Algebra of the Infrared: Massive d=2 N=(2,2) QFT

- or -

A short ride with a big machine

KITP, March, 2014

Gregory Moore, Rutgers University

collaboration with

Davide Gaiotto & Edward Witten

draft is ``nearly finished"…

Three Motivations

1. IR sector of ***massive*** 1+1 QFT with N =(2,2) SUSY

2. Knot homology.

3. Categorification of 2d/4d wall-crossing formula.

(A unification of the Cecotti-Vafa and Kontsevich-Soibelman formulae.)

d=2, N=(2,2) SUSY

$$\{Q_+, \overline{Q_+}\} = H + P$$

$$\{Q_-, \overline{Q_-}\} = H - P$$

$$\{Q_+, Q_-\} = \bar{Z}$$

$$[F, Q_+] = Q_+ \qquad [F, \bar{Q}_-] = \bar{Q}_-$$

We will be interested in situations where two supersymmetries are unbroken:

$$U(\zeta) := Q_+ - \zeta^{-1}\overline{Q_-}$$

$$\{U(\zeta), \overline{U(\zeta)}\} = 2\left(H - \mathrm{Re}(\zeta^{-1}Z)\right)$$

Outline

- Introduction & Motivations
- Some Review of LG Theory
- Goals, Results, Questions Old & New
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- Web-based Formalism
- Summary & Outlook

Example: LG Models - 1

$\phi, \psi_{\pm}, \bar{\psi}_{\pm}, \ldots$ Chiral superfield

$W(\phi)$ Holomorphic superpotential

$$S = \int d\phi * d\phi - |\nabla W|^2 + \cdots$$

Massive vacua are Morse critical points:

$$dW(\phi_i) = 0 \qquad W''(\phi_i) \neq 0$$

Example: LG Models -2

More generally,….

(X,ω): Kähler manifold. (Simplify: ω=dλ)

W: X $\longrightarrow \mathbb{C}$ Superpotential (A holomorphic Morse function)

$$\phi : D \times \mathbb{R} \to X$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

Boundary conditions for ϕ

Boundaries at infinity:

$$\phi \to \phi_i \qquad \phi \to \phi_j$$

$$x \to -\infty \qquad x \to +\infty$$

Boundaries at finite distance: Preserve ζ-susy:

$$\phi|_{x_\ell, x_r} \in \mathcal{L} \subset X$$

$$\iota^*_{\mathcal{L}}(\lambda) = dk$$

$$\pm \mathrm{Im}(\zeta^{-1} W) \geq \Lambda$$

Fields Preserving ζ-SUSY

U(ζ)[Fermi] =0 implies the ζ-*instanton* equation:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

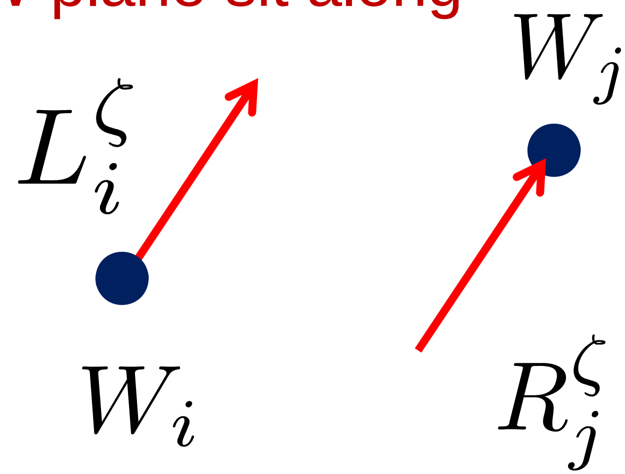
Time-independent: ζ-*soliton* equation:

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

Lefshetz Thimbles

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

The projection of solutions to the complex W plane sit along straight lines of slope ζ



If D contains x $\longrightarrow$ -∞ $\quad \phi \to \phi_i$

If D contains x $\longrightarrow$ +∞ $\quad \phi \to \phi_j$

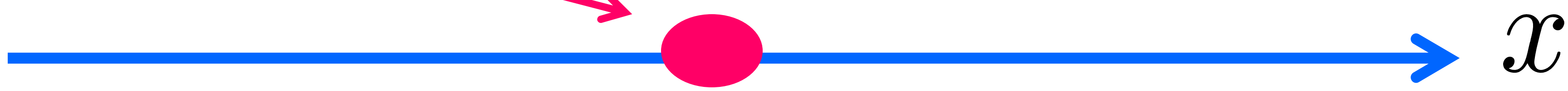
Inverse image in X of all solutions defines left and right Lefshetz thimbles

They are Lagrangian subvarieties of X

Solitons For D=$\mathbb{R}$

Scale set by W

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

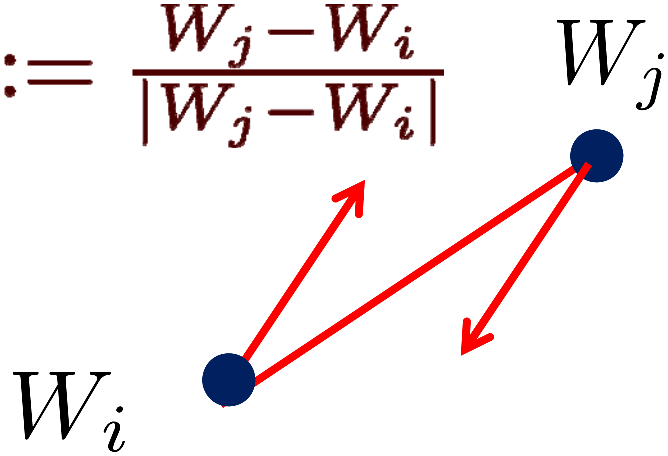


$\phi \cong \phi_i$ $\phi \cong \phi_j$

For general ζ there is no solution.

$$\zeta = \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

But for a suitable phase there is a solution



This is the classical soliton. There is one for each intersection (Cecotti & Vafa)

$$p \in L_i^{\zeta} \cap R_j^{\zeta}$$

(in the fiber of a regular value)

Witten Index

Some classical solitons are lifted by instanton effects, but the Witten index:

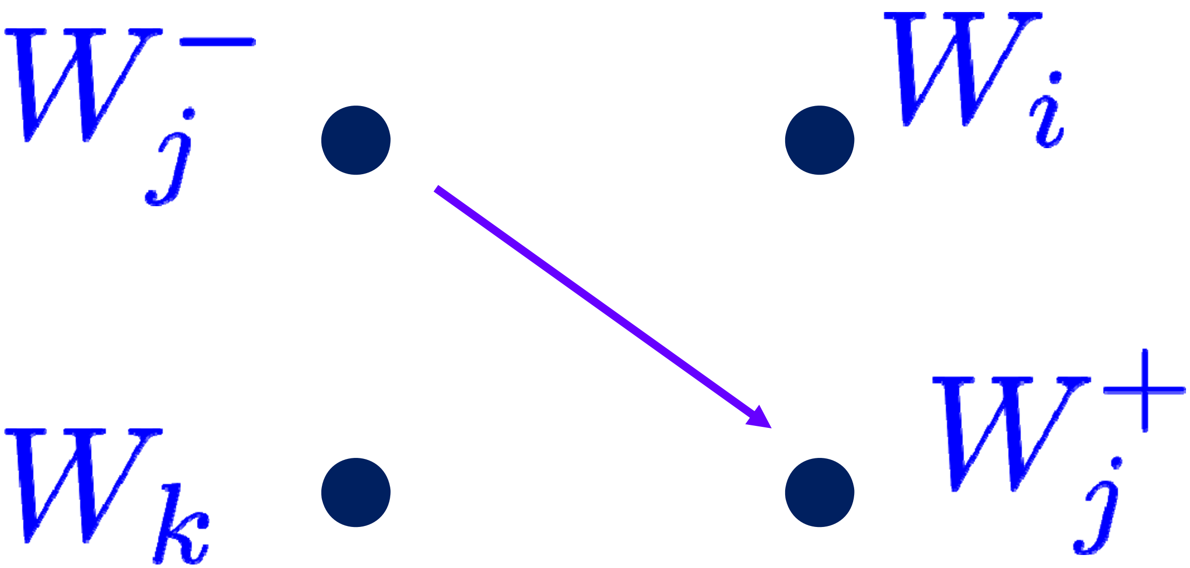
$$\mu_{ij} := \mathrm{Tr}_{\mathcal{H}_{ij}^{BPS}}(-1)^F$$

can be computed with a signed sum over classical solitons:

$$\mu_{ij} = \sum_{p \in L_i^{\zeta} \cap R_j^{\zeta}} (-1)^{\iota(p)}$$

These BPS indices were studied by [Cecotti, Fendley, Intriligator, Vafa and by Cecotti & Vafa]. They found the wall-crossing phenomena:

Given a one-parameter family of W's:



$$\mu_{ik}^- \to \mu_{ik}^+ = \mu_{ik}^- + \mu_{ij}\mu_{jk}$$

Outline

- Introduction & Motivations
- Some Review of LG Theory
- Goals, Results, Questions Old & New
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- Web-based Formalism
- Summary & Outlook

Goals & Results - 1

Goal: Say everything we can about the theory in the far IR.

Since the theory is massive this would appear to be trivial.

Result: When we take into account the BPS states there is an extremely rich mathematical structure.

We develop a formalism – which we call the ``web-based formalism" – (that's the ``big machine") - which shows that:

Goals & Results - 2

BPS states have ``interaction amplitudes'' governed by an L_∞ algebra

(That is, using just IR data we can define an L_∞ - algebra and there are ``interaction almplitudes'' of BPS states that define a solution to the Maurer-Cartan equation of that algebra.)

There is an A_∞ category of branes/boundary conditions, with amplitudes for emission of BPS particles from the boundary governed by an A_∞ algebra.

(A_∞ and L_∞ are mathematical structures which play an important role in open and closed string field theory, respectivey. Strangely, they show up here.)

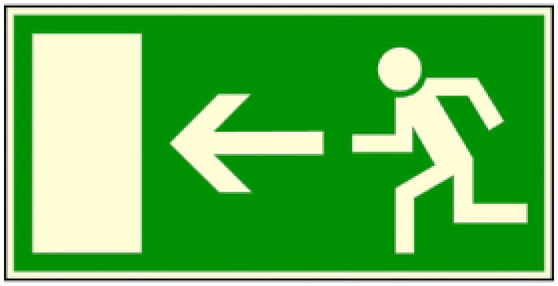
Goals & Results - 3

If we have continuous families of theories
(e.g. a continuous family of LG superpotentials)
then we can construct half-supersymmetric interfaces between the theories.

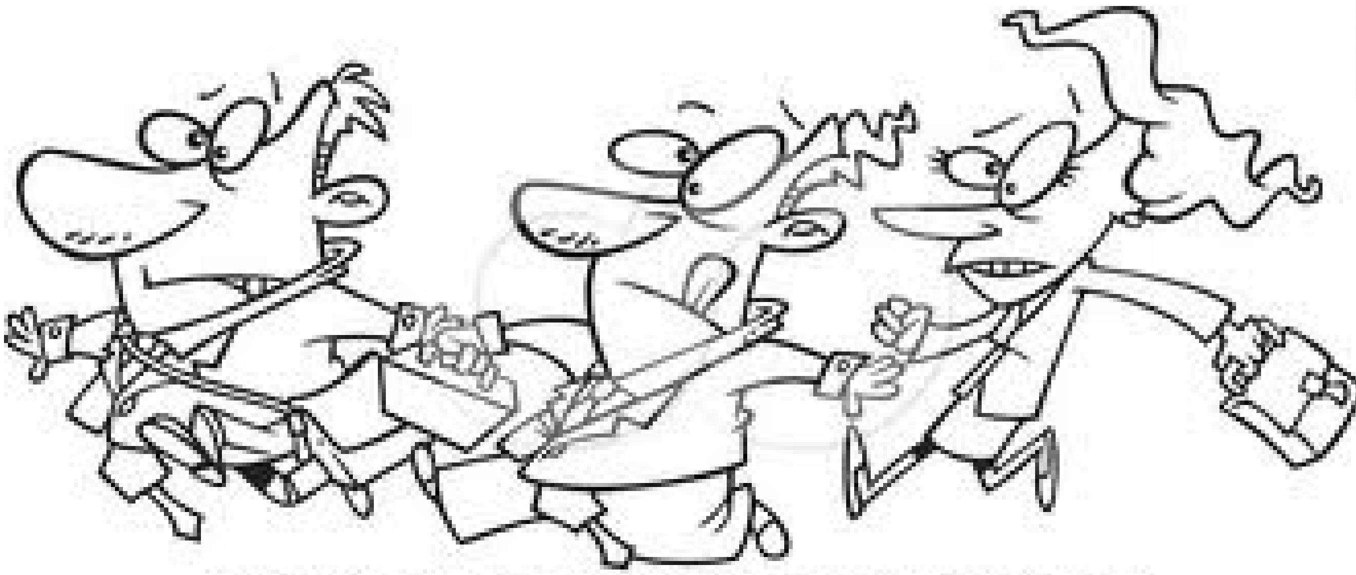
These interfaces can be used to ``implement'' wall-crossing.

Half-susy interfaces form an A_∞ 2-category, and to a continuous family of theories we associate a flat parallel transport of brane categories.

The flatness of this connection implies, and is a categorification of, the 2d wall-crossing formula.




EMERGENCY
EXIT ONLY



Some Old Questions

What are the BPS states on $\mathbb{R}$ in sector ij ? $\mathcal{H}_{ij}^{\mathrm{BPS}}$

Fendley & Intriligator; Cecotti, Fendley, Intriligator, Vafa; Cecotti & Vafa c. 1991

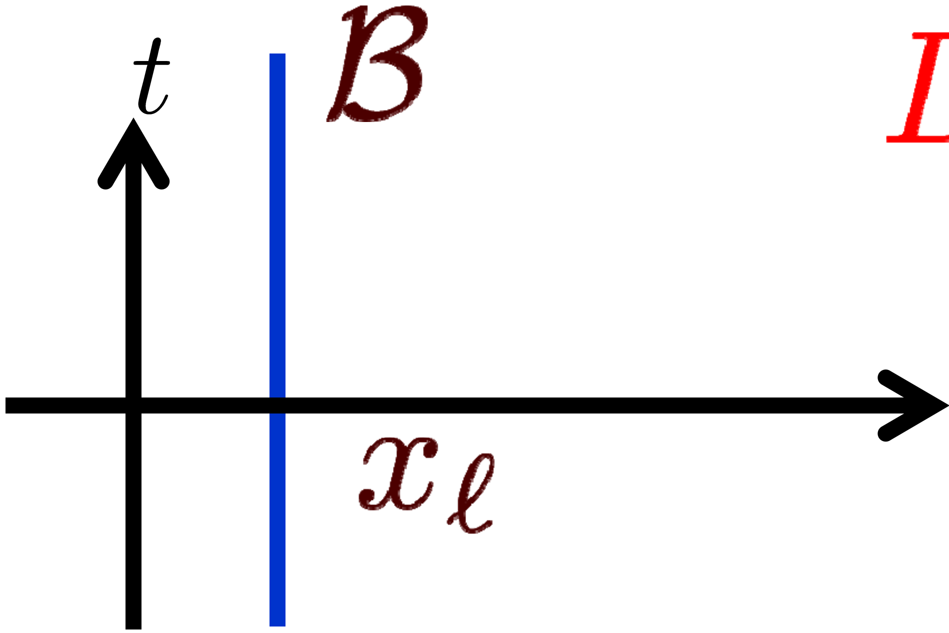
Some refinements. Main new point: L_∞ structure

What are the branes/half-BPS boundary conditions ? $\mathcal{B}$

Hori, Iqbal, Vafa c. 2000 & Much mathematical work on A-branes and Fukaya-Seidel categories.

We clarify the relation to the Fukaya-Seidel category & construct category of branes from IR.

Some New Questions -1



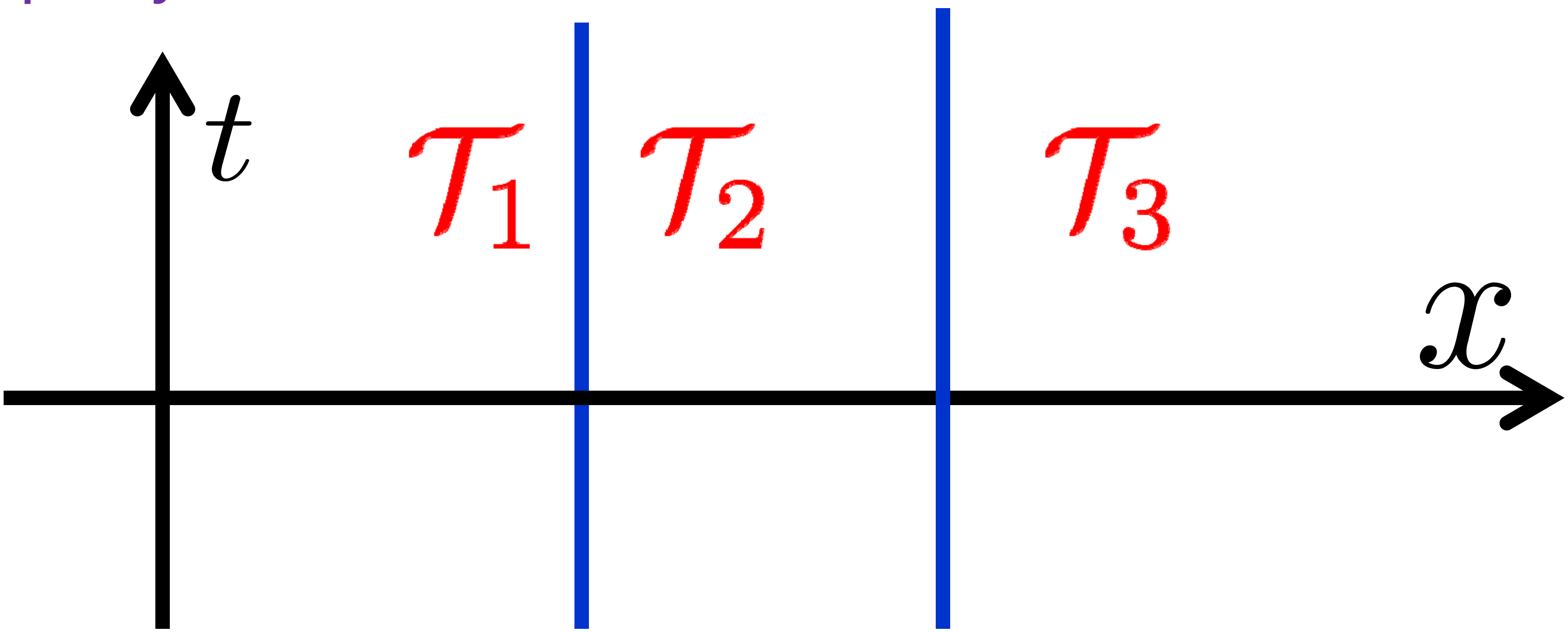
$$D = [x_\ell, \infty)$$

What are the BPS states on the half-line ?

$$\mathcal{H}^{\mathrm{BPS}}_{\mathcal{B},j}$$

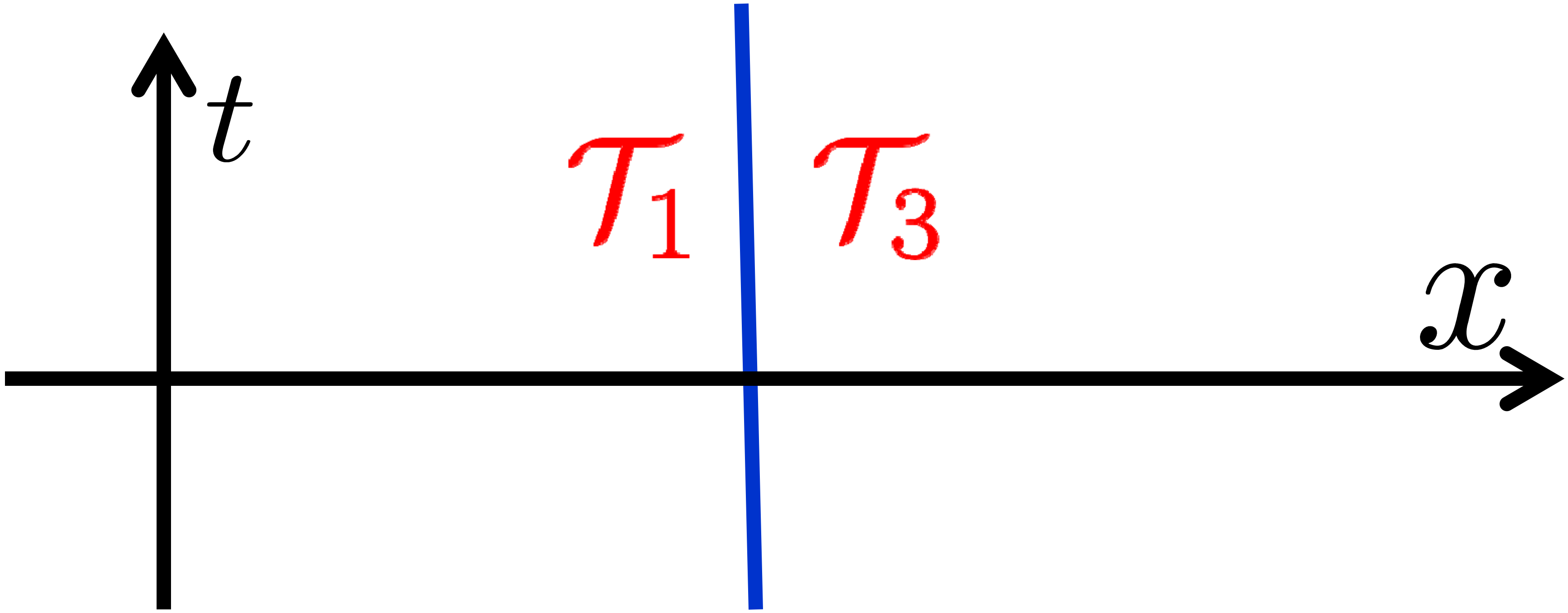
Some New Questions - 2

Given a pair of theories $\mathcal{T}_1, \mathcal{T}_2$ what are the supersymmetric interfaces?



