambda)$

6d Sugra - 2

Can write multiplets, Lagrangian, equations of motion.[Riccioni, 2001]

Fermions are chiral (symplectic Majorana-Weyl)

2-form fieldstrengths are (anti-)self dual

Multiplet	Field
Gravity	$(g_{\mu\nu},$
Tensor	$(B^{-}_{\mu\nu},$
Vector	$(A_{\mu}, \lambda$

The Anomaly Polynomial

Chiral fermions & (anti-)self-dual tensor fields ⇒ gauge & gravitational anomalies.

From G and $\mathcal{R}$ we compute, following textbook procedures,

$$I_8 \sim \left(\dim_{\mathbb{H}}(\mathcal{R}) - \dim(G) + 29\,T - 273\right) Tr(R^4) + \cdots$$

$$+ (9 - T)(Tr\, R^2)^2 + (F^4{-}type) + \cdots$$

6d Green-Schwarz mechanism requires (Sagnotti)

$$I_8 = \frac{1}{2} Y^2 \qquad Y \in H^4(\mathcal{W}; \Lambda \otimes \mathbb{R})$$

Standard Anomaly Cancellation

Interpret Y as background magnetic current for the tensor-multiplets $\Rightarrow$

$$dH = Y$$

$\Rightarrow B$ transforms under diff & VM gauge transformations…

Add counterterm to sugra action

$$\mathrm{e}^{\mathrm{i}S} \to e^{iS} e^{-2\pi i \frac{1}{2}\int BY}$$

So, What's The Big Deal?

Definition Of Anomaly Coefficients

Let's try to factorize:

$$I_8 = \frac{1}{2} Y^2 \qquad Y \in H^4(\mathcal{W}; \Lambda \otimes \mathbb{R})$$

$$\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{g}_{Abel} \cong \oplus_i \mathfrak{g}_i \oplus_I \mathfrak{u}(1)_I$$

General form of Y: $Y = \frac{a}{4} p_1 - \sum_i b_i c_2^i + \frac{1}{2} \sum_{IJ} b_{IJ} c_1^I c_1^J$

Anomaly coefficients:

$$p_1 := \frac{1}{8\pi^2} Tr_{vec} R^2$$

$$a, b_i, b_{IJ} \in \Lambda \otimes \mathbb{R}$$

$$c_2^i := \frac{1}{16\pi^2 h_i^\vee} Tr_{adj} F_i^2$$

The Data Of 6d Sugra

The very *existence* a factorization $I_8 = \frac{1}{2} Y^2$ puts strong constraints on $(G, \mathcal{R}, \Lambda)$. These have been well-explored. See Taylor's TASI lectures.

Factorization $\Rightarrow$ constraints on $a, b_i \; b_{IJ}$

Example: $a^2 = 9 - T$

There can be multiple choices of anomaly coefficients (a, b_i, b_{IJ}) factoring the same I_8

Full data for 6d sugra:

$(G, \mathcal{R}, \Lambda)$ AND $a, b_i, b_{IJ} \in \Lambda \otimes \mathbb{R}$

Standard Anomaly Cancellation -2/2

For any $(G, \mathcal{R}, \Lambda, a, b)$ adding the GS term cancels all perturbative anomalies.

All is sweetness and light…

There are solutions of the factorizations conditions that cannot be realized in F-theory!

Global anomalies ?

Does the GS counterterm even make mathematical sense ?

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The New Constraints

Global anomalies have been considered before.

Examples:

Λ is self-dual [Seiberg & Taylor - 1103.0019]

$a \cdot b_i, \quad b_i \cdot b_j \in \mathbb{Z}$ [Kumar, Morrison, Taylor- 1008.1062]

We have just been a little more systematic.

