10:04:30

$\boxed{\text{II} - 1}$

TALK II:

Before talk: On BB: Coh TFT

$$\begin{tikzcd} E \arrow[d] \\ M \arrow[u, bend left, "s"] \arrow[r, "\mathcal{G}"'] & \overline{M} \end{tikzcd}$$

$$0 \longrightarrow \mathrm{Lie}\,\mathcal{G} \xrightarrow{V_m} T_m M \xrightarrow{\nabla s} E_m$$

$$\mathbb{F} = \nabla s \oplus V^{\dagger}$$

$$Z = \int_{\mathrm{Lie}\mathcal{G}} [d\phi] \int_{\hat{E} \times W(\mathrm{Lie}\mathcal{G})^{\vee}} \mathrm{Ber} \; \mathcal{O} \; e^{Q\Psi}$$

Q-closed & $\mathcal{G}$-inv t

$$= \int_{\mathcal{M} = Z(s)/\mathcal{G}} \omega_{\mathcal{O}} \wedge \mathrm{Eul}(\mathrm{cok}\,\mathbb{F})$$

$$\mathcal{O} \longleftrightarrow \omega_{\mathcal{O}}$$

$$\mathcal{O} \in H^*_Q \longleftrightarrow H^*(\mathcal{M})$$

$Z \neq 0$ only for $gh\#(\mathcal{O})$ $\cancel{= \mathrm{Index}(\mathbb{F})}$

$$= \deg \omega_{\mathcal{O}}$$

$$= \mathrm{Index}(\mathbb{F})$$

Two corrections from last time

Before talk: On BB

$$P \xrightarrow{G} X$$

$$M = \mathrm{Conn}(P) = \mathcal{A}$$

Donaldson-Witten:

$$E = \mathcal{A} \times \Omega^{2,+}(\mathrm{ad}P)$$

$$s(A) = F + *F$$

$$\mathcal{G} = \mathrm{Aut}(P)$$

$N=2$ SP^0 $\qquad su(2)_- \oplus su(2)_+ \oplus su(2)_R \oplus U(1)_R$

VM:	A_μ	$(2,2,1)^0$	$\longrightarrow$	$A \in \mathcal{A}$
	$\bar{\Psi}_{\dot\alpha}^{A}$	$(1,2,2)^{-1}$	$\longrightarrow$	$\chi_{\mu\nu}, \eta$
	ψ_α^{A}	$(2,1,2)^{+1}$	$\longrightarrow$	ψ_μ
	ϕ	$(1,1,1)^2$	$\longrightarrow$	ϕ
	$\bar\phi$	$(1,1,1)^{-2}$	$\longrightarrow$	$\bar\phi$
	D	$(1,1,3)^0$	$\longrightarrow$	$D^+ \in \Gamma(\mathrm{ad}P \otimes \Lambda^+)$

$$Z(\omega^-, \omega^+, \omega_R) = \int [dA\, d\psi \cdots]\, e^{S_{phys}}$$

Top Tw: $\omega^+ = \omega_R$: $\boxed{Q = \delta^{\dot\alpha}_{A} \bar{Q}_{\dot\alpha}^{A} \qquad Q^2 = 0}$

$$= \sum_{[P]} \int_{\mathcal{M}(P,g)} \omega_0 \; \mathrm{cok}(\mathbb{F})$$

Don't erase!

$$\boxed{S_{phys}} = \int \frac{1}{g_0^2} \mathrm{tr}\left(F*F + D\phi * D\phi^* - [\phi,\phi^*]^2\right) + \frac{\theta_0}{8\pi^2} \mathrm{tr}\, F^2 + \cdots$$

$$= Q\Psi + \mathrm{cnst} \int \mathrm{Tr}\, F^2$$

10:12 8M

$$\mathcal{M}(P,g) = \{A \mid F^+(A) = 0\} / \mathcal{G}.$$

~~First order~~ Def. cplx AHS:

$$0 \to \Omega^0(\mathrm{ad}P) \xrightarrow{\nabla_A} \Omega^1(\mathrm{ad}P) \xrightarrow{\nabla_A^+} \Omega^{2,+}(\mathrm{ad}P)$$

$$\not{D}_E : \Gamma(S^- \otimes E) \longrightarrow \Gamma(S^+ \otimes E)$$

$$E = S^+ \otimes \mathrm{ad}P$$

$$\mathrm{vdim}\,\mathcal{M}(P,g) = 4h^\vee k - \dim G \,\frac{\chi+\sigma}{2}$$

Thm (Gen. Metrics) $G = SU(2), SO(3)$ $g_{\mu\nu}$ generic

$H^0, H^2 = 0. \Rightarrow \mathcal{M}(P,g)$ smooth

$G = SU(N)$ Kronheimer

<u>CohTFT</u>: cplx = AHS cplx

$$\mathbb{F} = \not{D}_E$$

gh# = $U(1)_R$ charge

Anomaly in $U(1)_R$ = Index $\not{D}_E$

<u>Q-fixed</u>:

$$0 = Q\chi = F^+ - D$$

$$0 = Q\psi = D_A\phi$$

pure VM: $D = 0$

NOT true VM+HM

$\mathcal{O} \longrightarrow \omega_{\mathcal{O}}$

$Q\phi = 0$ $\quad \mathcal{P}$ invt poly on $\mathfrak{g}$.