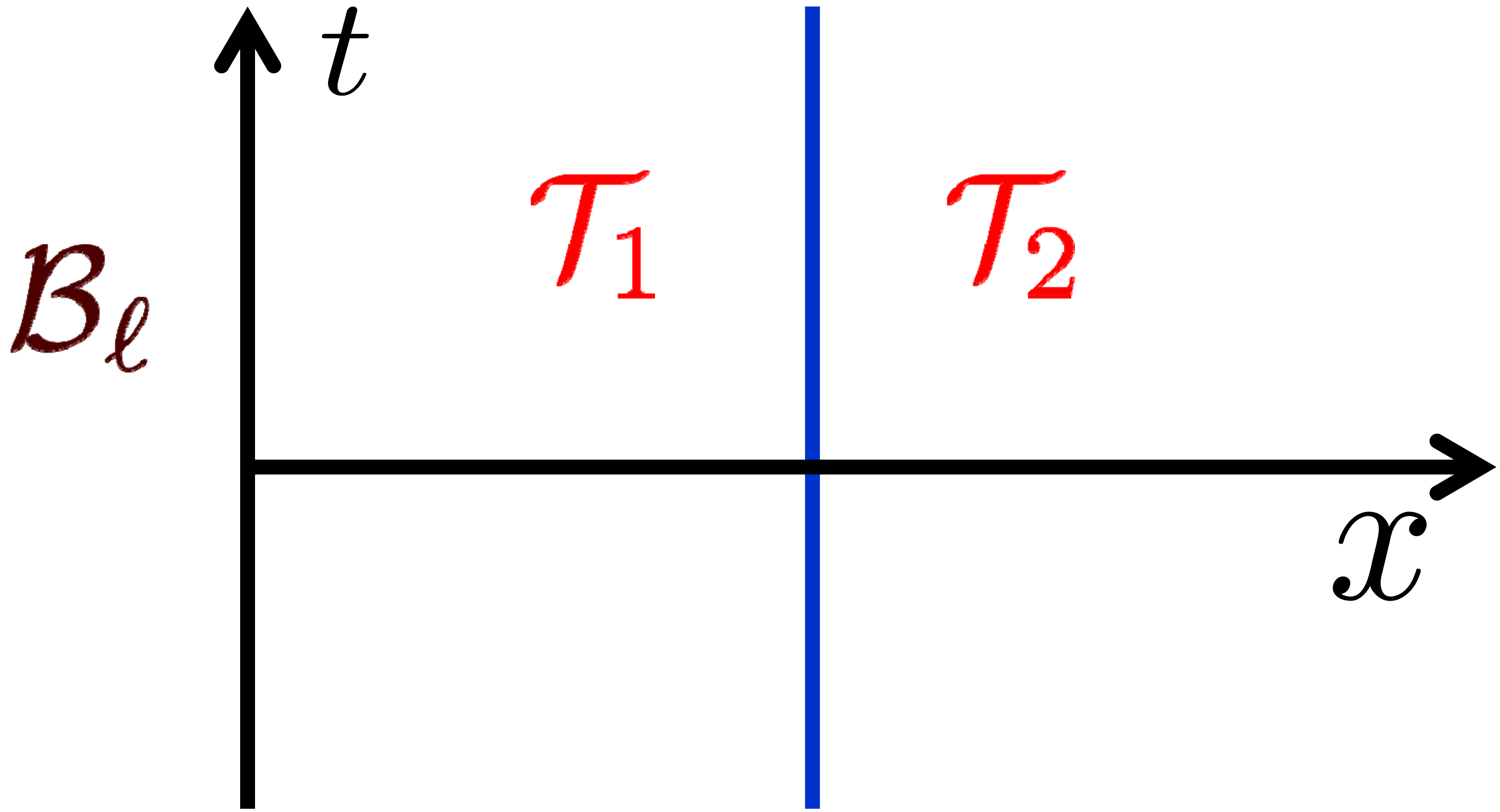
Is there an (associative) way of ``multiplying'' interfaces to produce new ones? And how do you compute it?

Some New Questions - 3



We give a method to compute the product. It can be considered associative, once one introduces a suitable notion of ``homotopy equivalence'' of interfaces.

Some New Questions - 4



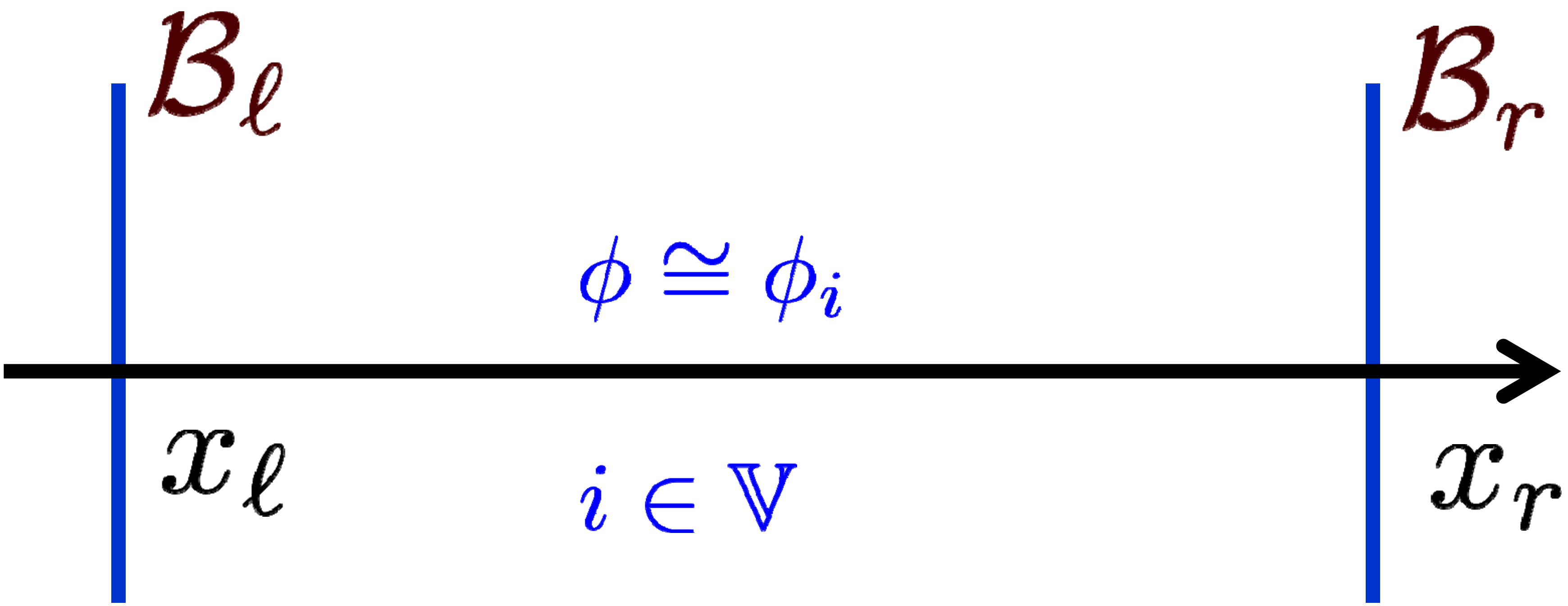
There is a way of using interfaces to ``map" branes in theory $\mathcal{T}_1$, to branes in theory $\mathcal{T}_2$?

Example of a surprise:

What is the space of BPS states on an interval ?

The theory is massive:

For a susy state, the field in the middle of a large interval is close to a vacuum:



Does the Problem Factorize?

For the Witten index: Yes

$$\mu_{\mathcal{B}_\ell,i} = \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,i}} (-1)^F e^{-\beta H}$$

$$\boxed{\mu_{\mathcal{B}_\ell,\mathcal{B}_r} = \sum_{i\in\mathbb{V}} \mu_{\mathcal{B}_\ell,i} \cdot \mu_{i,\mathcal{B}_r}}$$

For the BPS states?

$$\mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,\mathcal{B}_r} \neq \sum_{i\in\mathbb{V}} \mathcal{H}^{\mathrm{BPS}}_{\mathcal{B}_\ell,i} \otimes \mathcal{H}^{\mathrm{BPS}}_{i,\mathcal{B}_r}$$

No!

Enough with vague generalities!

Now I will start to be more systematic.

The key ideas behind everything we do come from Morse theory.

Outline

- Introduction & Motivations
- Some Review of LG Theory
- Goals, Results, Questions Old & New
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- Web-based Formalism
- Summary & Outlook

SQM & Morse Theory (Witten: 1982)

M: Riemannian; h: *M* $\longrightarrow \mathbb{R}$, Morse function

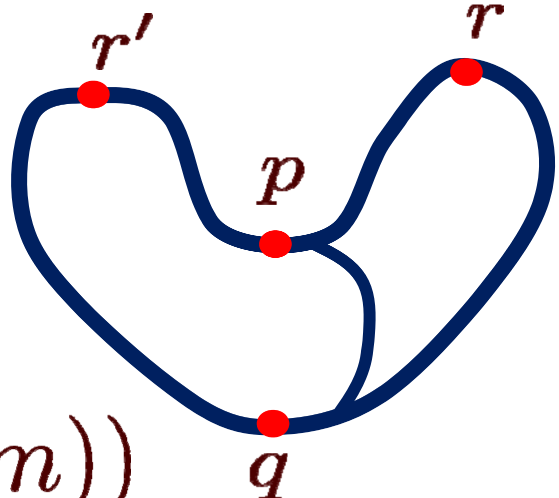
SQM: $q : \mathbb{R}_{\mathrm{time}} \to M \quad \chi \in \Gamma(q^*(TM \otimes \mathbb{C}))$

$$L = g_{IJ}\dot{q}^I \dot{q}^J - g^{IJ}\partial_I h \partial_J h$$

$$+ g_{IJ}\bar{\chi}^I D_t \chi^J - g^{IJ} D_I D_J h \bar{\chi}^I \chi^J \; - R_{IJKL}\bar{\chi}^I \chi^J \bar{\chi}^K \chi^L$$

Perturbative vacua: $dh(m) = 0$

$\Longrightarrow \Psi(m)$



$$F(\Psi(m)) = \tfrac{1}{2}(d_{\uparrow}(m) - d_{\downarrow}(m))$$

Instantons & MSW Complex

Instanton equation: $$\frac{d\phi}{d\tau} = -g^{IJ}\frac{\partial h}{\partial \phi^J}$$

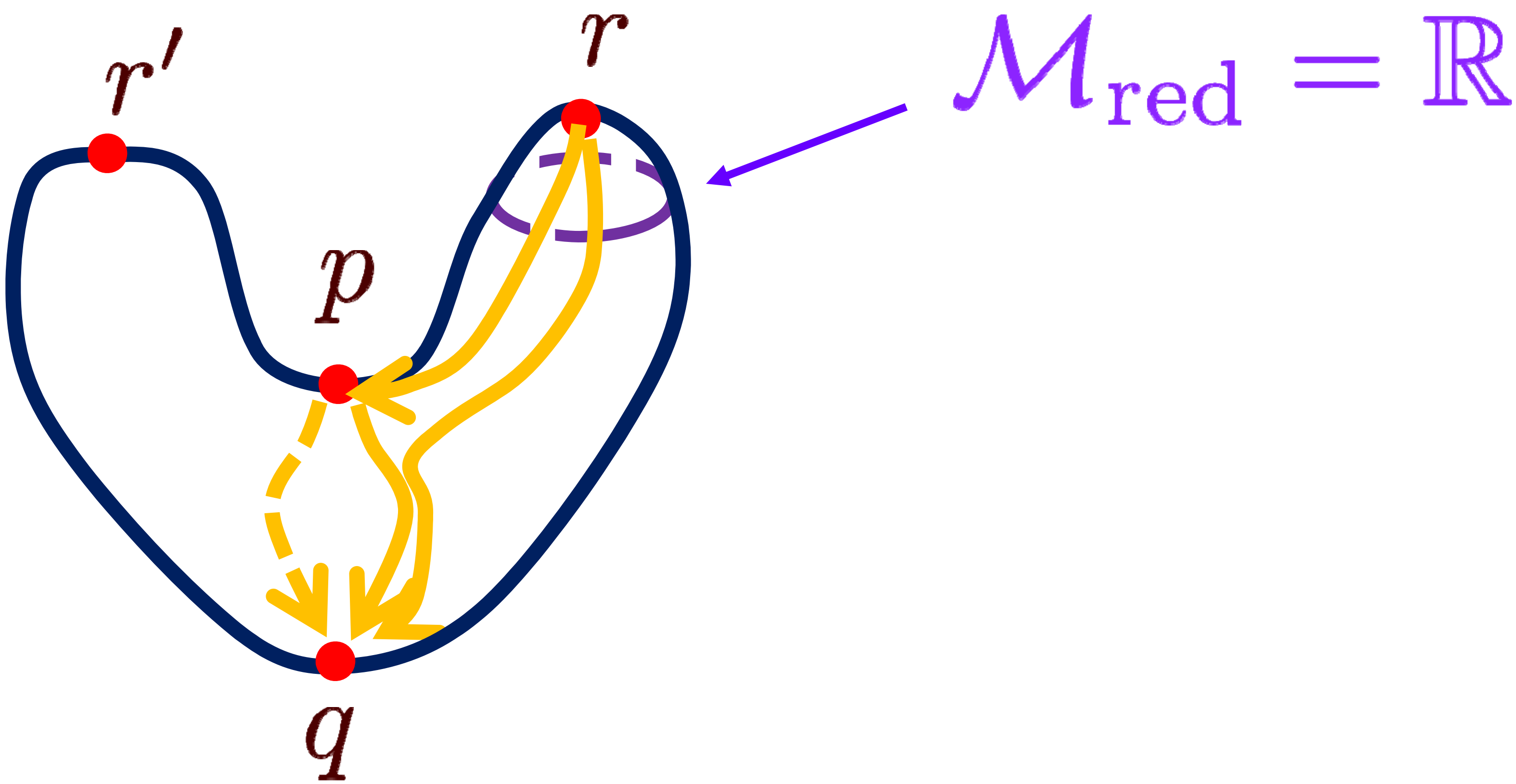
``Rigid instantons'' - with zero reduced moduli – will lift some perturbative vacua. To compute exact vacua:

MSW complex: $$\mathbb{M}^{\bullet} := \oplus_{p:dh(p)=0}\mathbb{Z}\cdot\Psi(p)$$

$$d(\Psi(p)) = \sum_{p':F(p')-F(p)=1} n(p,p')\Psi(p')$$

Space of groundstates (BPS states) is the *cohomology*.

Why $d^2 = 0$



Ends of the moduli space correspond to broken flows which cancel each other in computing $d^2 = 0$.
A similar argument shows independence of the cohomology from h and g_{IJ}.

1+1 LG Model as SQM

Target space for SQM:

$$M = \mathrm{Map}(D, X) = \{\phi : D \to X\}$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

$$h = \int_D \left(\phi^* \lambda + \mathrm{Re}(\zeta^{-1} W) dx\right)$$

$$d\lambda = \omega \qquad \lambda = pdq$$

Recover the standard 1+1 LG model with superpotential: Two –dimensional ζ-susy algebra is manifest.

Families of Theories

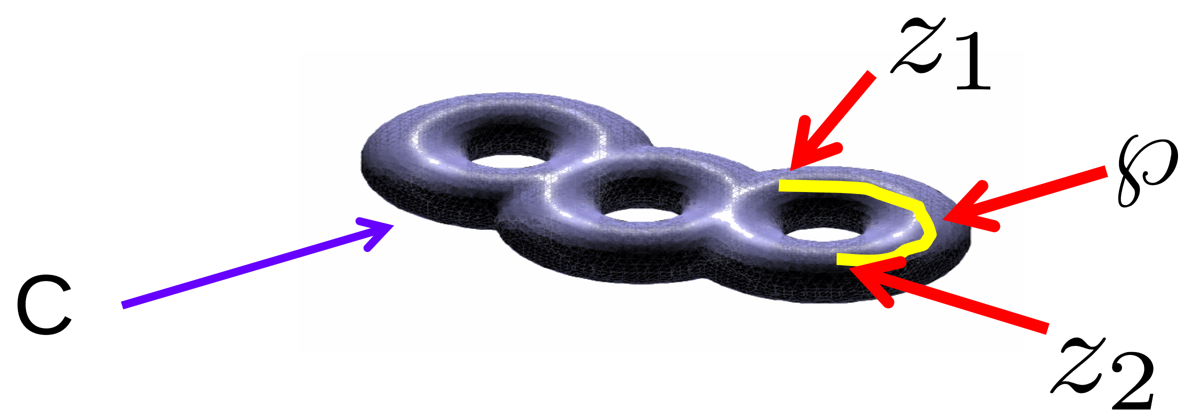
This presentation makes construction of half-susy interfaces easy:

Consider a *family* of Morse functions

$$W(\phi; z) \quad z \in C$$

Let $\wp$ be a path in C connecting z_1 to z_2.

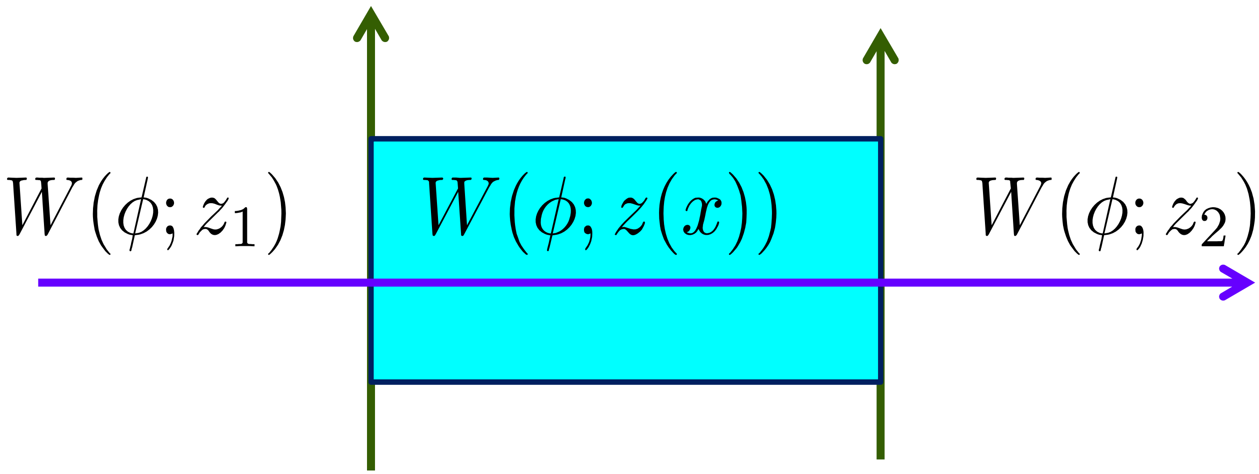
View it as a map z: $[x_l, x_r] \longrightarrow C$ with $z(x_l) = z_1$ and $z(x_r) = z_2$



Domain Wall/Interface

Using z(x) we can still formulate our SQM!

$$h = \int_D \left(\phi^*(pdq) + \mathrm{Re}(\zeta^{-1} W(\phi; z(x)) dx \right)$$



From this construction it manifestly preserves two supersymmetries.