Example of new constraints:

$$a, b_i, \frac{1}{2} b_{II}, b_{IJ} \in \Lambda$$

But these are not the strongest constraints…

Quadratic Forms On $\mathfrak{g}$ And $H^4(BG)$

To state the best result we note that $b = \left(b_i, b_{IJ}\right)$ determines a $\Lambda \otimes \mathbb{R}$ –valued quadratic form on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{g}_{Abel} \cong \oplus_i \mathfrak{g}_i \oplus_I \mathfrak{u}(1)_I$$

$$b(f,f) = \sum_i b_i \frac{1}{2h_i^\vee} Tr_{\mathfrak{g}_i} f_i^2 + \frac{1}{2} \sum_{IJ} b_{IJ} f^I f^J \in \Lambda \otimes \mathbb{R}$$

$$1 \to G_1 \to G \to \pi_0(G) \to 1$$

$G_1 \cong \tilde{G}_1/\Gamma$ Γ finite group $\tilde{G}_1 := \tilde{G}_{1,ss} \times U(1)^r$

Vector space Q of $\Lambda \otimes \mathbb{R}$ -valued quadratic forms on $\mathfrak{g}$: $Q \cong H^4(BG_1; \Lambda \otimes \mathbb{R}) \cong H^4\left(B\tilde{G}_1; \Lambda \otimes \mathbb{R}\right)$

Strongest Constraints - 2

$$H^4(BG_1;\Lambda) \subset H^4\left(B\tilde{G}_1;\Lambda\right) \subset \mathcal{Q}$$

Lattices!

$$b_i, \frac{1}{2}b_{II}, b_{IJ} \in \Lambda \quad \Leftrightarrow \quad \frac{1}{2}b \in H^4\left(B\tilde{G}_1;\Lambda\right)$$

$$\frac{1}{2}b \in H^4(BG_1;\Lambda)$$

First Derivation

Assume Completeness Hypothesis:

A consistent sugra can be put on an arbitrary spin 6-fold with arbitrary gauge bundle.

Cancellation of background string charge in compact Euclidean spacetime $\Rightarrow \forall\, \Sigma \in H_4(\mathcal{M}_6; \mathbb{Z})$

$$\int_\Sigma Y \in \Lambda$$

Because the background string charge must be cancelled by strings.

Suffices to consider gauge bundles on $\mathcal{M}_6 = \mathbb{CP}^3$. Note that G_1 bundles that do not lift to $\tilde{G}_1$-bundles give the ``most fractional'' c_2^i.

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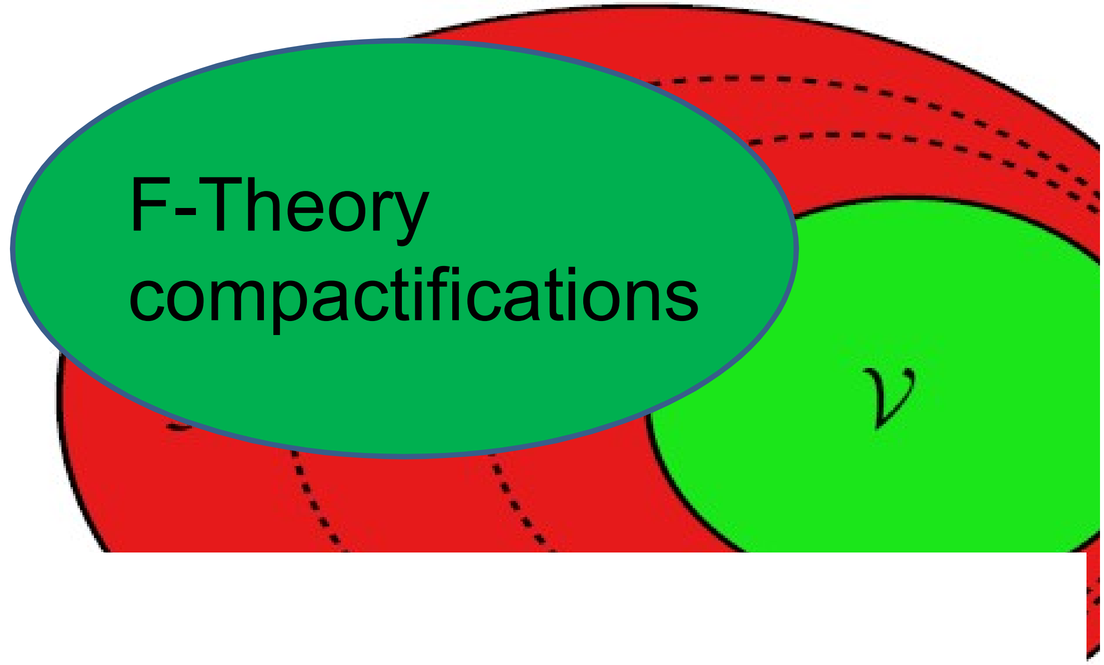
And What About F-Theory ?



?

F-Theory: Some Definitions

$\pi: \hat{X} \to B$ Smooth resolution of a Weierstrass model over 2-fold B

$\mathfrak{g}$ is determined from the discriminant locus [Morrison & Vafa 96]

$\Lambda = H_2^{free}(B; \mathbb{Z})$ Wrapped 3-branes give self-dual strings .

$$\tilde{b}: H_4^{free}(\hat{X}; \mathbb{Z}) \to H_2(B; \mathbb{Z})$$

$\tilde{b}(x_1, x_2) = -\pi_*(x_1 \cap x_2)$ Can show: $\tilde{b}$ is even.

F-Theory: More Definitions

Define a lattice: $\mathcal{H} \subset H_4^{free}(\hat{X};\mathbb{Z})$

$$\mathcal{H} := \{ x \mid x \cap f = 0 \ \& \ x|_Z = 0 \}$$
$$= \tilde{b} \perp to \ Span\{ Z, \pi^{-1}(\Sigma_i) \}$$

$$\mathcal{H} \otimes \mathbb{R} \cong \mathfrak{t} \subset \mathfrak{g}$$

D. Park: 1111.2351

$$b := \tilde{b} \big|_{\mathcal{H} \otimes \mathbb{R}}$$

In order to check $\frac{1}{2} b \in H^4(BG_1;\mathbb{Z})$ we clearly need to know G_1.

Bonus: Global Form Of F-Theory Gauge Group

All we know from the Lie algebra $\mathfrak{g}$ is that

$$G_1 = (\tilde{G}_{1,ss} \times T)/\Gamma$$

where $\Gamma \subset Z(\tilde{G}_{1,ss})$ is a finite subgroup of the center of the universal cover of the ss part.

Knowing Γ is same as knowing $\Lambda_{cocharacter}(G_1)$

Claim: $\mathcal{H} = \Lambda_{cocharacter}(G_1)$

Proof is omitted here, but we believe a very similar argument also determines the (identity component of) the gauge group in 4d F-theory compactifications.

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6d Green-Schwarz Mechanism Revisited

Goal: Understand Green-Schwarz anomaly cancellation in precise mathematical terms.

Benefit: We recover the constraints:

$$a \in \Lambda \qquad \frac{1}{2} b \in H^4(BG_1; \Lambda)$$

and derive a new constraint:

a is a ***characteristic vector***:

$$\forall v \in \Lambda \qquad v \cdot v = v \cdot a \quad mod\, 2$$

What's Wrong With Textbook Green-Schwarz Anomaly Cancellation?

What does B even mean when $\mathcal{M}_6$ has nontrivial topology? (H is not closed!)

How are the periods of dB quantized?

Does the GS term even make sense?

$$\frac{1}{2}\int_{\mathcal{M}_6} B\,Y = \frac{1}{2}\int_{\mathcal{U}_7} dB\,Y$$

must be independent of extension to $\mathcal{U}_7$!

But it isn't ….

Even for the difference of two B-fields,

$$d(H_1 - H_2) = 0$$

we can quantize $[H_1 - H_2] \in H^3(\mathcal{U}_7; \Lambda)$

$$\exp(2\pi i \frac{1}{2} \int_{\mathcal{U}_7} (H_1 - H_2) Y)$$

is not well-defined because of the factor of $\frac{1}{2}$.

Geometrical Anomaly Cancellation - 1/2

If $D: V \to V$ is a linear transformation then $\det(D)$ is a well-defined number.

If $D: V \to W$ is a linear transformation then $\det(D)$ is not a well-defined number!

Rather, it is an element of a line (= one-dimensional vector space):

$$\det(D) \in DET(D) := \Lambda^{top} W \otimes \Lambda^{top} V^{\vee}$$

Determinants of chiral Dirac operators $D: \Gamma(S^{-} \otimes E) \to \Gamma(S^{+} \otimes E)$ are sections of line bundles over spaces of metrics and connections.