MSW Complex

Now return to a single W. Another good thing about this presentation is that we can discuss ij solitons in the framework of Morse theory:

$$\frac{\delta h}{\delta \phi} = 0$$

Equivalent to the ζ-soliton equation

$$\mathbb{M}_{ij} = \oplus_{\text{solitons}} \mathbb{Z} \cdot \Psi_{ij}$$

(Taking some shortcuts here….)

$$D = \sigma^3 \mathrm{i} \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\zeta^{-1}}{2} W'' + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\zeta}{2} \bar{W}''$$

$$F = -\tfrac{1}{2} \eta (D - \epsilon)$$

Instantons

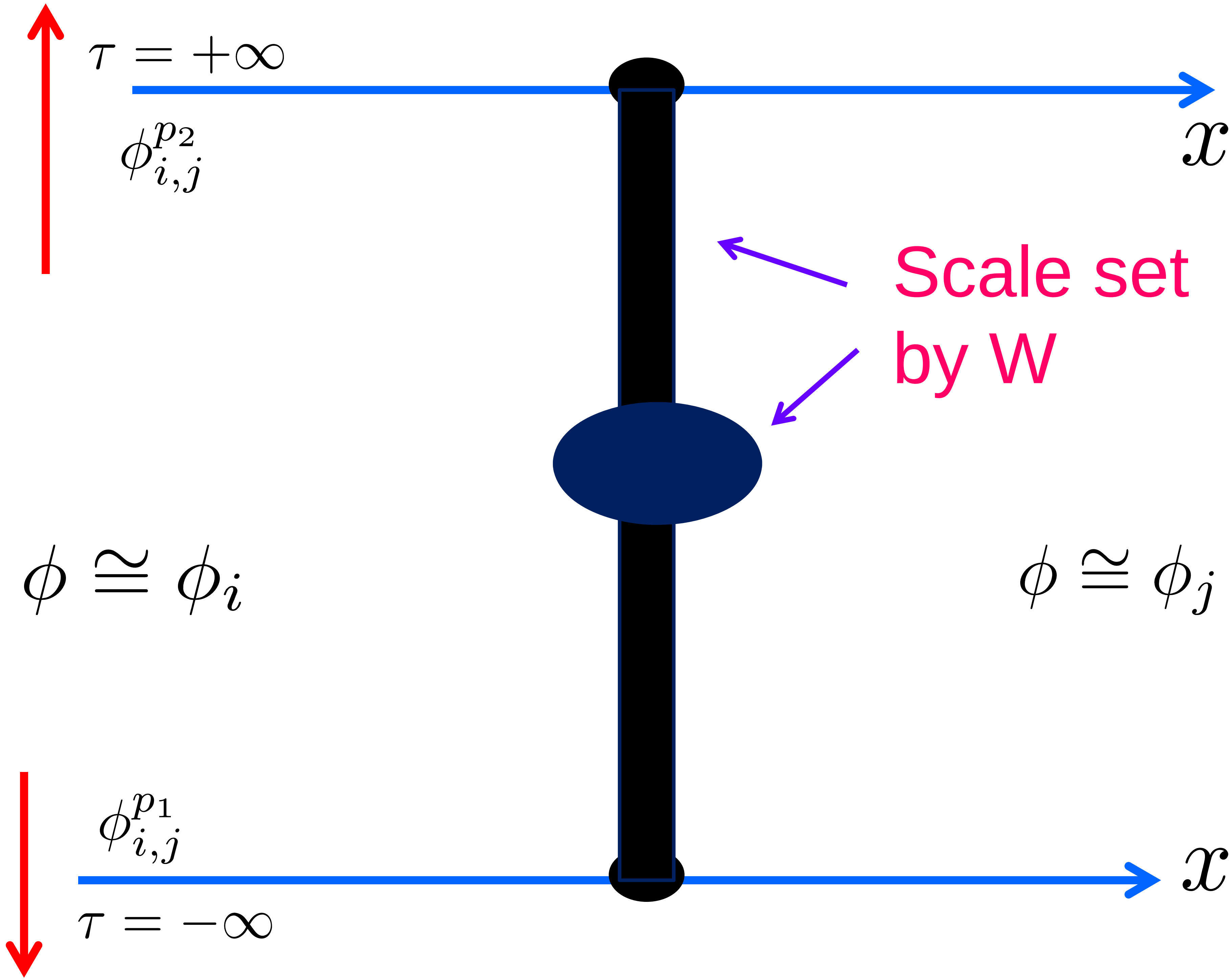
Instanton equation $$\frac{d\phi}{d\tau} = -\frac{\delta h}{\delta \bar{\phi}}$$

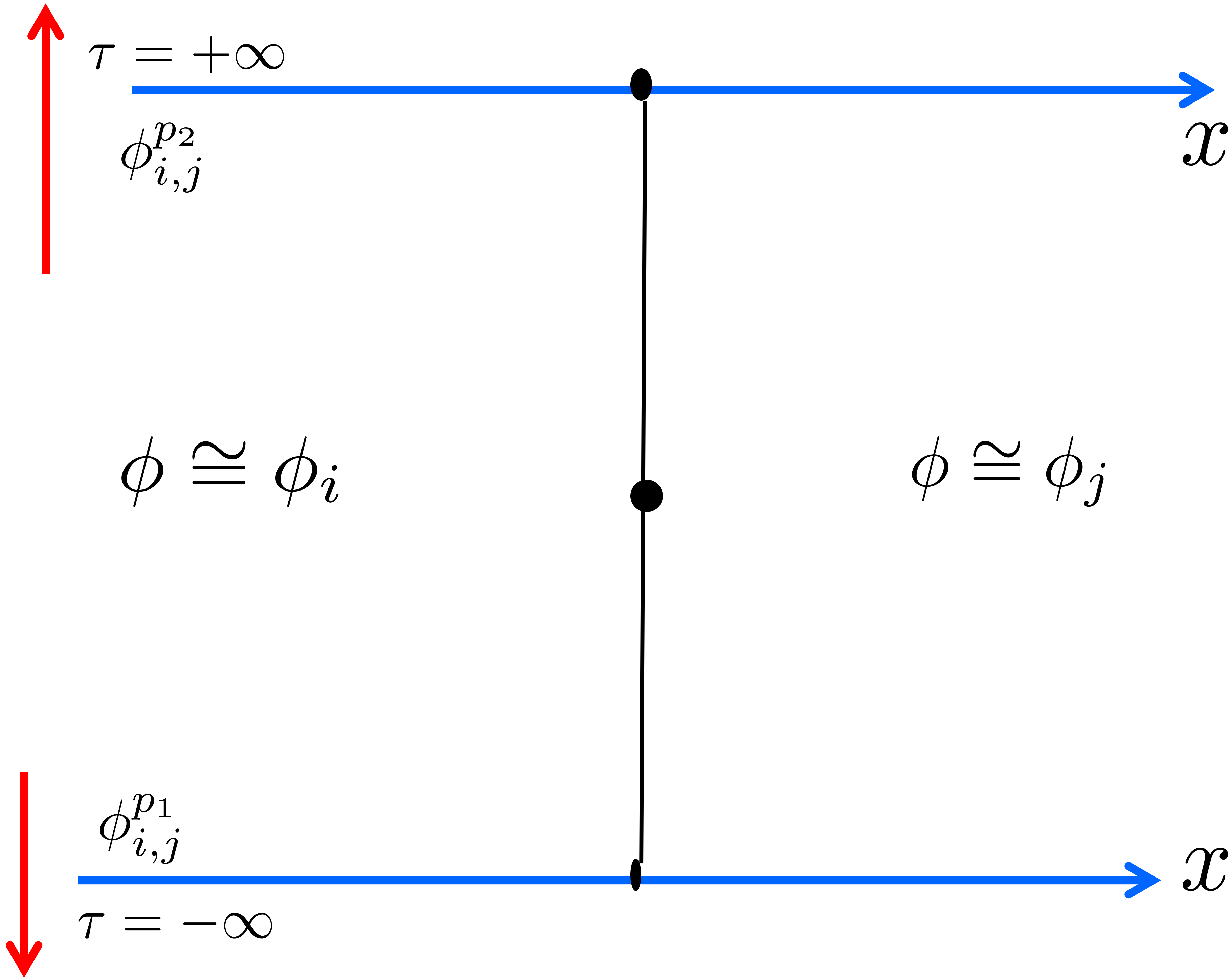
$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$$\bar{\partial}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

At short distance scales W is irrelevant and we have the usual holomorphic map equation.

At long distances the theory is almost trivial since it has a mass scale, and it is dominated by the vacua of W.


$\tau = +\infty$
$\phi_{i,j}^{p_2}$
x
Scale set
by W
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{i,j}^{p_1}$
x
$\tau = -\infty$



$\tau = +\infty$
x
$\phi^{p_2}_{i,j}$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi^{p_1}_{i,j}$
x
$\tau = -\infty$

BPS Solitons on half-line D:

Semiclassically:

Q_ζ -preserving BPS states must be solutions of differential equation

$$\frac{\partial \phi^I}{\partial x} = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$$\phi\big|_{x_\ell} \in \mathcal{L}$$

$$\phi \to \phi_j$$
$$x \to \infty$$

Classical solitons on the positive half-line are labeled by: $p \in \mathcal{L} \cap R_j^\zeta$

Quantum Half-Line Solitons

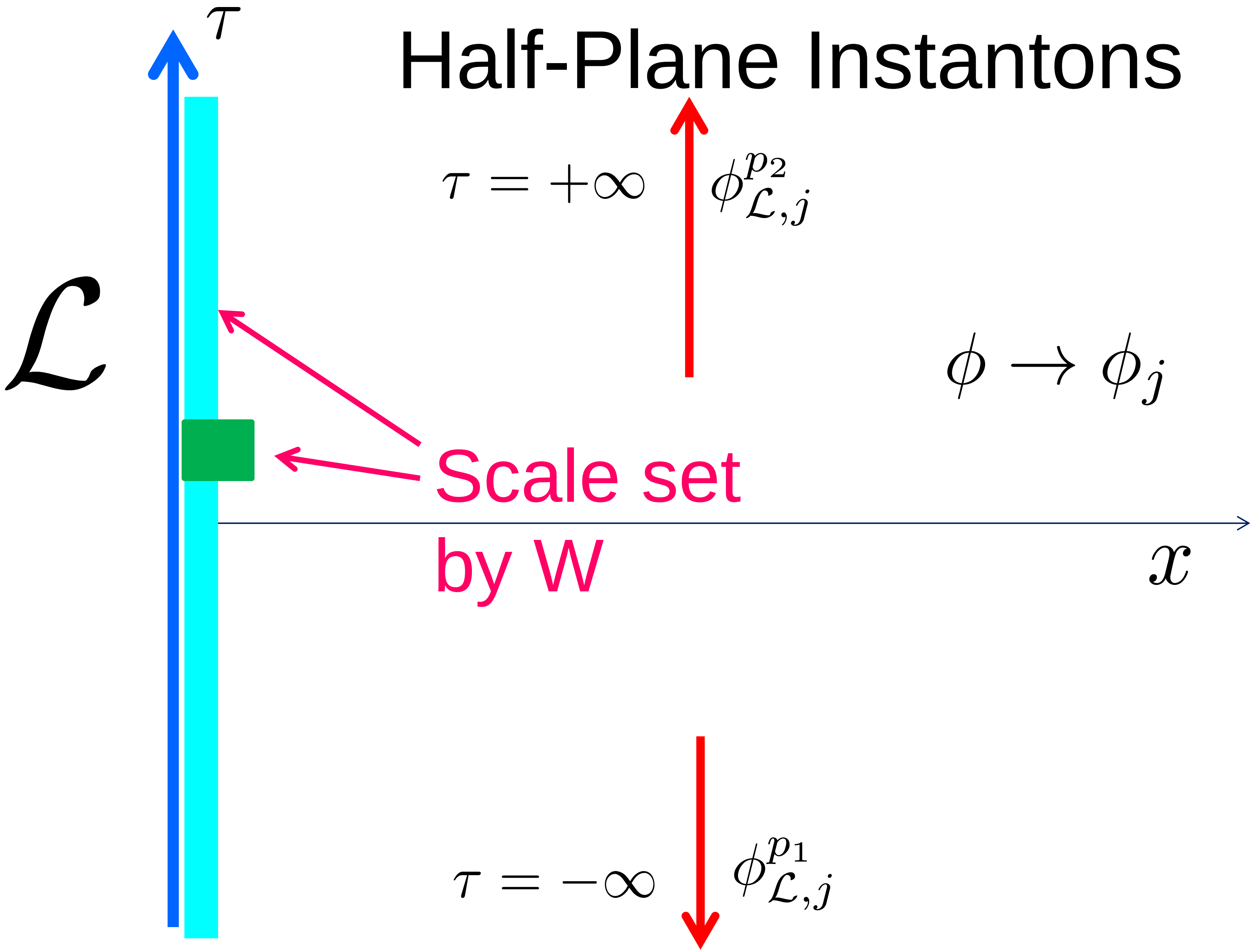
MSW complex: $\mathbb{M}_{\mathcal{L},j} = \oplus_p \mathbb{Z} \cdot \Psi_{\mathcal{L},j}(p)$

Grading the complex: Assume X is CY and that we can find a logarithm:

$$w = \operatorname{Im} \log \frac{\iota^*(\Omega^{d,0})}{\operatorname{vol}(\mathcal{L})}$$

Then the grading is by $f = \eta(D) - w$

Half-Plane Instantons



Solitons On The Interval

Now return to the puzzle about the finite interval $[x_l, x_r]$ with boundary conditions $\mathcal{L}_l$, $\mathcal{L}_r$

When the interval is much longer than the scale set by W the MSW complex is

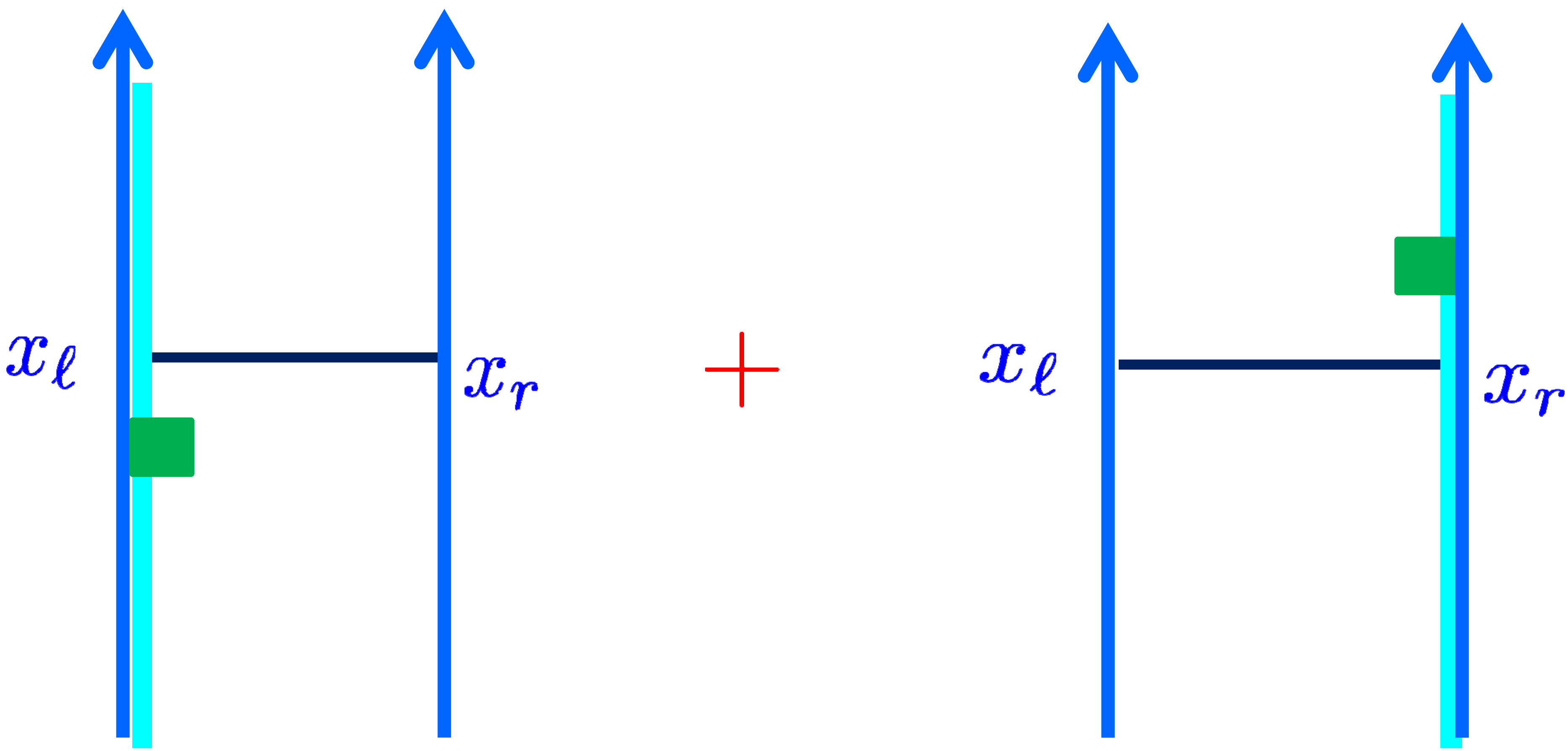
$$\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r} = \oplus_{i\in\mathbb{V}}\mathbb{M}_{\mathcal{L}_\ell,i} \otimes \mathbb{M}_{i,\mathcal{L}_r}$$

The Witten index factorizes nicely: $\mu_{\mathcal{L}_\ell,\mathcal{L}_r} = \sum_i \mu_{\mathcal{L}_\ell,i}\mu_{i,\mathcal{L}_r}$

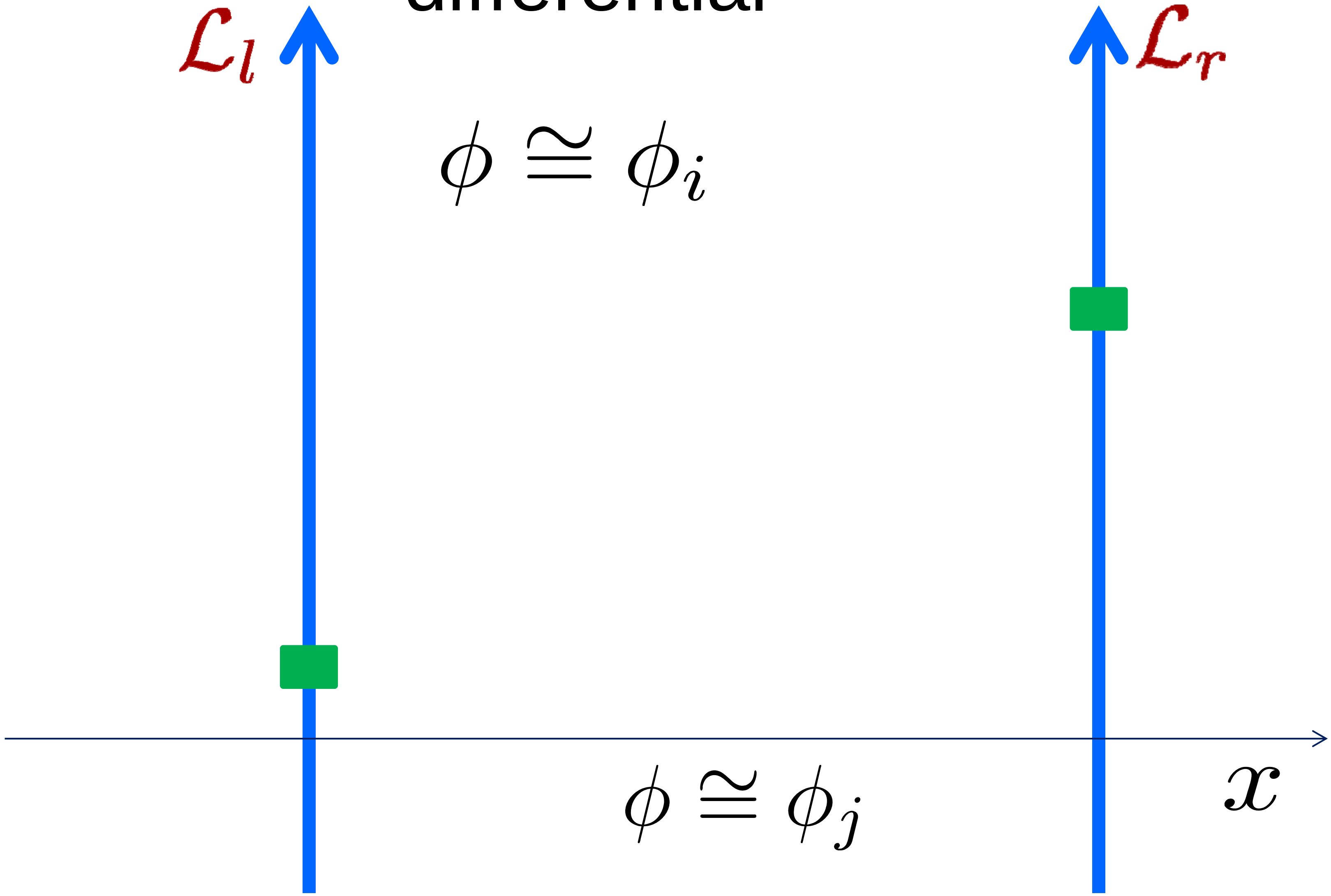
But the differential $d_{\mathcal{L}_\ell,i} \otimes 1 + 1 \otimes d_{i,\mathcal{L}_r}$

is too naïve !

$$d_{\mathcal{L}_\ell, i} \otimes 1 + 1 \otimes d_{i, \mathcal{L}_r}$$



Instanton corrections to the naïve differential



Outline

- Introduction & Motivations
- Some Review of LG Theory
- Goals, Results, Questions Old & New
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- Web-based Formalism
- Summary & Outlook

The Boosted Soliton - 1

We are interested in the ζ-instanton equation for a fixed generic ζ

We can still use the soliton to produce a solution for phase ζ

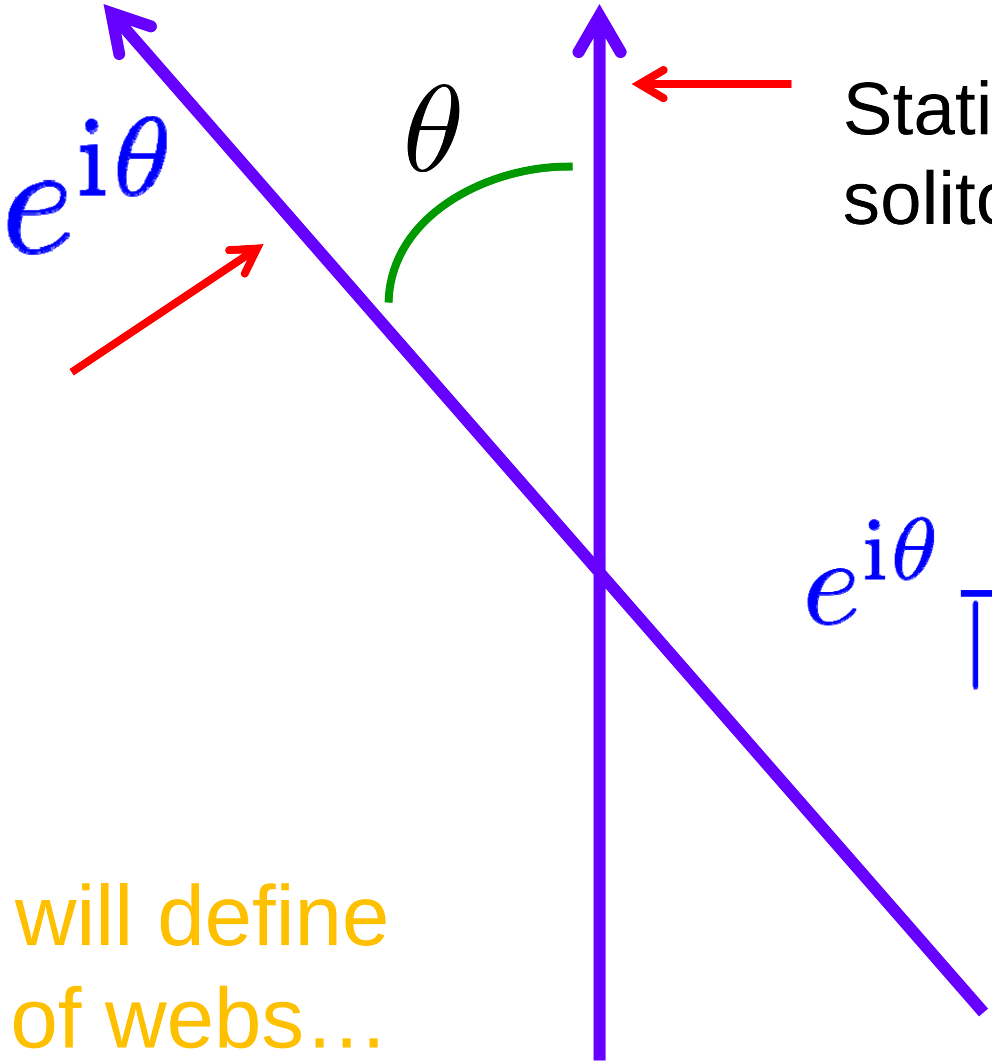
$$\phi_{ij}^{\mathrm{inst}}(x,\tau) := \phi_{ij}^{\mathrm{sol}}(\cos\theta x + \sin\theta\tau)$$

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi_{ij}^{\mathrm{inst}} = e^{\mathrm{i}\theta}\zeta_{ji}\frac{\partial \bar{W}}{\partial \bar{\phi}}$$

Therefore we produce a solution of the instanton equation with phase ζ if

$$\zeta = e^{\mathrm{i}\theta}\zeta_{ji} \qquad\qquad \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

The Boosted Soliton -2



Stationary soliton

$\phi \cong \phi_j$

``Boosted soliton''

$\phi \cong \phi_i$

$$e^{i\theta}\frac{W_{ji}}{|W_{ji}|} = \zeta$$

These will define edges of webs…

The Boosted Soliton - 3

Put differently, the stationary soliton in Minkowski space preserves the supersymmetry:

$$Q_+ - \zeta_{ij}^{-1} \overline{Q_-}$$

So a boosted soliton preserves supersymmetry :

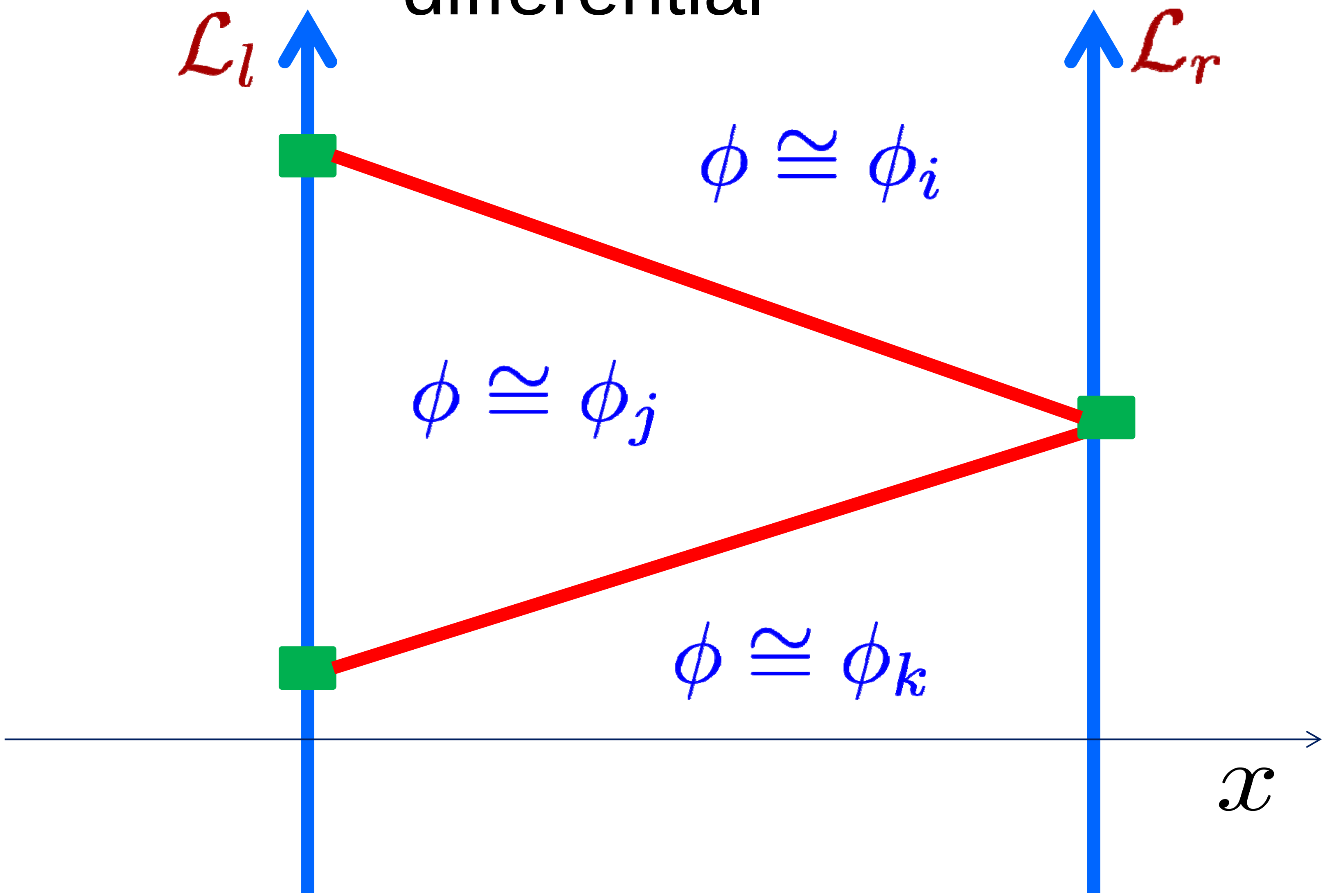
$$e^{\beta/2} Q_+ - \zeta_{ij}^{-1} e^{-\beta/2} \overline{Q_-}$$

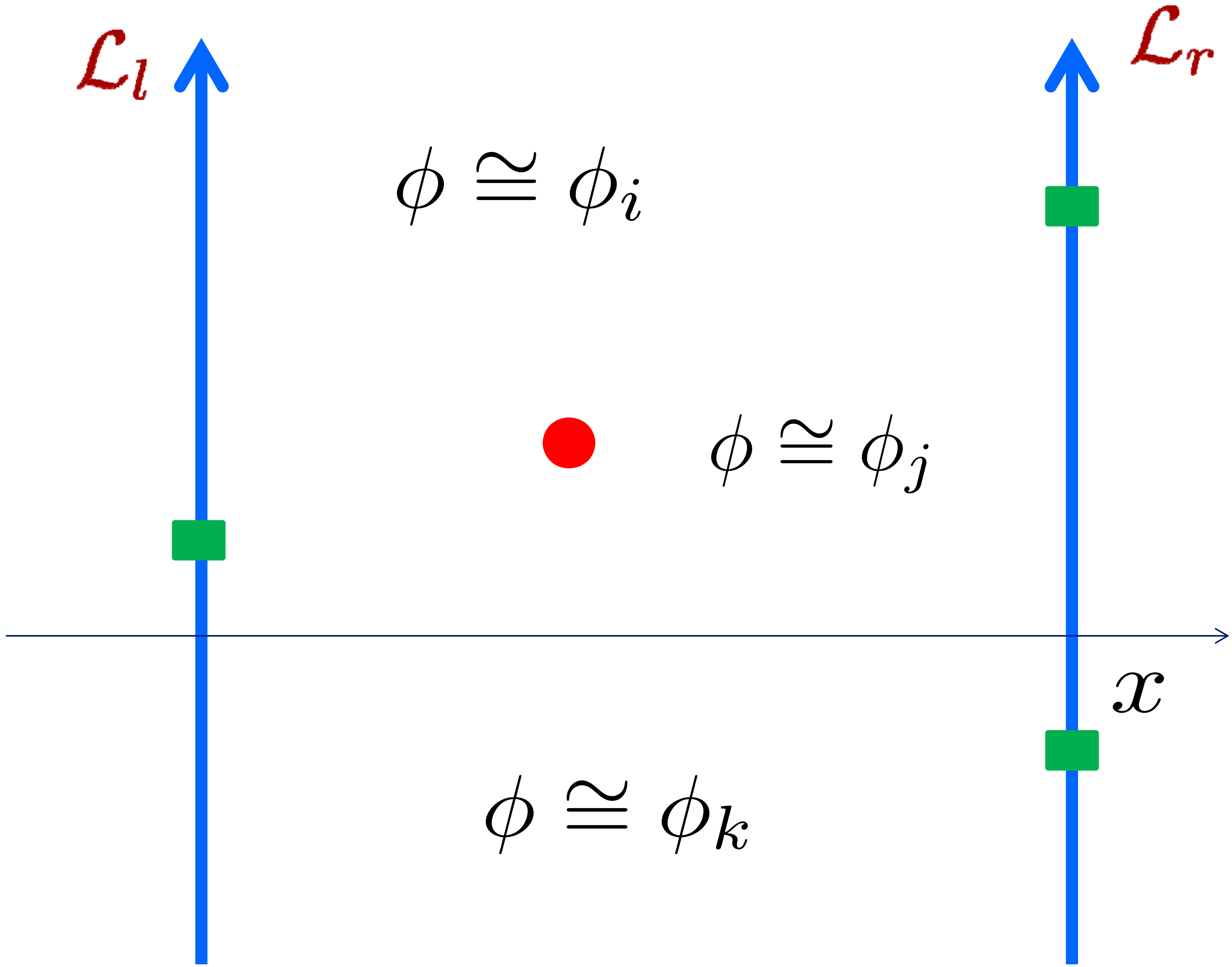
β is a real boost. In Euclidean space this becomes a rotation:

$$e^{\mathrm{i}\theta/2} Q_+ - \zeta_{ij}^{-1} e^{-\mathrm{i}\theta/2} \overline{Q_-}$$

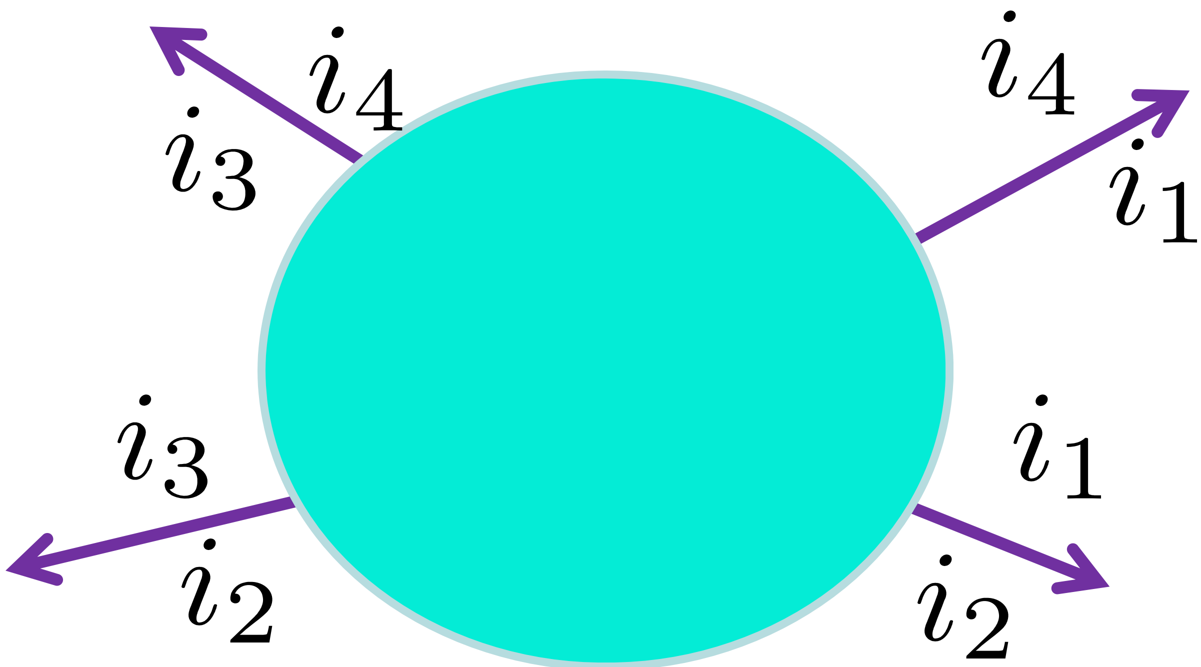
And for suitable θ this will preserve ζ-susy

More corrections to the naïve differential




$\mathcal{L}_l$
$\mathcal{L}_r$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
x
$\phi \cong \phi_k$

Path integral on a large disk



Choose boundary conditions preserving ζ-supersymmetry:

Consider a cyclic ``fan of vacua" $I = \{i_1, \ldots, i_n\}$.

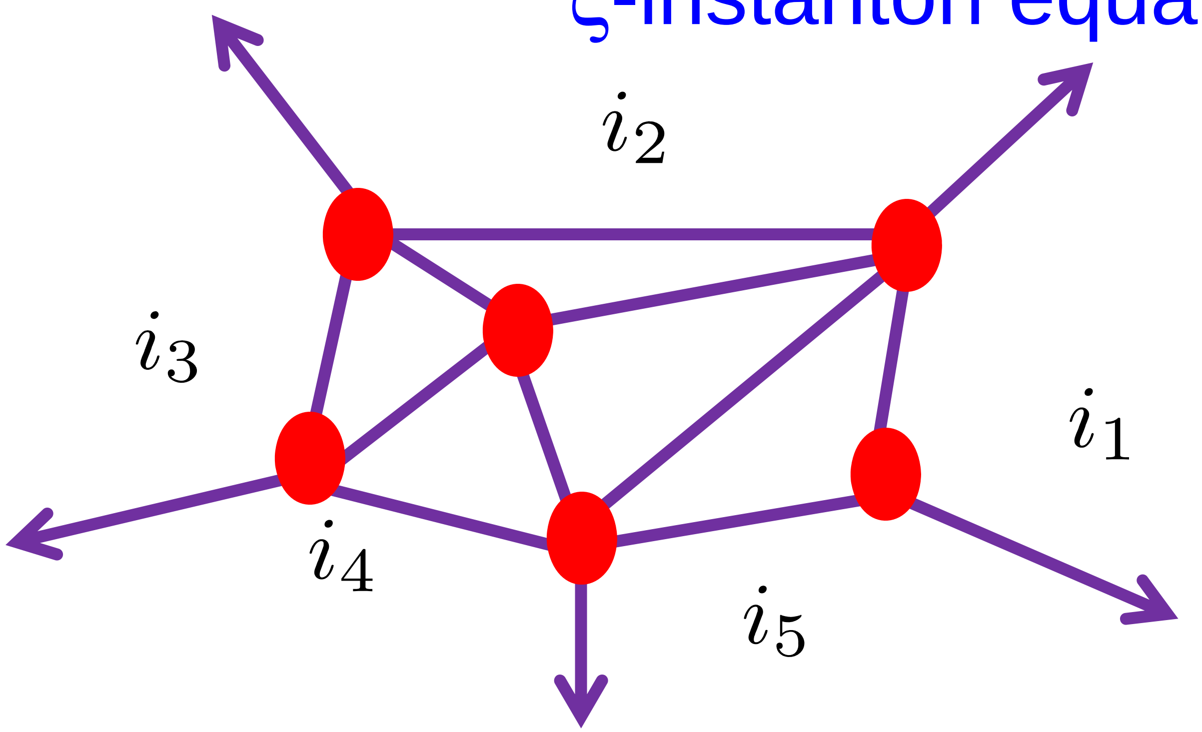
$$\mathcal{F} = \phi^{\text{inst}}_{i_1 i_2} \otimes \cdots \otimes \phi^{\text{inst}}_{i_n i_1}$$

Ends of moduli space

Path integral localizes on moduli space of ζ-instantons with these boundary conditions:

$$\mathcal{M}(\mathcal{F})$$

This moduli space has several "ends" where solutions of the ζ-instanton equation look like



We call this picture a *web*: $\mathfrak{w}$

Path Integral With Fan Boundary Conditions

Just as in the Morse theory proof of $d^2=0$ using ends of moduli space corresponding to broken flows, here the broken flows correspond to webs $\mathfrak{w}$

Label the ends of $\mathcal{M}(\mathcal{F})$ by webs $\mathfrak{w}$. Each end produces a wavefunction $\Psi(\mathfrak{w})$ associated to a web $\mathfrak{w}$.