Geometrical Anomaly Cancellation -2/2

Space of all fields in 6d sugra is fibered over nonanomalous fields:

$$\mathcal{B} = Met(\mathcal{M}_6) \times Conn(\mathcal{P}) \times \{Scalar\ fields\}$$

Partition function: $\int_{\frac{\mathcal{B}}{\mathcal{G}}} \int_{Fermi+B} e^{S_0 + S_{Fermi+B}}$

$$\Psi_{Anomaly}(A, g_{\mu\nu}, \phi) := \int_{Fermi+B} e^{S_{Fermi+B}}$$

is a section of a line bundle over $\mathcal{B}/\mathcal{G}$

You cannot integrate a section of a line bundle over $\mathcal{B}/\mathcal{G}$ unless it is trivialized.

Approach Via Invertible Field Theory

Definition: An invertible d – dimensional field theory Z is a ``map'' (functor)

$$\mathcal{C}_{\leq d} = \{\leq d\ dimensional\ - manifolds\ with\ structure\ \mathcal{S}\}$$

$$Z: \mathcal{C}_d \rightarrow U(1) \subset \mathbb{C}^* \subset \mathbb{C}$$

$$Z: \mathcal{C}_{d-1} \rightarrow \{dim = 1\ \ Hermitian\ vector\ spaces\}$$

$Z: \mathcal{C}_{d-2} \rightarrow \cdots$ satisfying natural gluing rules.

The Anomaly Field Theory

$Z_{Anomaly}$ is a 7D invertible field theory constructed from $(G, \mathcal{R}, \Lambda)$+ $\mathcal{B}$

Structure $\mathcal{S}$: G-bundles $\mathcal{P}$ with gauge connection, Riemannian metric, spin structure $\mathfrak{s}$

Varying metric and gauge connection ⇒

$Z_{Anomaly}(\mathcal{M}_6, \mathcal{P}, \mathfrak{s})$ is a LINE BUNDLE

$\Psi_{Anomaly}$ is a SECTION of $Z_{Anomaly}(\mathcal{M}_6, \mathcal{P}, \mathfrak{s})$

Anomaly Cancellation In Terms of Invertible Field Theory

This field theory must be trivialized by a ``counterterm'' 7D invertible field theory Z_{CT}

$$Z_{CT}(\mathcal{M}_6, \mathcal{S}) \cong Z_{Anomaly}(\mathcal{M}_6, \mathcal{S})^*$$

and, using just the data of the local fields in six dimensions we construct a section:

$$\Psi_{CT} \in Z_{CT}(\mathcal{M}_6, \mathcal{S})$$

so that $$\int_{Fermi+B} e^{S_{Fermi+B}} \Psi_{CT}$$

is canonically a ***<u>function</u>*** on $\mathcal{B}/\mathcal{G}$

APS η-Invariant

Recall APS $\eta -$ invariant of Dirac operator.

$$\eta(D) = \sum_{E \neq 0} sign(E) \qquad \xi(D) := \frac{\eta(D) + \dim \ker(D)}{2}$$

$e^{2\pi i\, \xi(D)}$ is relevant to anomalies: It is the holonomy of an anomalous path integral around loops in $\mathcal{B}/\mathcal{G}$ [Witten, 1982]

We'll say later just HOW it fits in with the anomaly field theory.

Important Facts About ξ

If D is a Dirac operator on a odd-diml manifold $\mathcal{U}$, with $\partial\mathcal{U} = \emptyset$ then $\xi(D)$ is a number

- in general impossible to compute explicitly ...

But, if $\mathcal{U}$, and D extend to $\mathcal{W}$ with $\partial\mathcal{W} = \mathcal{U}$ then $e^{2\pi i\, \xi(D)}$ is given by APS index theorem:

$$e^{2\pi i\, \xi(D)} = e^{2\pi\, i\, \int_{\mathcal{W}} Index\ density}$$

Dai-Freed Field Theory

$e^{2\pi i \xi(D)}$ defines an invertible field theory

[Dai & Freed, 1994]

If $\partial \mathcal{U} = \emptyset$ $\qquad Z_{DaiFreed}(\mathcal{U}) = e^{2\pi i \xi(D)}$

If $\partial \mathcal{U} = \mathcal{M} \neq \emptyset$ then $e^{2\pi i\, \xi(D)}$ is a ***section of a line bundle*** over the space of boundary data.