The total wavefunction is Q-invariant

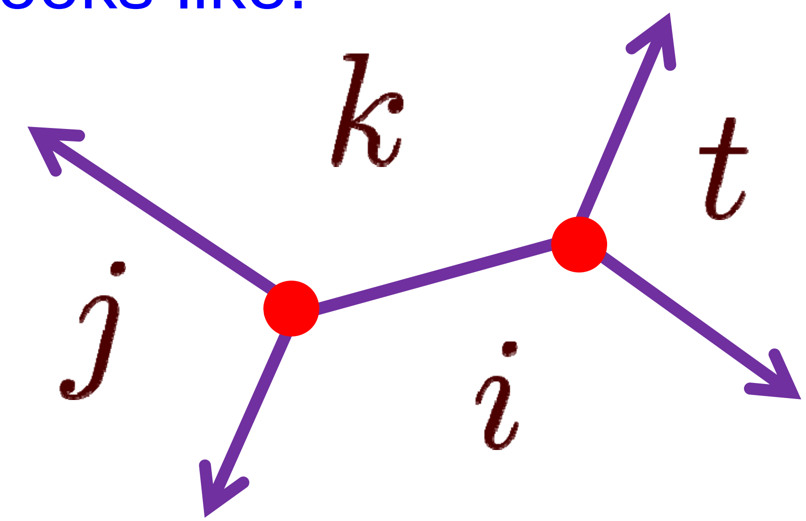
$$Q \sum_{\mathfrak{w}} \Psi(\mathfrak{w}) = 0$$

The wavefunctions $\Psi(\mathfrak{w})$ are themselves constructed by gluing together wavefunctions $\Psi(\mathfrak{r})$ associated with rigid webs $\mathfrak{r}$

$\Longrightarrow$ L_∞ identities on the interior amplitude

Example:

Consider a fan of vacua {i,j,k,t}. One end of the moduli space looks like:



The red vertices are path integrals with rigid webs. They have amplitudes β_{ikt} and β_{ijk}.

$$\mathcal{M} = \mathbb{R}^2_{transl} \times \mathbb{R}^+_{scale} \quad ?$$

Ends of Moduli Spaces in QFT

In LG theory (say, for X= $\mathbb{C}^n$) the moduli space cannot end like that.

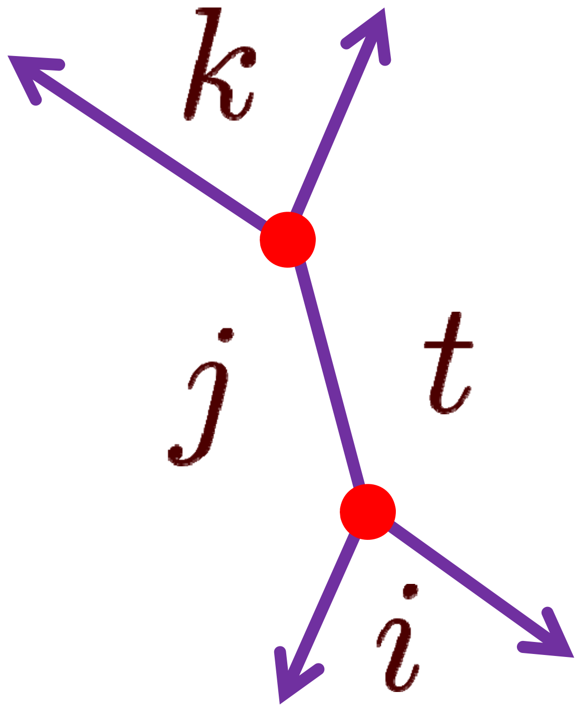
In QFT there can be three kinds of ends to moduli spaces of PDE's:

UV effect: Example: Instanton shrinks to zero size; bubbling in Gromov-Witten theory

Large field effect: Some field goes to ∞

Large distance effect: Something happens at large distances.

None of these three things can happen here.
So, there must be another end:



Amplitude: $\beta_{jkt}\beta_{ijt}$

The boundaries where the internal distance shrinks to zero must cancel leading to identities on the amplitudes like:

$$\beta_{ijk}\beta_{ikt} - \beta_{jkt}\beta_{ijt} = 0$$

This set of identities turns out to be the Maurer-Cartan equation for an L_∞ - algebra.

This is really a version of the argument for $d^2 = 0$ in SQM.

At this point it is useful to introduce a formalism that facilitates writing the identities satisfied by the various amplitudes - the "web-based formalism"

Outline

- Introduction & Motivations
- Some Review of LG Theory
- Goals, Results, Questions Old & New
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- Web-based Formalism
- Summary & Outlook

Definition of a Plane Web

We now give a purely mathematical construction.

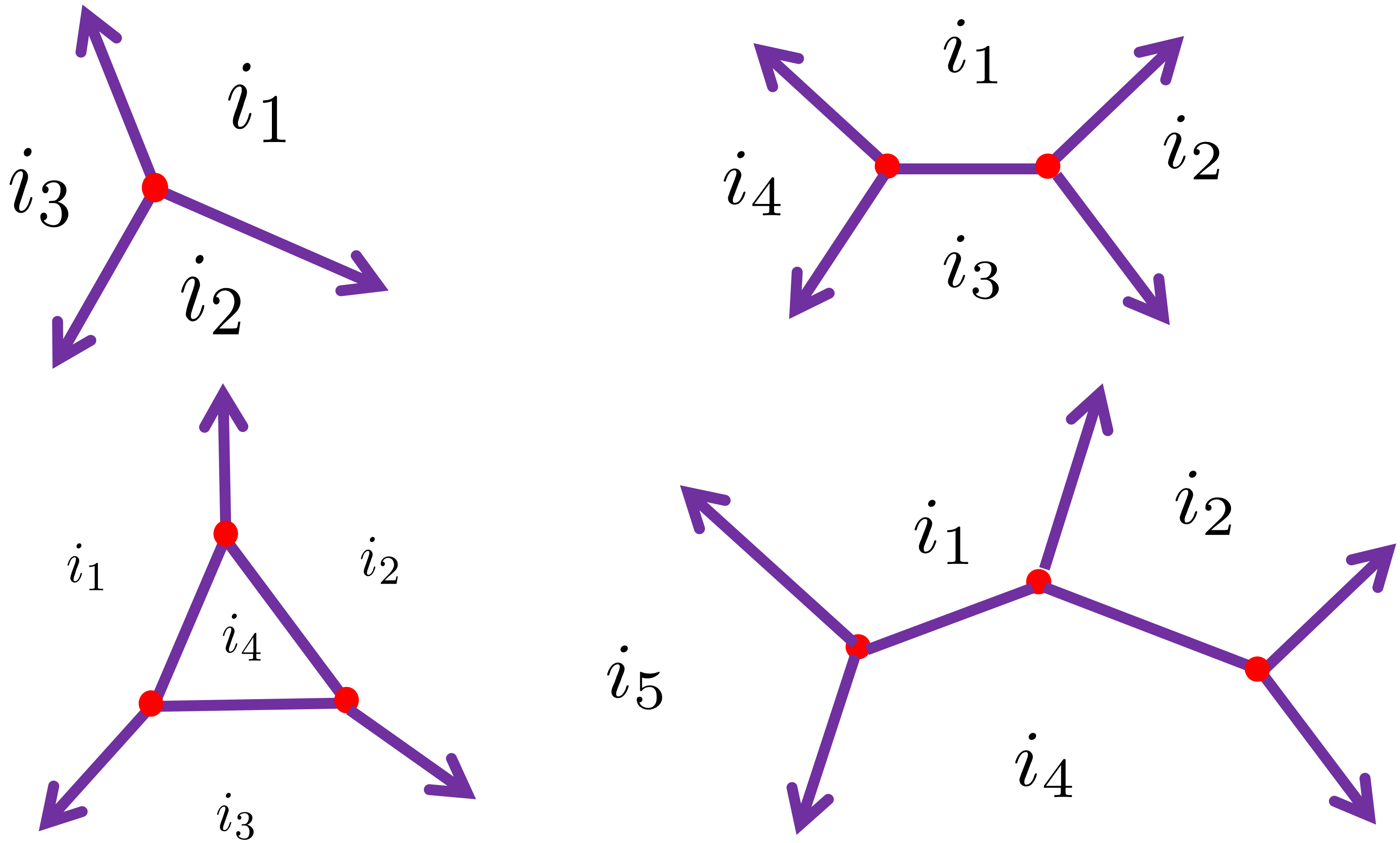
It is motivated from LG field theory.

Vacuum data:

1. A finite set of ``vacua'': $i, j, k, \cdots \in \mathbb{V}$

2. A set of weights $z : \mathbb{V} \to \mathbb{C}$

Definition: A *plane web* is a graph in $\mathbb{R}^2$, together with a labeling of faces by vacua so that across edges labels differ and if an edge is oriented so that *i* is on the left and *j* on the right then the edge is parallel to $z_{ij} = z_i - z_j$. (Option: Require all vertices at least 3-valent.)

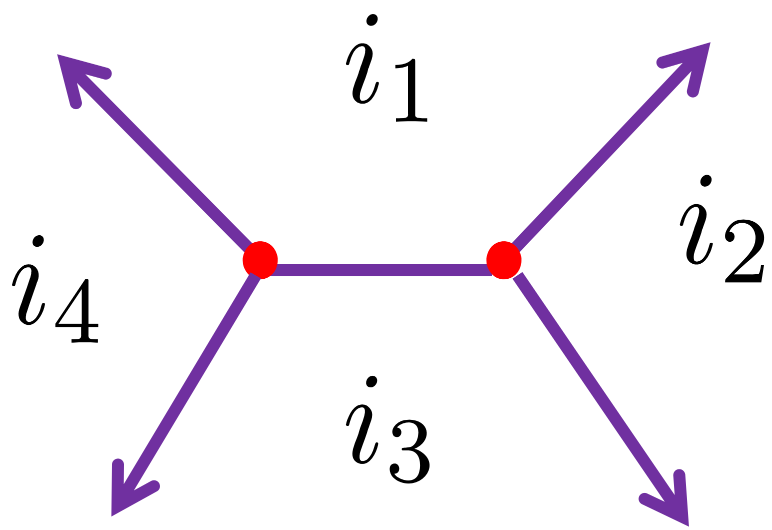


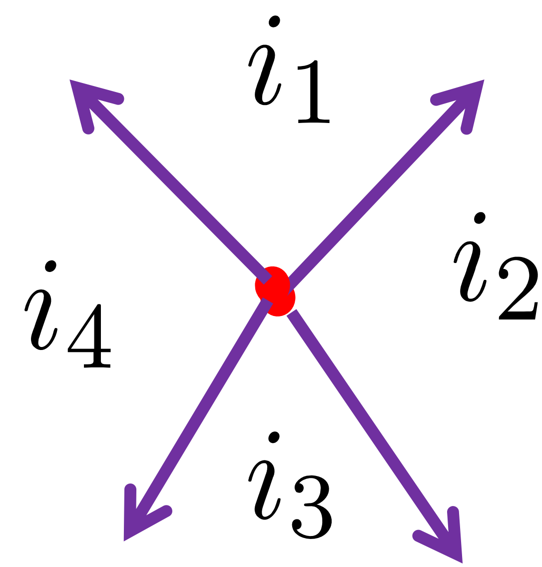
Useful intuition: We are joining together straight strings under a tension z_{ij}. At each vertex there is a no-force condition:

$$z_{i_1,i_2} + z_{i_2,i_3} + \cdots z_{i_n,i_1} = 0$$

Deformation Type

Equivalence under translation and stretching (but not rotating) of strings subject to no-force constraint defines ***deformation type***.





Moduli of webs with fixed deformation type

$$\dim \mathcal{D}(\mathfrak{w}) = 2V(\mathfrak{w}) - E(\mathfrak{w})$$

$V(\mathfrak{w}), E(\mathfrak{w})$ Number of vertices, internal edges.

(z_i in generic position)

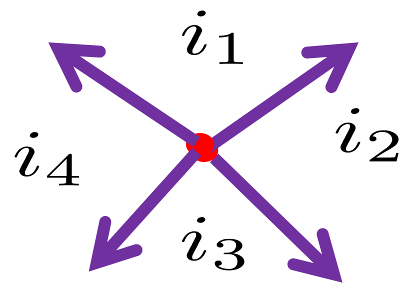
$$\mathcal{D}^{\rm red}(\mathfrak{w}) = \mathcal{D}(\mathfrak{w})/\mathbb{R}^2_{\rm transl}$$

$$\dim \mathcal{D}^{\rm red}(\mathfrak{w}) = d(\mathfrak{w})$$

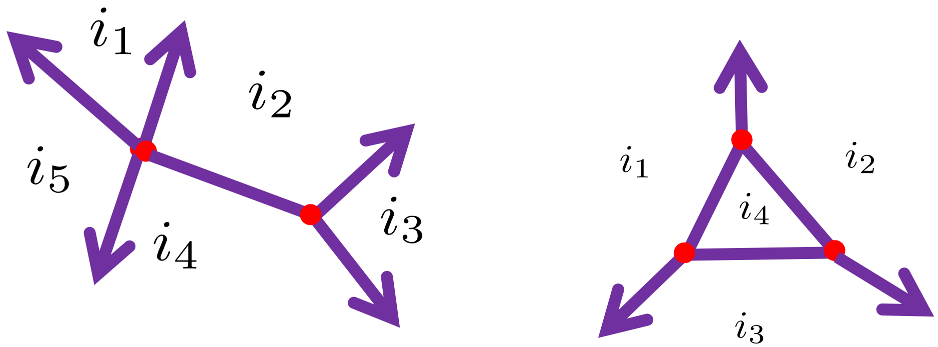
$$d(\mathfrak{w}) := 2V(\mathfrak{w}) - E(\mathfrak{w}) - 2$$

Rigid, Taut, and Sliding

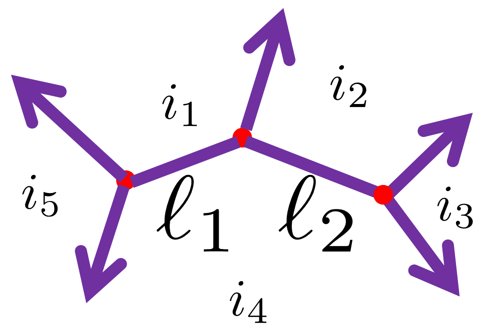
A *rigid web* has d($\mathfrak{w}$) = 0.
It has one vertex:



A *taut web* has d($\mathfrak{w}$) = 1:



A *sliding web* has d($\mathfrak{w}$) = 2



Cyclic Fans of Vacua

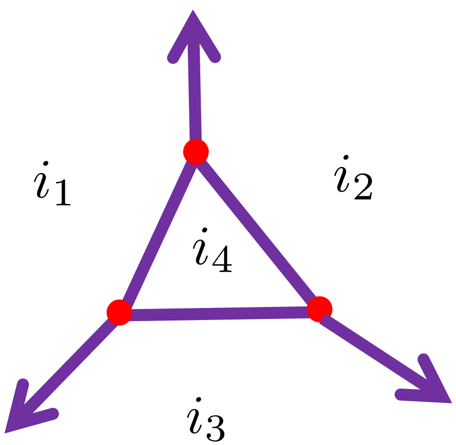
Definition: A cyclic fan of vacua is a cyclically-ordered set

$$I = \{i_1, \dots, i_n\}$$

so that the rays $z_{i_k, i_{k+1}} \mathbb{R}_+$ are ordered clockwise

Local fan of vacua at a vertex *v*:

$$I_v(\mathfrak{w})$$



and at ∞

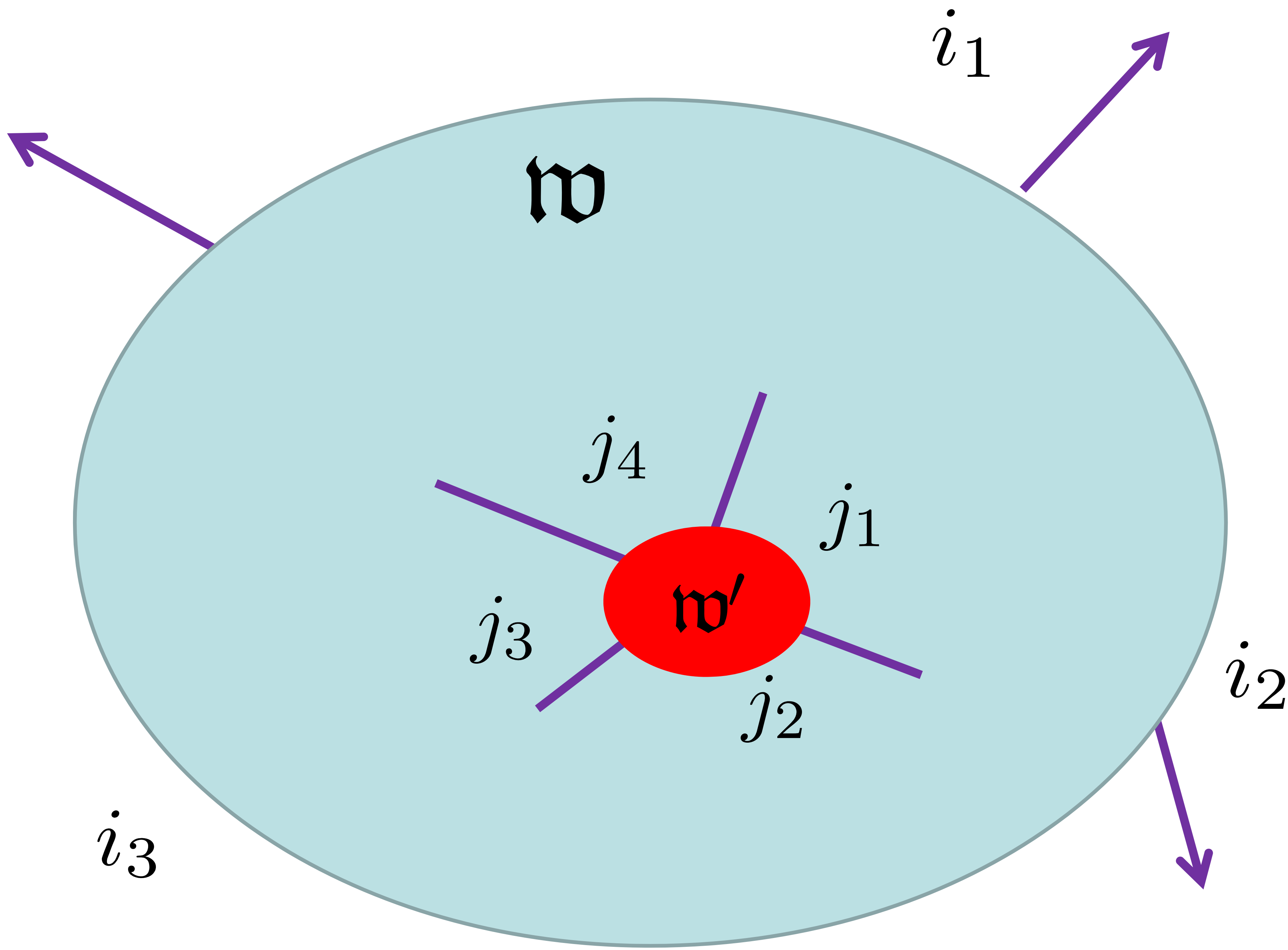
$$I_\infty(\mathfrak{w})$$

Convolution of Webs

Definition: Suppose $\mathfrak{w}$ and $\mathfrak{w}'$ are two plane webs and $v \in \mathcal{V}(\mathfrak{w})$ such that

$$I_v(\mathfrak{w}) = I_\infty(\mathfrak{w}')$$

The *convolution of $\mathfrak{w}$ and $\mathfrak{w}'$*, denoted $\mathfrak{w} *_v \mathfrak{w}'$ is the deformation type where we glue in a copy of $\mathfrak{w}'$ into a small disk cut out around *v*.


i_1
$\mathfrak{w}$
j_4
j_1
j_3
$\mathfrak{w}'$
j_2
i_2
i_3

The Web Ring

$\mathcal{W}$ Free abelian group generated by oriented deformation types of plane webs.

``oriented'': Choose an orientation o($\mathfrak{w}$) of $\mathcal{D}^{\mathrm{red}}(\mathfrak{w})$

$$* : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$$

$$I_v(\mathfrak{w}_1) \neq I_\infty(\mathfrak{w}_2) \Rightarrow \mathfrak{w}_1 *_v \mathfrak{w}_2 = 0$$

$$\mathfrak{w}_1 * \mathfrak{w}_2 := \sum_{v \in \mathcal{V}(\mathfrak{w}_1)} \mathfrak{w}_1 *_v \mathfrak{w}_2$$

$$o(\mathfrak{w} *_v \mathfrak{w}') = o(\mathfrak{w}) \wedge o(\mathfrak{w}')$$

The taut element

Definition: The taut element $\mathfrak{t}$ is the sum of all taut webs with standard orientation

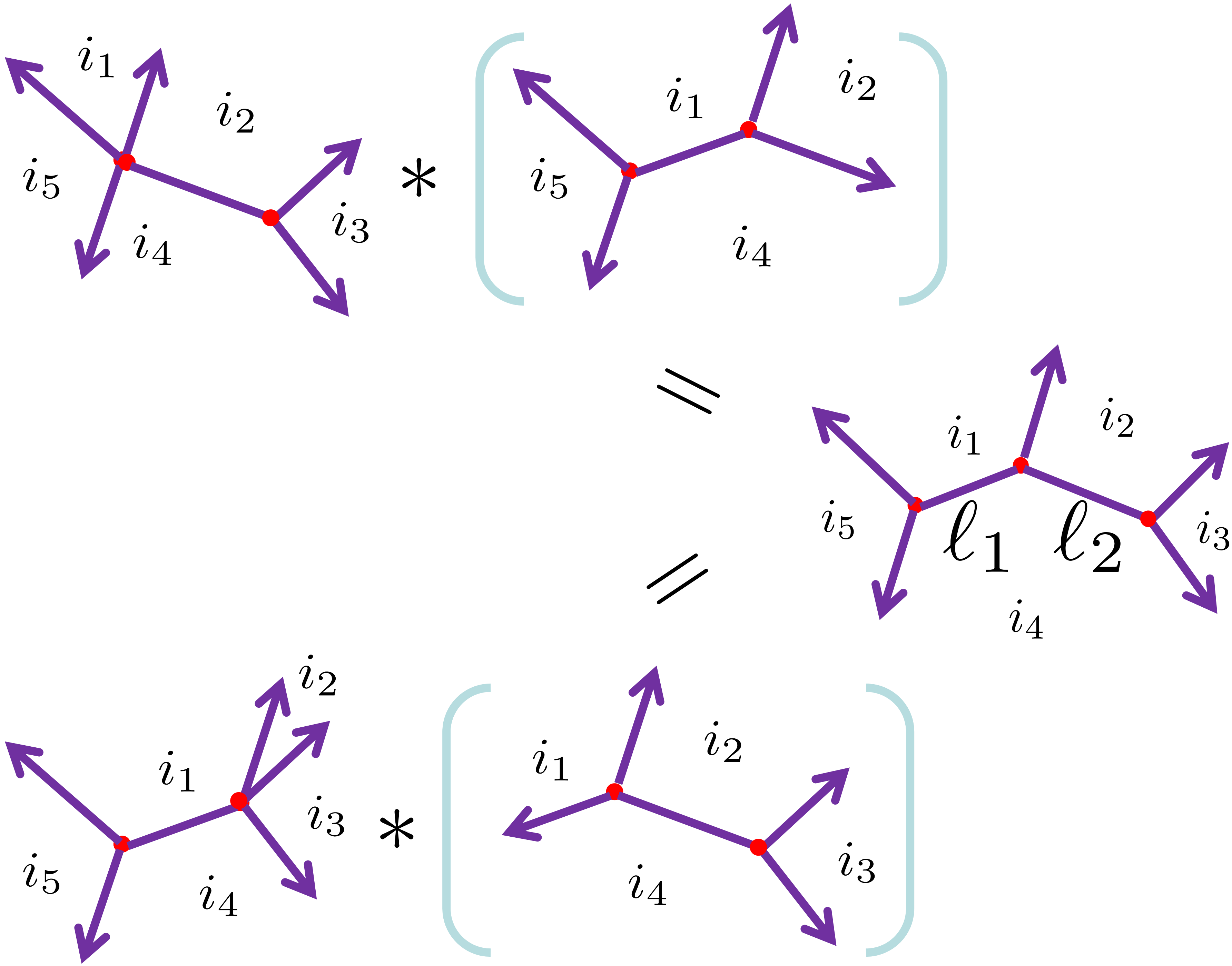
$$\mathfrak{t} := \sum_{d(\mathfrak{w})=1} \mathfrak{w}$$

Theorem: $\boxed{\mathfrak{t} * \mathfrak{t} = 0}$

Proof: The terms can be arranged so that there is a cancellation of pairs:

$$\mathfrak{w}_1 * \mathfrak{w}_2 \qquad \mathfrak{w}_3 * \mathfrak{w}_4$$

They represent the two ends of a one-dimensional (doubly reduced) sliding moduli space.



i_1
i_2
i_3
i_4
i_5
$*$
$=$
ℓ_1
ℓ_2

SKIP TO WEB REPRESENTATIONS & INTERIOR AMPLITUDES: SLIDES 87- 93

Extension to the tensor algebra

Define an operation by taking an unordered set $\{v_1, \ldots, v_m\}$ and an ordered set $\{\mathfrak{w}_1, \ldots, \mathfrak{w}_m\}$ and saying

$$\mathfrak{w} *_{\{v_1,\ldots,v_m\}} \{\mathfrak{w}_1, \ldots, \mathfrak{w}_m\}$$

- vanishes unless there is some ordering of the v_i so that the fans match up.
- when the fans match up we take the appropriate convolution.

$$T\mathcal{W} := \mathcal{W} \oplus \mathcal{W}^{\otimes 2} \oplus \mathcal{W}^{\otimes 3} \oplus \cdots$$

$$T(\mathfrak{w}) : T\mathcal{W} \to \mathcal{W}$$

$$T(\mathfrak{w})[\mathfrak{w}_1 \otimes \cdots \otimes \mathfrak{w}_n] := \mathfrak{w} *_{\mathcal{V}(\mathfrak{w})} \{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\}$$

Convolution Identity on Tensor Algebra

$$\mathfrak{t} * \mathfrak{t} = 0 \quad \Longrightarrow \quad T(\mathfrak{t})$$ satisfies L_∞ relations

$$\sum_{\mathrm{Sh}_2(S)} \epsilon\, T(\mathfrak{t})[T(\mathfrak{t})[S_1], S_2] = 0.$$

$$S = \{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\}$$

Two-shuffles: $\mathrm{Sh}_2(S)$ $\quad S = S_1 \amalg S_2$

This makes $\mathcal{W}$ into an L_∞ algebra

Half-Plane Webs

Same as plane webs, but they sit in a half-plane $\mathcal{H}$.

Some vertices (but no edges) are allowed on the boundary.

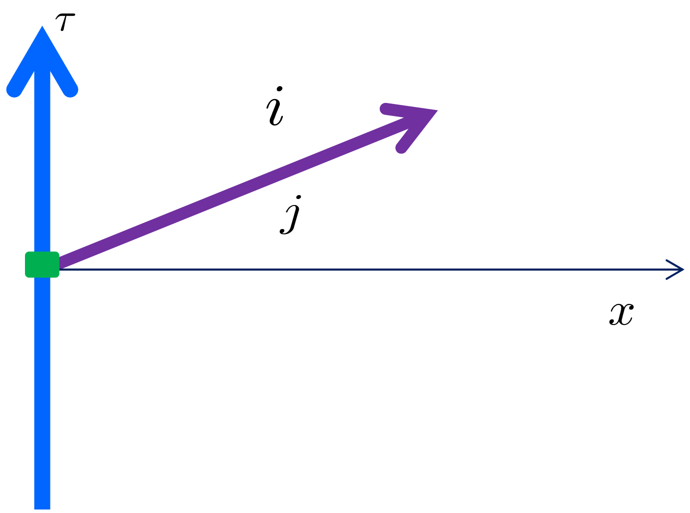
$\mathcal{V}_i(\mathfrak{u})$ Interior vertices

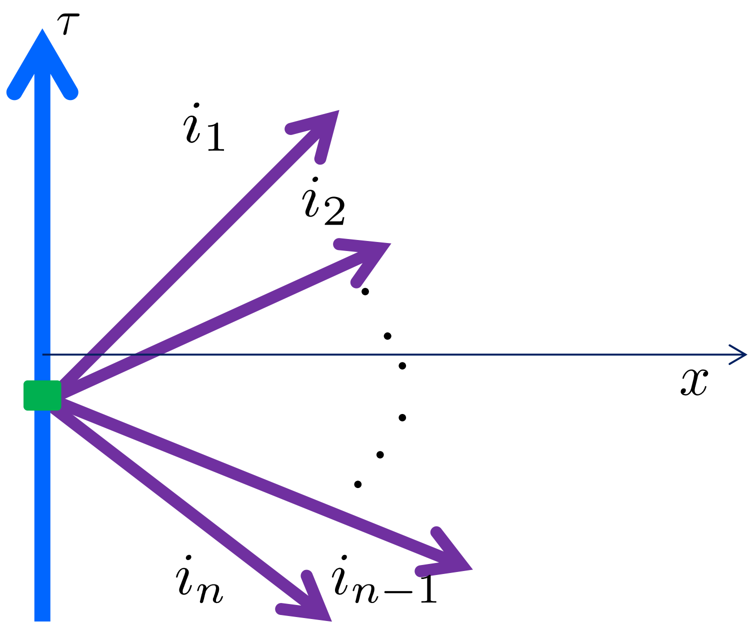
$\mathcal{V}_\partial(\mathfrak{u}) = \{v_1, \dots, v_n\}$ ***time-ordered*** boundary vertices.

deformation type, reduced moduli space, etc. ….

$$d(\mathfrak{u}) := 2V_i(\mathfrak{u}) + V_\partial(\mathfrak{u}) - E(\mathfrak{u}) - 1$$

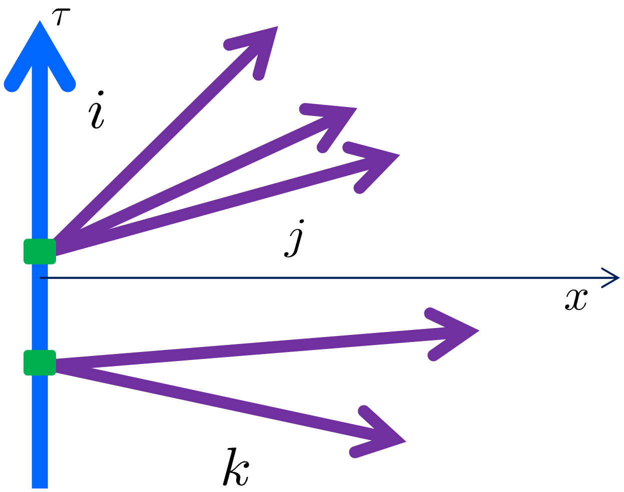
Rigid Half-Plane Webs

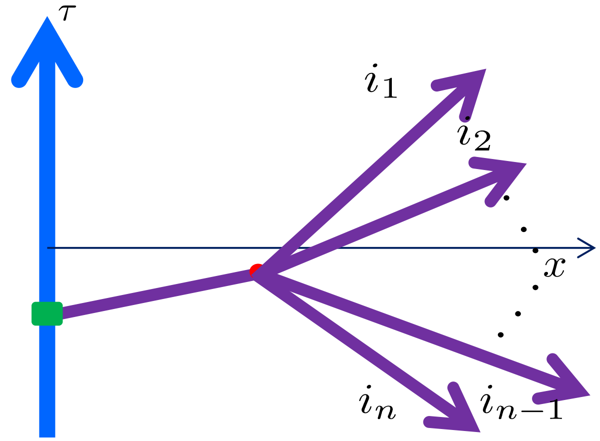




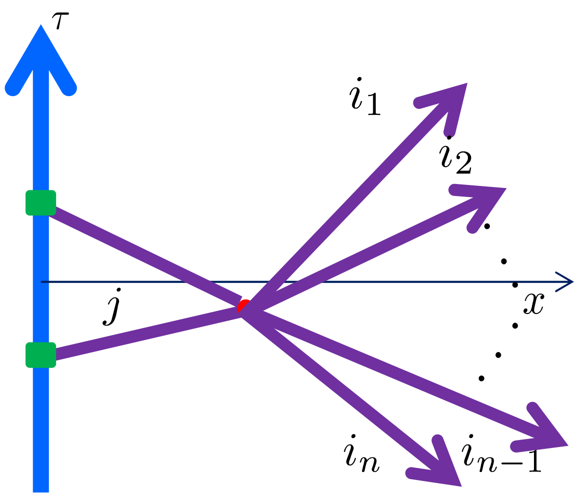
$$d(\mathfrak{u}) = 0$$

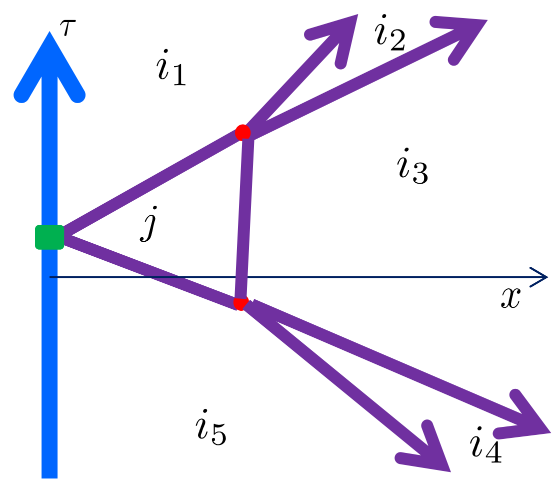
Taut Half-Plane Webs



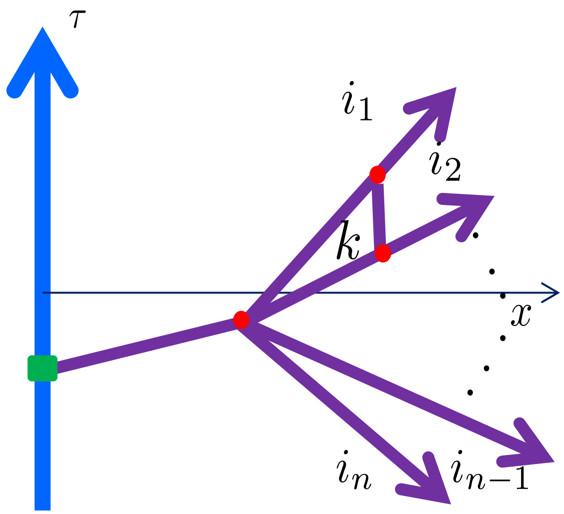


$$d(\mathfrak{u}) = 1$$

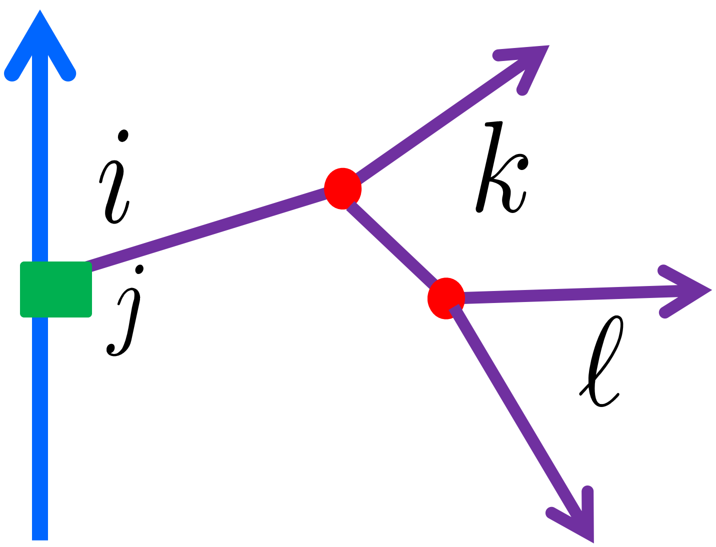


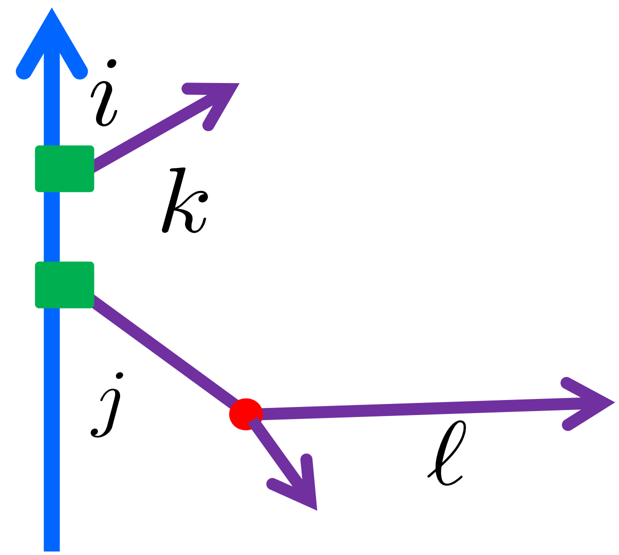


Sliding Half-Plane webs



$$d(\mathbf{u}) = 2$$





Half-Plane fans

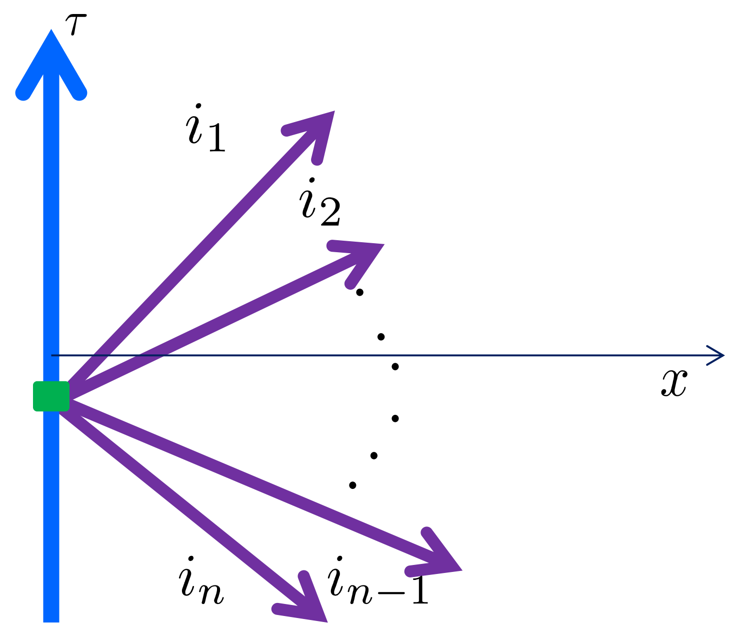
A half-plane fan is an ordered set of vacua,

$$J = \{ i_1, \dots, i_n \}$$

such that successive vacuum weights:

$$z_{i_s, i_{s+1}}$$

are ordered clockwise:



Convolutions for Half-Plane Webs

We can now introduce a convolution at boundary vertices:

Local half-plane fan at a boundary vertex *v*: $J_v(\mathfrak{u})$

Half-plane fan at infinity: $J_\infty(\mathfrak{u})$

$\mathcal{W}_{\mathcal{H}}$ Free abelian group generated by oriented def. types of half-plane webs

There are now two convolutions:

$$\mathcal{W}_{\mathcal{H}} \times \mathcal{W}_{\mathcal{H}} \to \mathcal{W}_{\mathcal{H}}$$

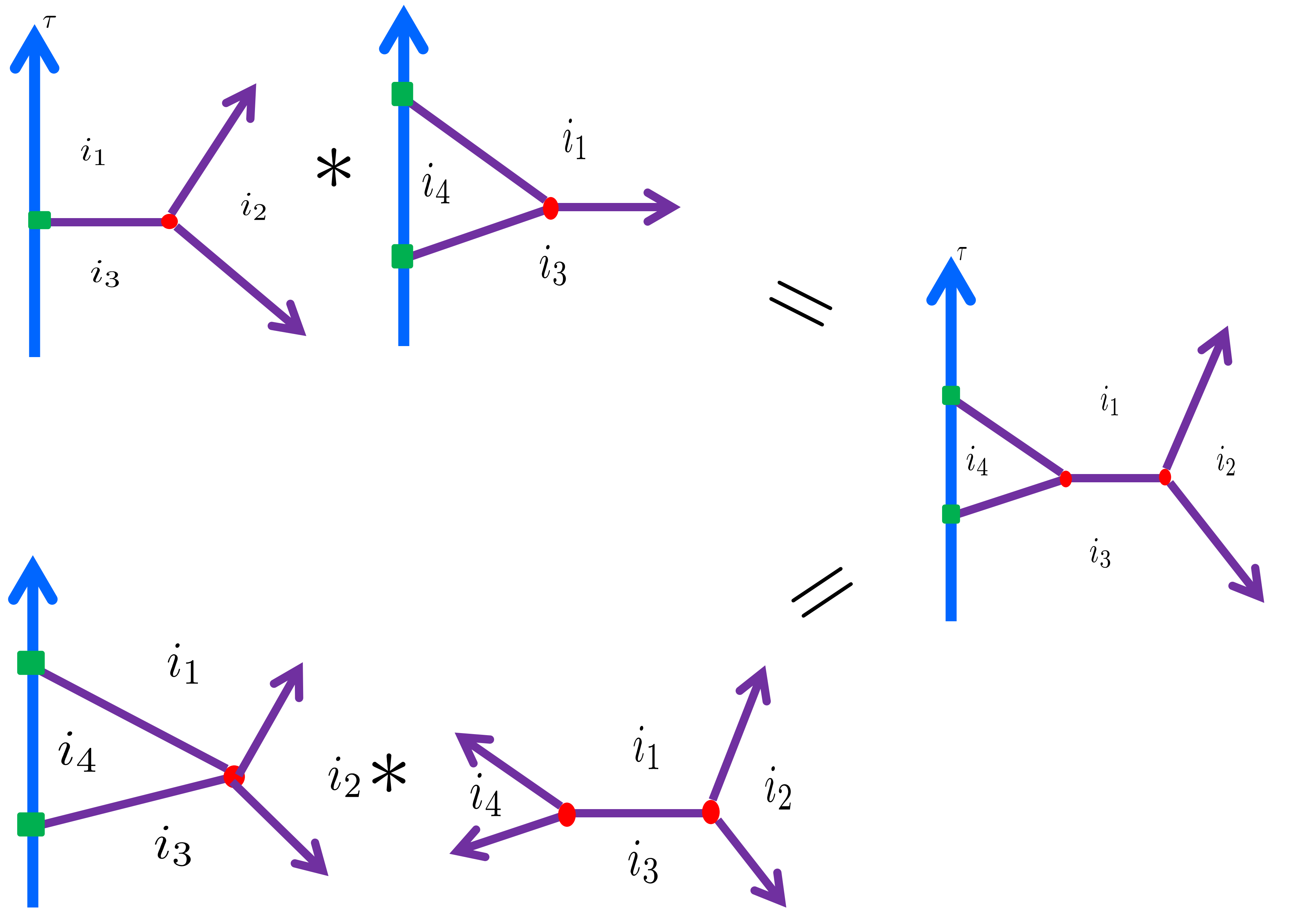
$$\mathcal{W}_{\mathcal{H}} \times \mathcal{W} \to \mathcal{W}_{\mathcal{H}}$$

Convolution Theorem

Define the half-plane taut element: $$\mathfrak{t}_{\mathcal{H}} := \sum_{d(\mathfrak{u})=1} \mathfrak{u}$$

Theorem: $$\mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathcal{H}} + \mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{p} = 0$$

Proof: A sliding half-plane web can degenerate (in real codimension one) in two ways: Interior edges can collapse onto an interior vertex, or boundary edges can collapse onto a boundary vertex.