Suitable gluing properties hold.

Chern-Simons On Manifold With Boundary

Simpler example: Consider gauge theory on a two-dimensional manifold with G-bundle $\mathcal{P} \to \mathcal{M}_2$

$$\Psi(\tilde{A}_\partial) := \exp 2\pi i \int_{\mathcal{U}_3} CS(\tilde{A}_\partial)$$

where $(\mathcal{U}_3, \tilde{A}_\partial)$ extends $(\mathcal{M}_2, A_\partial)$

Consider the set of all functions Ψ

{extensions of $(\mathcal{M}_2, A_\partial)$ to$(\mathcal{U}_3, \tilde{A}_\partial)$ } $\to \mathbb{C}$,

s.t. if we glue two extensions to get $(\bar{\mathcal{U}}, \tilde{A})$ then

$$\frac{\Psi(Extension1)}{\Psi(Exension2)} = \exp 2\pi i \int_{\bar{\mathcal{U}}} CS(\tilde{A})$$

Chern-Simons On Manifold With Boundary - 2

This set of functions is
ONE DIMENSIONAL for each A_∂

Therefore we have defined a line bundle over the space of connections $Conn(\mathcal{P} \to \mathcal{M}_2)$

A choice of extensions gives a section of that line bundle.

Similarly, $e^{2\pi i \, \xi(D)}$ is a section of a line bundle on odd-dimensional $\mathcal{U}$ with bdry

Anomaly Field Theory For 6d Sugra

On 7-manifolds $\mathcal{U}_7$ with $\partial\mathcal{U}_7 = \emptyset$

$$Z_{Anomaly}(\mathcal{U}_7, \mathcal{S}_7) \\ = \exp[2\pi i \left(\xi(D_{Fermi}) + \xi\left(D_{B-field}\right)\right)]$$

On 7-manifolds with $\partial\mathcal{U}_7 = \mathcal{M}_6$:

The sum of ξ –invariants defines a unit vector $\widehat{\Psi}_{Anomaly}$ in a line $\mathrm{Z}_{\mathrm{Anomaly}}(\mathcal{M}_6, \mathcal{S}_6)$

Simpler Expression When $(\mathcal{U}_7, \mathcal{S}_7)$ Extends To Eight Dimensions

In general it is essentially impossible to compute η-invariants in simpler terms.

But if the matter content is such that $I_8 = \frac{1}{2} Y^2$

AND if $(\mathcal{U}_7, \mathcal{S}_7)$ is bordant to zero:

$$\Psi_{Anomaly}(\mathcal{U}, \mathcal{P}) = \exp(2\pi\, i\, (\frac{1}{2}\int_{\mathcal{W}_8} Y^2 - \frac{sign(\Lambda)\sigma(\mathcal{W}_8)}{8}))$$

When can you extend $\mathcal{U}_7$ and its gauge bundle to a spin 8-fold $\mathcal{W}$??

Spin Bordism Theory

$\Omega_7^{spin} = 0$:

Can always extend spin $\mathcal{U}_7$ to spin $\mathcal{W}_8$

$\Omega_7^{spin}(BG)$: Can be nonzero: There can be obstructions to extending a G-bundle $\mathcal{P} \to \mathcal{U}_7$ to a G-bundle $\tilde{\mathcal{P}} \to \mathcal{W}_8$

But $\Omega_7^{spin}(BG) = 0$ for many groups, e.g. products of U(n), SU(n), Sp(n). Also E8

We will return to this key point later.

When 7D data extends to $\mathcal{W}_8$ the formula

$$\Psi_{Anomaly}(\mathcal{U},\mathcal{P}) = \exp\left(2\pi\, i \left(\frac{1}{2}\int_{\mathcal{W}_8} Y^2 - \frac{sign(\Lambda)\sigma(\mathcal{W}_8)}{8}\right)\right)$$

gives a clue to construct the counterterm invertible field theory, Z_{CT}. We can write it as:

$$\exp\left(2\pi i \int_{\mathcal{W}} \frac{1}{2} X(X+\lambda')\right) \qquad X = Y - \frac{1}{2}\lambda'$$

$$\lambda' = a \otimes \lambda \qquad a^2 = sign(\Lambda)\, mod\, 8$$

$$\lambda = \frac{1}{2} p_1 \qquad \lambda^2 = \sigma(\mathcal{W})\, mod\, 8$$

$X \in \Omega^4(\mathcal{W};\Lambda)$ has quantized coho class in $H^4(\mathcal{W};\Lambda)$

$$\exp\left(2\pi i \int_{\mathcal{W}} \frac{1}{2} X(X + \lambda') \right)$$

is independent of extension ONLY if

$a \in \Lambda$ is a characteristic vector:

$$\forall\, v \in \Lambda \qquad v^2 = v \cdot a \;\; mod\; 2$$

λ is a characteristic vector of $H^4(\mathcal{W}_8; \mathbb{Z})$

Happily, a characteristic vector always satisfies $a^2 = sign(\Lambda) \;\; mod\; 8$

This action is the partition function of a 7-dimensional topological field theory known as ``Wu-Chern-Simons theory.'' WCS generalizes spin Chern-Simons theory to higher form fields.

Spin Chern-Simons Theory 1/2

If A is a connection on a $U(1)$ bundle $\mathcal{P} \to \mathcal{U}_3$ then we can define a (level one) Chern-Simons invariant:

$$CS(A) = \int_{\mathcal{W}_4} X^2 \bmod \mathbb{Z} \in \mathbb{R}/\mathbb{Z} \qquad X = \frac{F}{2\pi}$$

We want to divide by 2 but this is not well-defined.

If we give $\mathcal{U}_3$ a spin structure we can make sense of level ½ Chern-Simons: Choose integral lift of $w_2(\mathcal{W}_4)$

$$CS^{spin}(A) = \int_{\mathcal{W}_4} \frac{1}{2} X\,(X + \widehat{w}_2)\; mod\; \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$$

Spin Chern-Simons Theory – 2/2

Makes sense because $\widehat{w}_2$ is a characteristic vector on $H^4(\mathcal{W}_4; \mathbb{Z})$

A choice of spin structure, ω on $\mathcal{U}_3$ can be viewed as a trivialization $\widehat{w}_2 = d\,\eta$ at the boundary of $\mathcal{W}_4$

Change of spin structure: $\omega \to \omega + [\epsilon],\ [\epsilon] \in H^1\left(\mathcal{U}_3; \frac{\frac{1}{2}\mathbb{Z}}{\mathbb{Z}}\right)$

$$\eta \to \eta + \epsilon \qquad \epsilon \in Z^1\left(\mathcal{U}_3; \frac{1}{2}\mathbb{Z}\right)$$

The value of the spin Chern-Simons action changes: $CS^{\omega+[\epsilon]}(A) = CS^{\omega}(A) + \int_{\mathcal{U}_3} X\,\epsilon$

Wu-Chern-Simons Theory

Generalizes spin-Chern-Simons to p-form gauge fields.

Developed in detail in great generality by
Samuel Monnier arXiv:1607.0139

Our case: 7D TFT Z_{WCS} of a (locally defined) 3-form gauge potential C with fieldstrength $X = dC$

$$[X] \in H^4(\cdots ; \Lambda)$$

Instead of spin structure we need a ``Wu-structure": A trivialization of:

$$\nu_4 = w_4 + w_3\, w_1 + w_2^2 + w_1^4$$

Wu-Chern-Simons

On a spin manifold $w_1 = 0$ and $w_2 = 0$ and $p_1(TM)$ has a canonical quotient by 2 : $\lambda = \frac{1}{2} p_1$.

Moreover, λ is an integral lift of w_4.

$$e^{iS_{WCS}} = \exp\left(-2\pi i \int_W \frac{1}{2} X(X + \lambda')\right)$$

is the action when $\mathcal{U}_7$ is spin-bordant to zero.

$$\lambda' = a \otimes \lambda$$

a must be a characteristic vector of Λ

Dependence On Wu Structure

Action and partition function depend on choice of Wu structure.