τ
i_1
i_2
i_3
$*$
i_1
i_4
i_3
$=$
τ
i_1
i_4
i_2
i_3
$=$
i_1
i_4
i_2
i_3
$*$
i_1
i_4
i_2
i_3

Tensor Algebra Relations

Extend $\mathfrak{t}_{\mathcal{H}}$* to tensor algebra operator

$$T(\mathfrak{t}_{\mathcal{H}}) : T\mathcal{W}_{\mathcal{H}} \otimes T\mathcal{W} \to \mathcal{W}_{\mathcal{H}}$$

$$\sum \epsilon\, T(\mathfrak{t}_{\mathcal{H}})[P_1, T(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$+ \sum \epsilon\, T(\mathfrak{t}_{\mathcal{H}})[P; T(\mathfrak{t}_p)[S_1], S_2] = 0.$$

$$S = \{\mathfrak{w}_1, \ldots, \mathfrak{w}_n\} \quad P = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_m\}$$

Sum over ordered partitions:

$$P = P_1 \amalg P_2 \amalg P_3$$

Conceptual Meaning

$\mathcal{W}_{\mathcal{H}}$ is an $L\infty$ module for the L_∞ algebra $\mathcal{W}$

$\mathcal{W}_{\mathcal{H}}$ is an $A\infty$ algebra

There is an L_∞ morphism from the L_∞ algebra $\mathcal{W}$ to the L_∞ algebra of the Hochschild cochain complex of $\mathcal{W}_{\mathcal{H}}$

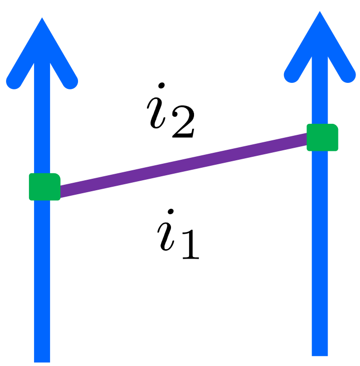
Strip-Webs

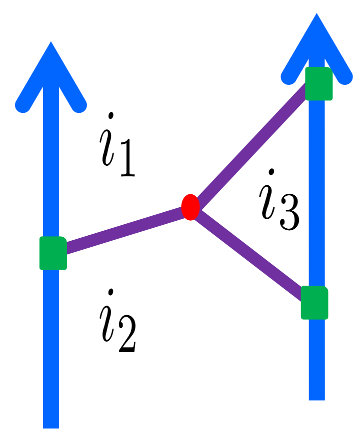
Now consider webs in the strip $\mathbb{R} \times [x_\ell, x_r]$

$$d(\mathfrak{s}) := 2V_i(\mathfrak{s}) + V_\partial(\mathfrak{s}) - E(\mathfrak{s}) - 1$$

Now ***taut*** and ***rigid strip-webs*** are the same, and have d(s)=0.

sliding strip-webs have d(s)=1.





Convolution Identity for Strip $\mathfrak{t}$'s

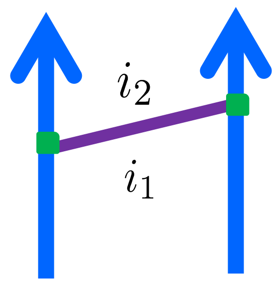
$$\mathfrak{t}_s := \sum_{d(\mathfrak{s})=0} \mathfrak{s}$$

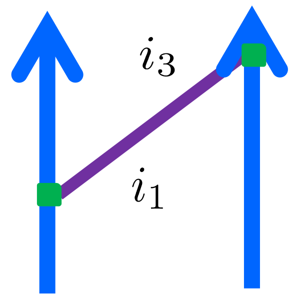
Convolution theorem:

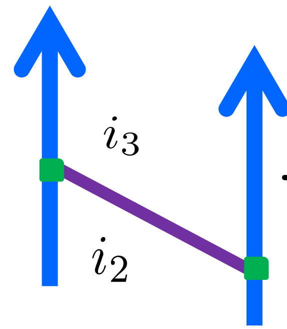
$$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_-} + \mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_+} + \mathfrak{t}_s * \mathfrak{t}_p + \mathfrak{t}_s \circ \mathfrak{t}_s = 0$$

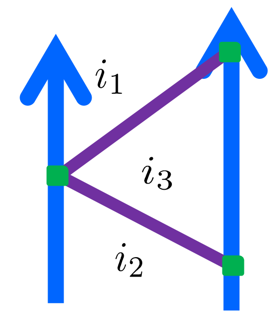
where for strip webs we denote time-concatenation by

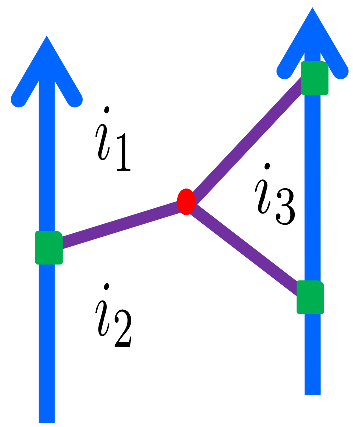
$$\mathfrak{s}_1 \circ \mathfrak{s}_2$$

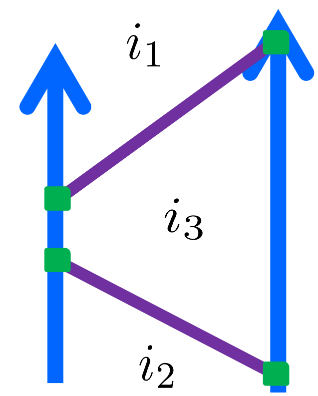
$\mathfrak{t}_s =$ 

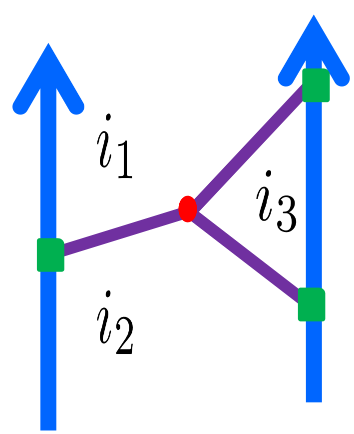
$+$ 

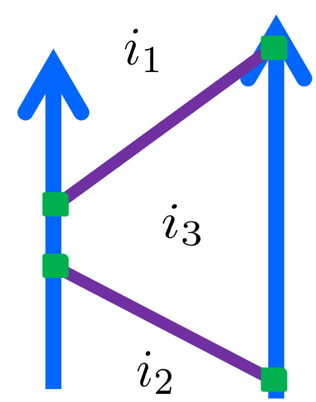
$+$ 

$+$ 

$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_-} =$ 

$\mathfrak{t}_s \circ \mathfrak{t}_s =$ 

$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_+} =$ 

$+$ 

Conceptual Meaning

$$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_-} + \mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_+} + \mathfrak{t}_s * \mathfrak{t}_p + \mathfrak{t}_s \circ \mathfrak{t}_s = 0$$

$\mathcal{W}_S$: Free abelian group generated by oriented def. types of strip webs.

There is a corresponding elaborate identity on tensor algebras …

$\mathcal{W}_S$ is an A_∞ bimodule

\+ … much more

Web Representations

Definition: A *representation of webs* is

a.) A choice of $\mathbb{Z}$-graded $\mathbb{Z}$-module R_{ij} for every ordered pair ij of distinct vacua.

b.) A degree = -1 pairing $$K : R_{ij} \otimes R_{ji} \to \mathbb{Z}$$

For every cyclic fan of vacua introduce a *fan representation*:

$$I = \{i_1, \dots, i_n\} \Rightarrow$$

$$R_I := R_{i_1,i_2} \otimes \cdots \otimes R_{i_n,i_1}$$

Web Rep & Contraction

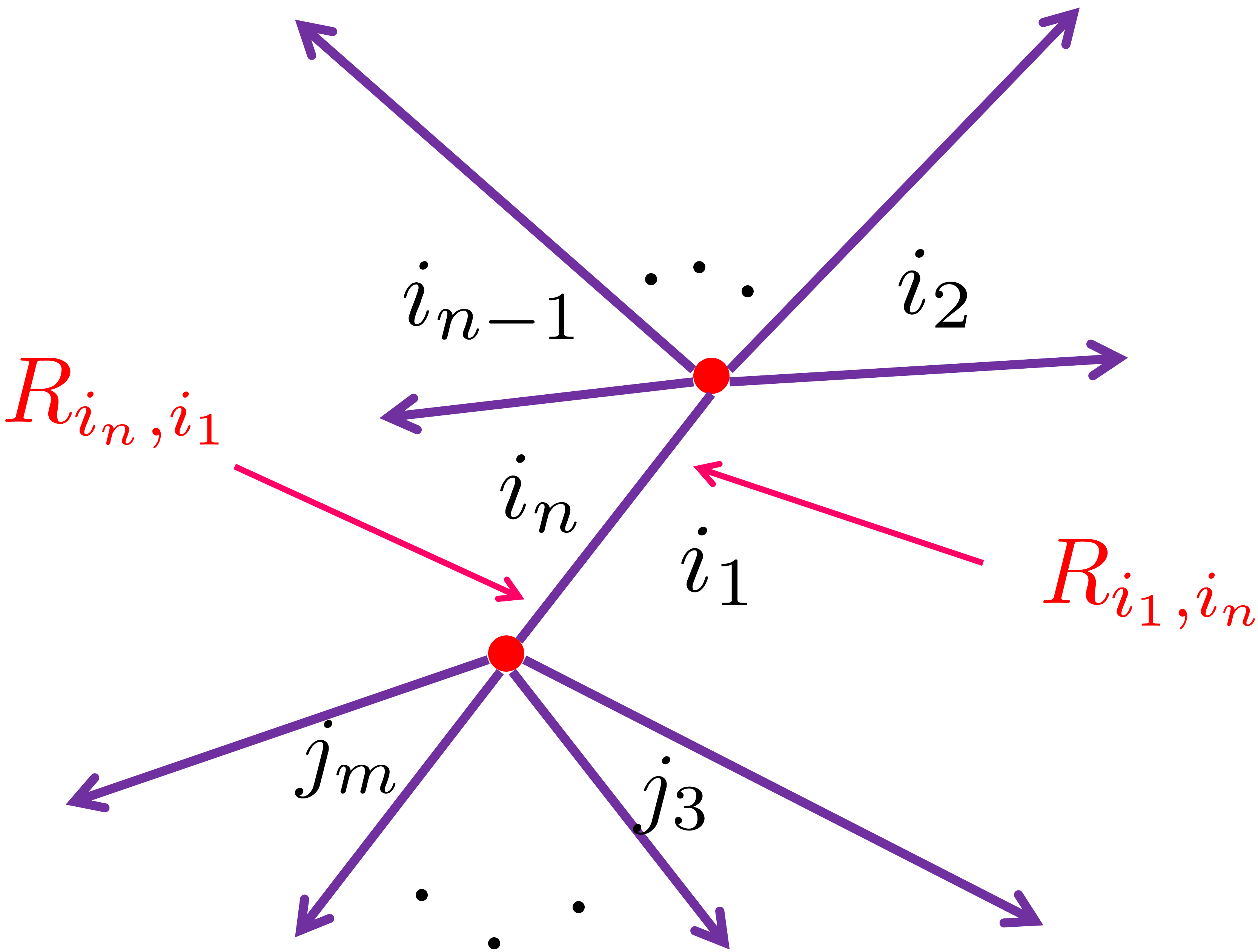
Given a rep of webs and a deformation type $\mathfrak{w}$ we define the <u>*representation of $\mathfrak{w}$*</u>:

$$R(\mathfrak{w}) := \otimes_{v \in \mathcal{V}(\mathfrak{w})} R_{I_v(\mathfrak{w})}$$

There is a natural contraction operator:

$$\rho(\mathfrak{w}) : R(\mathfrak{w}) \to R_{I_\infty(\mathfrak{w})}$$

by applying the contraction K to the pairs R_{ij} and R_{ji} on each internal edge:



i_{n-1}
i_2
R_{i_n,i_1}
i_n
i_1
R_{i_1,i_n}
j_m
j_3

L_∞ -algebras, again

$$R^{\mathrm{int}} := \oplus_I R_I$$ Rep of the rigid webs.

$$\rho(\mathfrak{t}_p) : TR^{\mathrm{int}} \to R^{\mathrm{int}}$$

$$\mathfrak{t}_p * \mathfrak{t}_p = 0 \Longrightarrow$$

$$\sum_{\mathrm{Sh}_2(S)} \epsilon\, \rho(\mathfrak{t}_p)[\rho(\mathfrak{t}_p)[S_1], S_2] = 0.$$

$$S = \{r_1, \dots, r_n\} \qquad r_i \in R^{\mathrm{int}}$$

$$S = S_1 \amalg S_2 \qquad \epsilon \in \{\pm 1\}$$

$L\infty$ and $A\infty$ Algebras - 1

If A is a vector space (or Z-module) then an ∞-algebra structure is a series of multiplications:

$$m_n(a_1, \ldots, a_n) \in A$$

Which satisfy quadratic relations:

$$S = \{a_1, \ldots, a_n\}$$

$$L_\infty : \quad \sum_{\mathrm{Sh}_2(S)} \epsilon m_{s_1+1}(m_{s_2}(S_2), S_1) = 0$$

$$A_\infty : \sum_{\mathrm{Pa}_3(S)} \epsilon m_{s_1+1+s_3}(S_1, m_{s_2}(S_2), S_3)) = 0$$

L∞ and A∞ Algebras - 2

$$V = V^i(x)\frac{\partial}{\partial x^i}$$

$$V^i(x) = m^i_j x^j + m^i_{j_1,j_2} x^{j_1,j_2} + m^i_{j_1,j_2,j_3} x^{j_1} x^{j_2} x^{j_3} + \cdots$$

$$V^2 = 0$$

A∞ if x^i noncommutative, V degree 1

L∞ if x^i graded-commutative, V degree 1

Consequence for LG Models

The main claim, in the context of LG models, is that counting solutions to the ζ-instanton equations with fan-boundary conditions and reduced dimension zero defines a solution to the L_∞ MC equation:

$$\rho(\mathbf{t}_p)(e^\beta) = 0$$

Half-Plane Contractions

A rep of a half-plane fan: $J = \{j_1, \dots, j_n\}$

$$R_J := R_{j_1, j_2} \otimes \cdots \otimes R_{j_{n-1}, j_n}$$

$\rho(\mathfrak{u})$ now contracts

$$\underbrace{\bigotimes_{v \in \mathcal{V}_\partial(\mathfrak{u})} R_{J_v(\mathfrak{u})}}_{\text{time ordered!}} \bigotimes_{v \in \mathcal{V}_i(\mathfrak{u})} R_{I_v(\mathfrak{u})}$$

$$\to R_{J_\infty(\mathfrak{u})}$$

The Vacuum A_∞ Category

(For the positive half-plane $\mathcal{H}_+$)

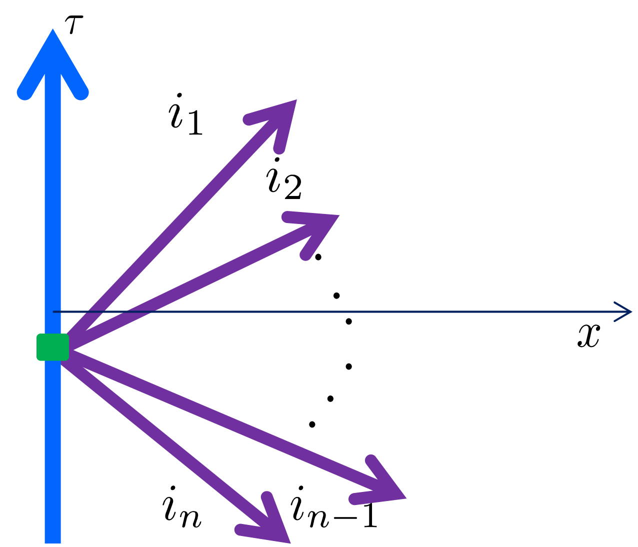
Objects: i $\in \mathbb{V}$.

Morphisms: $$\mathrm{Hom}(j,i) = \begin{cases} \widehat{R}_{ij} & \mathrm{Re}(z_{ij}) > 0 \\ \mathbb{Z} & i = j \\ 0 & \mathrm{Re}(z_{ij}) < 0 \end{cases}$$

$$\widehat{R}_{i_1,i_n} := \oplus_J R_J$$

$$J = \{i_1, \dots, i_n\}$$

$$\widehat{R}_{i_1,i_n} = R_{i_1,i_n} \oplus \cdots$$



Hint of a Relation to Wall-Crossing

The morphism spaces can be defined by a Cecotti-Vafa/Kontsevich-Soibelman-like product as follows:

Suppose $\mathbb{V} = \{ 1, \ldots, K\}$.
Introduce the elementary K x K matrices e_{ij}

$$1 + \oplus_{\mathrm{Re}(z_{ij})>0} \widehat{R}_{ij} e_{ij} = \underbrace{\otimes_{\mathrm{Re}(z_{ij})>0} (1 + R_{ij} e_{ij})}_{\text{phase ordered!}}$$

Defining A_∞ Multiplications

Sum over cyclic fans: $R^{\mathrm{int}} := \oplus_I R_I$

$$\rho(\mathfrak{t}_p) : TR^{\mathrm{int}} \to R^{\mathrm{int}}$$

<u>Interior amplitude:</u> $\beta \in R^{\mathrm{int}}$ Satisfies the L_∞ ``Maurer-Cartan equation''

$$\rho(\mathfrak{t}_p)(e^{\beta}) = 0$$

$$m_n^{\beta}[r_1, \ldots, r_n] := \rho(\mathfrak{t}_{\mathcal{H}})[r_1, \ldots, r_n; e^{\beta}]$$

$$r_s \in \mathrm{Hom}(i_{s-1}, i_s)$$

Proof of A_∞ Relations

$$\mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathcal{H}} + \mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{p} = 0 \Rightarrow$$

$$\sum \epsilon\, \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$+ \sum \epsilon\, \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_{p})[S_1], S_2] = 0.$$

$$S = \{r_1, \dots, r_m\} \qquad S = S_1 \amalg S_2$$

$$P = \{r_1^\partial, \dots, r_n^\partial\} \qquad P = P_1 \amalg P_2 \amalg P_3$$

$$r_a \in R^{\mathrm{int}} \qquad r_s^\partial \in \widehat{R}_{i_{s-1}, i_s}$$

$$\sum \epsilon \; \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$\cancel{+ \sum \epsilon \; \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_p)[S_1], S_2]} = 0.$$

$$S = \{\beta, \ldots, \beta\}$$

and the second line vanishes.