Set of Wu-structures are a torsor for $H^3\left(\mathcal{U}_7; \frac{\frac{1}{2}\mathbb{Z}}{\mathbb{Z}}\right)$

$$S_{WCS}^{\omega+\epsilon}(X) = S_{WCS}^{\omega}(X) + \int_{\mathcal{U}_7} \epsilon X$$

Z_{WCS} is an invertible 7D field theory: Evaluation on a six-manifolds $\mathcal{M}_6$ gives a line bundle over the space of boundary data for X

Defining Z_{CT} From Z_{WCS}

To define the counterterm line bundle Z_{CT} we want to evaluate Z_{WCS} on $(\mathcal{M}_6, Y)$.

Problem 1: Y is shifted: $[Y] = \frac{1}{2} a \otimes \lambda + [X]$

$$[X] = \sum b_i c_2^i + \frac{1}{2} \sum b_{IJ} c_1^I c_1^J \in H^4(\cdots; \Lambda)$$

Problem 2: Z_{WCS}^{ω} needs a choice of Wu-structure ω.

!! We do not want to add a choice of Wu structure to the defining set of sugra data

$(G, \mathcal{R}, \Lambda, a, b)$

Defining Z_{CT} From Z_{WCS}

Solution: Given a Wu-structure we can shift Y to $X = Y - \frac{1}{2}\nu(\omega)$, an unshifted field and then we show that $Z_{WCS}^{\omega}(Y - \frac{1}{2}\nu(\omega))$ is independent of ω

$$Z_{CT}(\cdots, Y) := Z_{WCS}^{\omega}\left(\cdots, Y - \frac{1}{2}\nu(\omega)\right)$$

Z_{CT} is actually independent of Wu structure: So no need to add this extra data to the definition of 6d sugra.

Z_{CT} transforms properly under B-field, diff, and VM gauge transformations: $Z_{CT}(\mathcal{M}_6, \mathcal{S}_6) \cong Z_{Anomaly}(\mathcal{M}_6, \mathcal{S}_6)^*$

Anomaly Cancellation

$Z_{Anomaly} \times Z_{CT}$ is a 7D ***topological field theory*** that is defined on spin bordism classes of G-bundles. It's 7D partition function is a homomorphism:

$$Z_{Anomaly} \times Z_{CT} \; : \Omega_7^{spin}(BG) \to U(1)$$

If this homomorphism is trivial then

$$Z_{Anomaly} \times Z_{CT}(\mathcal{M}_6) \cong 1$$

is canonically trivial.

Anomaly Cancellation

Suppose the 7D TFT is indeed trivializable

Now need a section, $\Psi_{CT}(\mathcal{M}_6, \mathcal{S}_6)$ which is local in the <u>*six-dimensional*</u> fields. This will be our Green-Schwarz counterterm:

$$\Psi_{Anomaly}(A, g_{\mu\nu})\Psi_{CT}(A, g_{\mu\nu}, B)$$

The product will be a function on $\mathcal{B}/\mathcal{G}$

Important Technicalities 1: Differential Cohomology

Precise formalism for working with p-form fields in general spacetimes.

Three independent pieces of gauge invariant information:

Wilson lines Fieldstrength Topological class

Differential cohomology is an infinite-dimensional Abelian group that precisely accounts for these data and nicely summarizes how they fit together.

Noncanonically: $\mathcal{H} \times U(1)^s \times H^p(\mathcal{M};\mathbb{Z})$

$\mathcal{H}$ is an infinite-dimensional Hilbert space of gauge-inequivalent local modes.

Differential Cochains For Torus-Valued p-Form Gauge Fields

Follow M. Hopkins & I. Singer:

Torus = V/Λ where $V = \Lambda \otimes \mathbb{R}$

$$\check{C}^p(M, V) \coloneqq C^p(M;V) \times C^{p-1}(M;V) \times \Omega^p(M;V)$$

$$\check{c} \coloneqq \left([\check{c}]_{ch}, [\check{c}]_{hol}, [\check{c}]_{fieldstrength}\right)$$

Differential: $d(a, h, F) \coloneqq (da, F - a - dh, dF\ )$

Shifted cocycles: Let $\nu \in H^p(M;\Lambda)$

$\check{Z}^p{}_\nu(M;\Lambda) \subset \widetilde{Z^p}(M)$ where $[a] = \nu \; mod \; \Lambda$

The background shifted differential cocycle

Given a Riemannian metric, gauge connection, and anomaly coefficients (a,b) we can canonically construct a differential cocycle $\check{Y}$

$$[\check{Y}]_{fieldstrength} = Y \quad = \frac{a}{4} p_1 - \sum_i b_i c_2^i + \frac{1}{2} \sum_{IJ} b_{IJ} c_1^I c_1^J$$

It is shifted by $\nu = \frac{1}{2} a \otimes \lambda$

$\check{Y}$ is NOT gauge invariant under diffeomorphisms and gauge transformations:

$$\check{Y} \rightarrow \check{Y} + d\,\check{V}$$

Model For The B-field

$$\check{Y} = d\,\breve{H} \qquad \breve{H} \in \check{C}^3(M;\Lambda)$$

B-field gauge transformations: $\breve{H} \to \breve{H} + d\check{R}$

Diffeomorphisms + VM Gauge transformations:

$$\check{Y} \to \check{Y} + d\check{V} \qquad \breve{H} \to \breve{H} + \check{V}$$

Important Technicalities 2: E-Theory

A second subtlety is that to define the WCS action precisely in ≤ 7 dimensions we actually have to work with a generalization of cohomology known as E-theory.

Example: WZW term in 4d pion Lagrangian for effective action of 4d $SU(N_c)$ YM + N_f flavors.

N_f odd: The action should depend on spin structure!!

Freed (2006): Actually, the well-defined integral is an integral in E-theory.

Construction Of The Green-Schwarz Counterterm:

$$\Psi_{CT} = \exp 2\pi i \int_{\mathcal{M}_6}^{E,\omega} gst$$

$$gst = \left(\frac{1}{2} \left[\left(\breve{H} - \frac{1}{2}\check{\eta} \right) \cup \left(\check{Y} + \frac{1}{2}\check{\nu} \right) \right]_{hol}, h_2 - \frac{1}{2}\eta \right)$$

Section of the right line bundle & independent of Wu structure ω.

Locally constructed in six dimensions, but makes sense in topologically nontrivial cases.

Locally reduces to the expected answer

1. Introduction & Summary Of Results
2. Six-dimensional Sugra & Green-Schwarz Mech.
3. Quantization Of Anomaly Coefficients.
4. F-Theory Check
5. Geometrical Anomaly Cancellation, η-Invariants & Wu-Chern-Simons
6. Technical Tools
7. **Conclusion & Discussion**

Conclusion: All Anomalies Cancel:

for $(G, \mathcal{R}, \Lambda, a, b)$ such that:

$$I_8 = \frac{1}{2}\ Y^2$$

$a \in \Lambda$ is a characteristic vector & $a^2 = 9 - T$

$$\frac{1}{2} b \in H^4(BG_1; \Lambda)$$

$$\Omega_7^{spin}(BG) = 0$$


Except,…

What If The Bordism Group Is Nonzero?

We would like to relax the last condition, but it could happen that

$$Z_{Anomaly} \times Z_{CT} \ : \Omega_7^{spin}(BG) \to U(1)$$

defines a nontrivial bordism invariant.

For example, if $G = O(N)$ then we could have $\exp 2\pi i \int_{\mathcal{U}_7} w_1^7$. In that case the theory is anomalous.

It is (just barely) conceivable that some more clever anomaly cancellation mechanism could be invented, but this seems extremely unlikely.

Future Directions

Global form of VM gauge group of
6- and 4- dimensional F-theory compactifications.

Understand the spin bordism theories we can get from $Z_{Anomaly} \times Z_{CT}$ for arbitrary 6d sugra data: $(G, \mathcal{R}, \Lambda, a, b)$

We have only shown that our quantization conditions on (a, b) are complete for G such that $\Omega_7^{spin}(BG) = 0$. When it is nonvanishing there will probably be new conditions.

Finding them looks like a very challenging problem…