Hence we obtain the A∞ relations for :

$$m^{\beta}[P] := \rho(\mathfrak{t}_{\mathcal{H}})[P; e^{\beta}]$$

Defining an A∞ category : $\mathfrak{Vac}(\mathbb{V}, z, R, K, \beta)$

Enhancing with CP-Factors

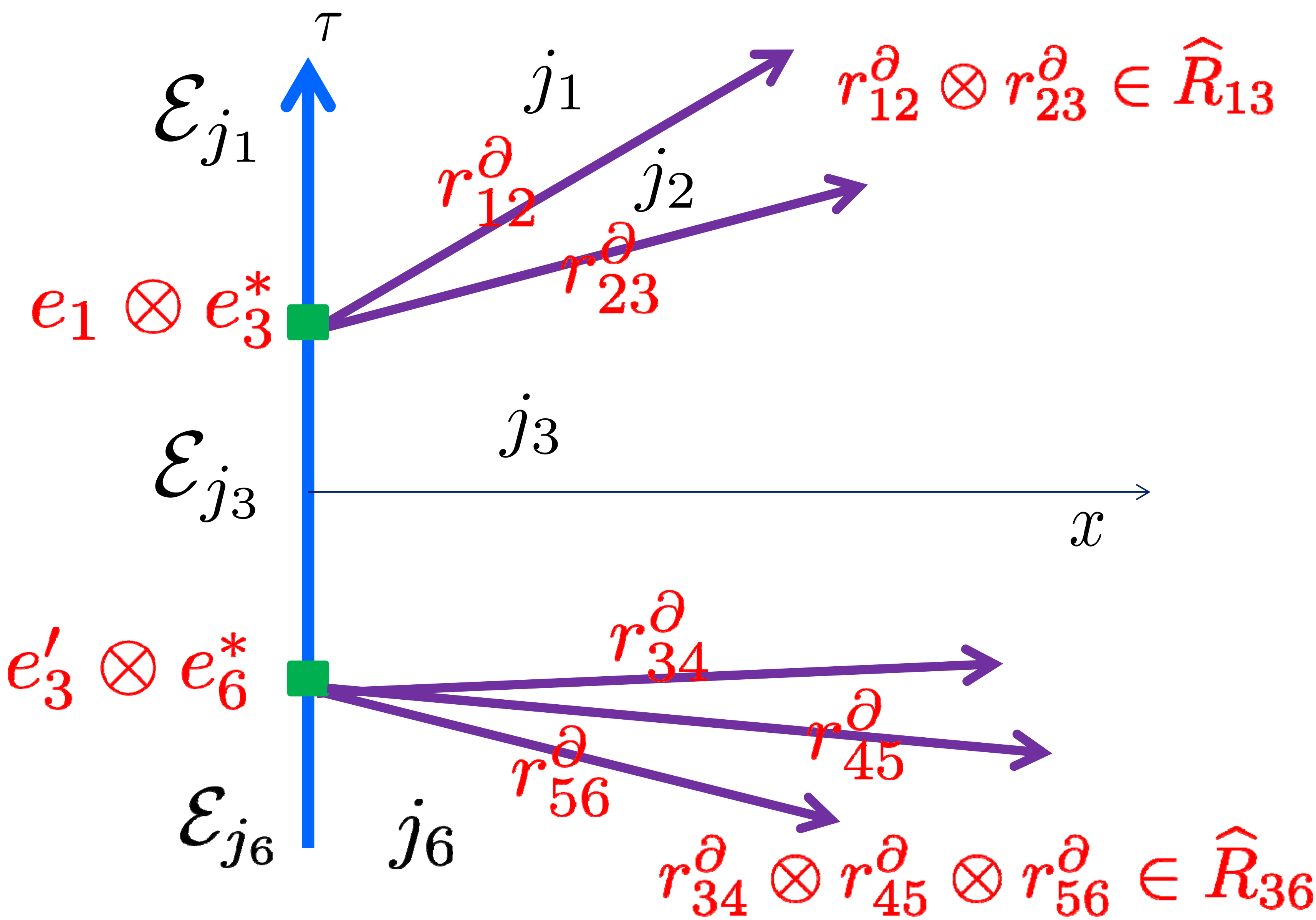
CP-Factors: $i \in \mathbb{V}$ $\longrightarrow$ $\mathcal{E}_i$ $\mathbb{Z}$-graded module

$$\mathrm{Hop}(i,j) \Longrightarrow \mathcal{E}_i \otimes \mathrm{Hop}(i,j) \otimes \mathcal{E}_j^*$$

$$m_n^\beta \Longrightarrow m_n^\beta \otimes m_2^{\mathrm{CP}}$$

Enhanced A∞ category : $\mathfrak{Vac}(\mathbb{V}, z, R, K, \beta; \mathcal{E}_*)$

Example: Composition of two morphisms



Boundary Amplitudes

A Boundary Amplitude $\mathcal{B}$ (defining a Brane) is a solution of the A_∞ MC:

$$\mathcal{B} \in \oplus_{i,j} \mathrm{Hop}(i,j)$$

$$\mathcal{B} \in \oplus_{\mathrm{Re}(z_{ij})>0} \mathcal{E}_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^*$$

$$\sum_{n=1}^{\infty} m_n^{\beta}[\mathcal{B}^{\otimes n}] = 0$$

$$\rho(\mathfrak{t}_{\mathcal{H}})[\frac{1}{1-\mathcal{B}}; e^{\beta}] = 0$$

Constructions with Branes

Strip webs with Brane boundary conditions help answer the physics question at the beginning.

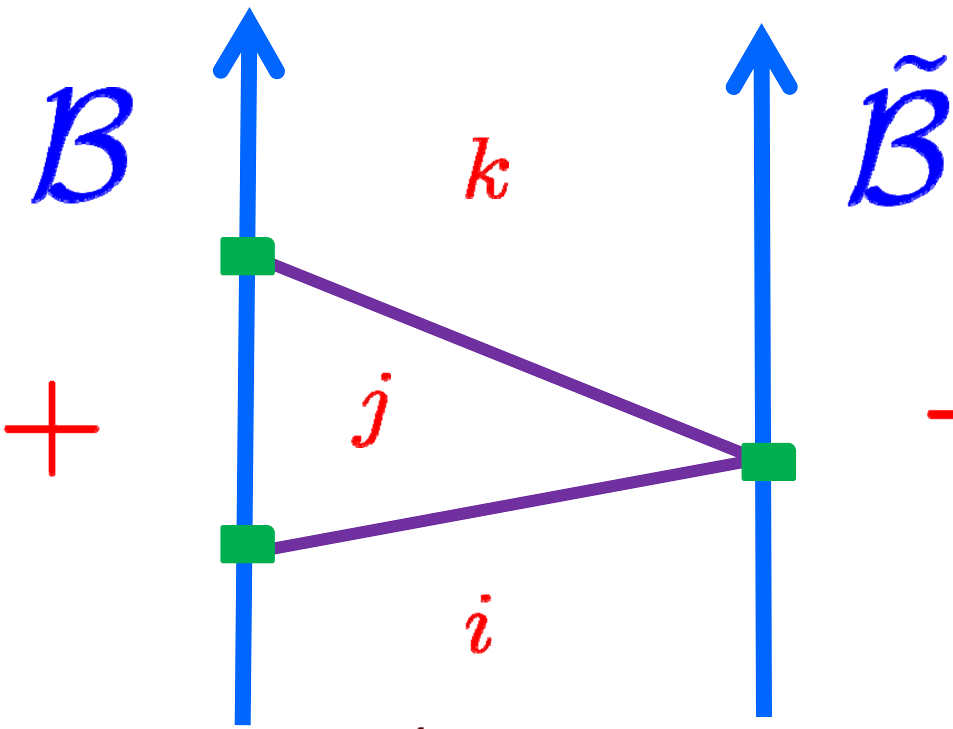
The Branes themselves are objects in an A_∞ category

$$\mathfrak{Br}(\mathbb{V}, z, R, K, \beta)$$

("Twisted complexes": Analog of the derived category.)

Given a (suitable) continuous path of data $(\mathbb{V}, z, R, K, \beta)(x)$ we construct an invertible functor between Brane categories, only depending on the homotopy class of the path. (Parallel transport of Brane categories.)

$$\mathfrak{t}_s = \cdots +$$



$$+ \cdots$$

$$d(v) := \rho(\mathfrak{t}_s)\left(\frac{1}{1-\mathcal{B}}; e^{\beta}; v; \frac{1}{1-\tilde{\mathcal{B}}}\right)$$

$$d : \oplus_{i \in \mathbb{V}} \mathcal{E}_i \otimes \widetilde{\mathcal{E}}_i \to \oplus_{i \in \mathbb{V}} \mathcal{E}_i \otimes \widetilde{\mathcal{E}}_i$$

$$\mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_-} + \mathfrak{t}_s * \mathfrak{t}_{\mathcal{H}_+} + \mathfrak{t}_s * \mathfrak{t}_p + \mathfrak{t}_s \circ \mathfrak{t}_s = 0$$

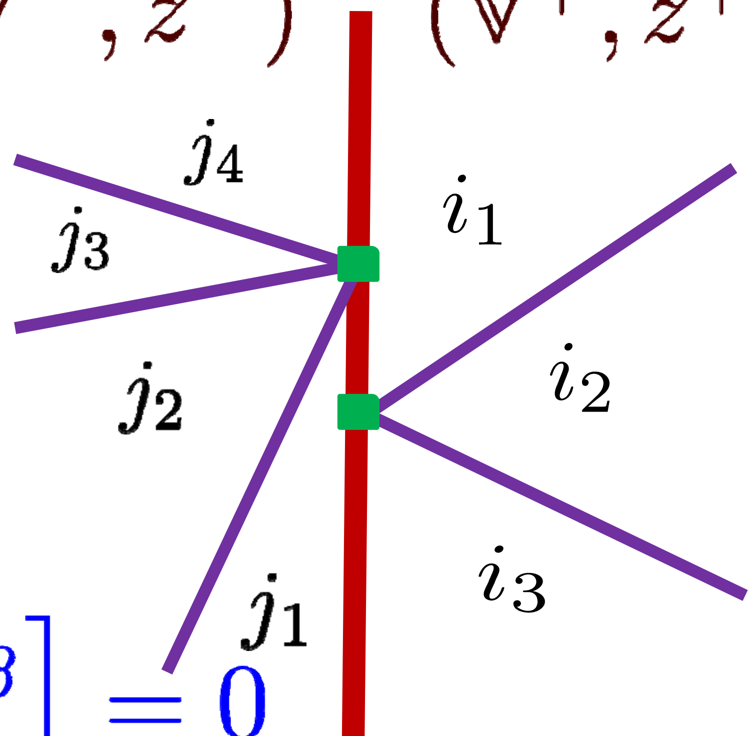
Convolution identity implies: $$d^2 = 0$$

Interfaces webs & Interfaces

Given data $(\mathbb{V}^{\pm}, z^{\pm}, R^{\pm}, K^{\pm}, \beta^{\pm})$

Introduce a notion of ``interface webs"

$(\mathbb{V}^{-}, z^{-})$ $(\mathbb{V}^{+}, z^{+})$



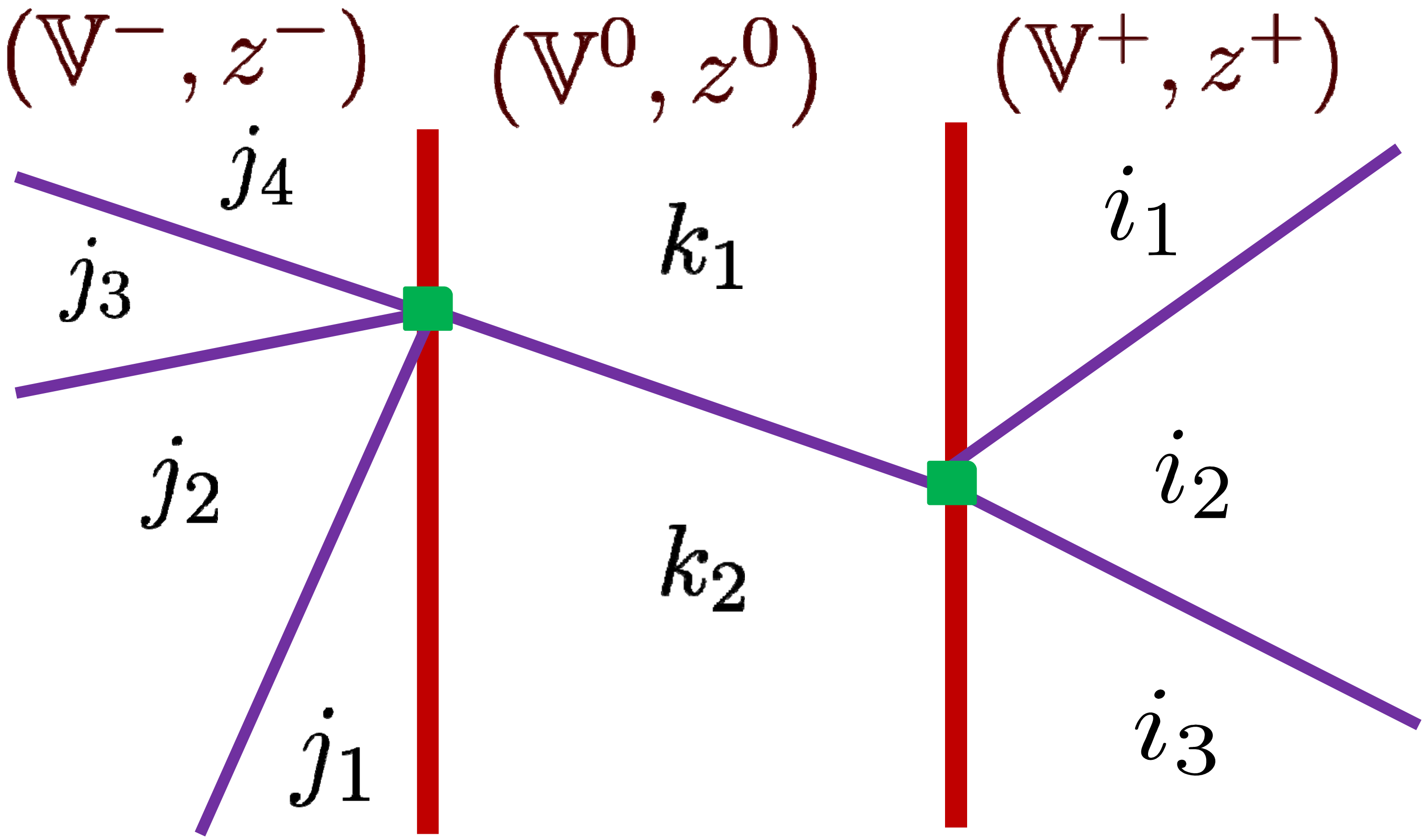
These behave like half-plane webs and we can define an *Interface Amplitude* to be a solution of the MC equation:

$$\rho(\mathfrak{t}^{-,+})\left[\frac{1}{1-\mathcal{B}^{-,+}}; e^{\beta}\right] = 0$$

Composite webs

Given data $(\mathbb{V}, z, R, K, \beta)^{-,0,+}$

Introduce a notion of ``composite webs''



Composition of Interfaces

A convolution identity implies:

$$\rho(\mathfrak{t}^{-,0,+})\left[\frac{1}{1-\mathcal{B}^{-,0}}, \frac{1}{1-\mathcal{B}^{0,+}}; e^{\beta}\right] \in \mathfrak{Br}^{-,+}$$

Defines a family of A_∞ bifunctors:

$$\mathfrak{Br}^{-,0} \times \mathfrak{Br}^{0,+} \to \mathfrak{Br}^{-,+}$$

$$\mathfrak{Br}^{-,0} \times \mathfrak{Br}^{0,1} \times \mathfrak{Br}^{1,+} \to \mathfrak{Br}^{-,+}$$

Product is associative up to homotopy

Composition of such bifunctors leads to categorified parallel transport

Physical ``Theorem”

Data

(X,ω): Kähler manifold (exact)

W: X $\longrightarrow \mathbb{C}$ Holomorphic Morse function

Finitely many critical points with critical values in general position.

We construct an explicit realization of above:

- Vacuum data.
- Interior amplitudes.
- Chan-Paton spaces and boundary amplitudes.
- “Parallel transport” of Brane categories.

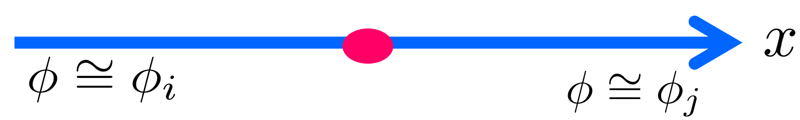
Vacuum data:

$\mathbb{V}$ Morse critical points ϕ_i $dW(\phi_i) = 0$

$$z_i \sim W_i := W(\phi_i) \quad \left[\text{Actually, } z_i = \mathrm{i}\zeta \overline{W}_i \right]$$

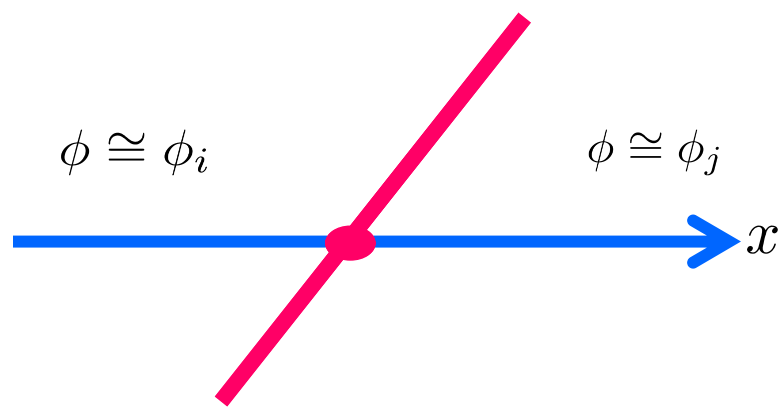
Connection to webs uses BPS states:

Semiclassically, they are solitonic particles.



Worldlines preserving "ζ-supersymmetry"
are solutions of the "ζ-instanton equation"

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}} \frac{\partial \overline{W}}{\partial \bar{\phi}^{\bar{J}}}$$



A Natural Conjecture

Following constructions used in the Fukaya category, Paul Seidel constructed an A_∞ category FS[X,W] associated to a holomorphic Morse function W: X to $\mathbb{C}$.

Tw[FS[X,W]] is meant to be the category of A-branes of the LG model.

But, we also think that Br[Vac[X,W]] is the category of A-branes of the LG model!

So it is natural to conjecture an equivalence of A_∞ categories:

$$\mathrm{Tw}[\mathrm{FS}[X,W]] \cong \mathrm{Br}[\mathrm{Vac}[X,W]]$$

"ultraviolet"

"infrared"

Parallel Transport of Categories

To $\wp$ we associate an A∞ functor

$$\mathbb{F}(\wp) : Br[Vac[W_1]] \to Br[Vac[W_2]]$$

(Relation to GMN: "Categorification of S-wall crossing")

To a composition of paths we associate a composition of A∞ functors:

$$\mathbb{F}(\wp_1 \circ \wp_2) = \mathbb{F}(\wp_1) \circ \mathbb{F}(\wp_2)$$

To a homotopy of $\wp_1$ to $\wp_2$ we associate an equivalence of A∞ functors. (Categorifies CVWCF.)

Outline

- Introduction & Motivations
- Some Review of LG Theory
- Goals, Results, Questions Old & New
- LG Theory as SQM
- Boosted Solitons & Soliton Fans
- Web-based Formalism
- Summary & Outlook

Summary

1. Instantons effects can be thought of in terms of an ``effective theory'' of BPS particles.

2. This naturally leads to L_∞ and A_∞ structures.

3. As an application, the set of BPS states on an interval does not satisfy the naïve clustering of classical BPS solitons.

4. When there are families of LG superpotentials there is a notion of parallel transport of the A_∞ categories.

Outlook

1. Relation to S-matrix singularities?

2. Are these examples of universal identities for massive 1+1 N=(2,2) QFT?

3. Generalization to 2d4d systems: Categorification of the 2d4d WCF.

4. Computability of Witten's approach to knot homology? Relation to other approaches to knot homology?