$\mathcal{O}^{(0)} = \mathcal{P}(\phi)$ evaluated at point $p \in X$.

$\mathcal{O}^{(0)}$: descent formalism:

$Q_{\alpha\dot{\alpha}} = K_{\mu}$ $\qquad \{Q, dx^{\mu} K_{\mu}\} = dx^{\mu}\partial_{\mu} = d.$

$\mathcal{O}^{(1)} = K\,\mathcal{O}^{(0)}$ $\qquad Q\int_{\gamma}\mathcal{O}^{(1)} = \mathcal{O}^{(0)}\Big|_{\partial\gamma}$

- $\partial\gamma = p_1 - p_2$ $\qquad \mathcal{O}^{(0)}(p_1) = \mathcal{O}^{(0)}(p_2) + \{Q, *\}$
- $\partial\gamma = \phi$ $\qquad \mathcal{O}(\gamma) = \int_{\gamma}\mathcal{O}^{(1)}$

  only depends on $[\gamma] \in H_1$
- $\partial\Sigma_j = \phi$ $\qquad \mathcal{O}(\Sigma_j) = \int_{\Sigma_j} K^j\,\mathcal{O}^{(0)}$

$G = SU(2)$ $\qquad \mathcal{O}^{(0)}(p) = \frac{1}{8\pi^2}\mathrm{Tr}_{\underline{2}}\,\phi^2(p) = \mathcal{O}$

$$\mathcal{O}(\Sigma) \cong \int_{\Sigma}\mathrm{Tr}(\phi F + \psi\psi)$$

"Witten polynomials" $\qquad P_W(p^l\Sigma^r) = \langle \mathcal{O}^l\,\mathcal{O}(\Sigma)^r\rangle_{\substack{SU(2)\,VM \\ \omega_R = \omega^+}}$

- $\mathbb{C}$- valued poly's on $H_*(X)$
- invts of smooth structure
- only instanton sector $4l + 2r = \mathrm{vdim}\,\mathcal{M}(P, g)$

Donaldson μ- map + poly's

$$\mu : H_*(X) \longrightarrow H^*_{cpt}(\mathcal{M})$$

$$\begin{array}{ccc}
P \times \mathcal{A} & & \\
\;\;\mathcal{G} \searrow & & \\
& P \times \mathcal{A}/\mathcal{G} & \\
& G \downarrow & \\
& X \times \mathcal{A}/\mathcal{G} & \xrightarrow{\;f\;} BG
\end{array}$$

$$\varpi \in H^{2d}(BG)$$

$$\mu_D(\Sigma_j) = \int_{\Sigma_j} f^*(\varpi) \in H^{2d-j}(\mathcal{A}/\mathcal{G})$$

$SU(2)$:

$$p \longrightarrow \mu_D(p) \in H^4(\mathcal{M})$$

$$S \longrightarrow \mu_D(S) \in H^2(\mathcal{M})$$

$\mathcal{M}$ can be oriented -

$$P_D(p^l S^r) = \int_{\mathcal{M}} \mu_D(p)^l \mu_D(S)^r$$

- $\mathbb{Q}$ - valued
- invts of smooth $b_2^+ > 1$

choose P

Claim 1:

$$\begin{tikzcd} H_*(X) \arrow[r, "\text{desc.}"] \arrow[dr, "\mu"'] & H_Q^{2d-*} \arrow[d] & \mathcal{O} \arrow[d] \\ & H^{2d-*}(\mathcal{M}) & \omega_{\mathcal{O}} \end{tikzcd}$$

Baulieu + Singer $\quad \mathcal{F} = \underset{(2,0)}{F} + \underset{(1,1)}{\psi} + \underset{(0,2)}{\phi}$

Claim 2:

keep: use right $\mathcal{B}_3$

$$Z^{\xi}_{DW}(p, s) = \left\langle e^{p\,\mathcal{O} + s^{\alpha}\mathcal{O}(\Sigma_\alpha)} \right\rangle_{\mathcal{T}(\Lambda)}$$

$$= \sum_{l, r \geq 0} \frac{p^l}{l!\, r!} \left\langle \mathcal{O}^l\, \mathcal{O}(\Sigma)^r \right\rangle_{\mathcal{T}(\Lambda)}$$

$$= \frac{1}{2}\, \Lambda^{-\frac{3}{4}(\chi+\sigma)} \sum_{l, r \geq 0} \frac{p^l \prod s_\alpha^{r_\alpha}}{l! \prod r_\alpha!} \Lambda^{2l+r}\, P_D\big(\rho^l \cancel{\Sigma^r} \;\; \textstyle\prod \Sigma_\alpha^{r_\alpha}\big)$$

- Λ
- $P \xrightarrow{SO(3)} X \quad w_2(P) = \xi \in H^2(X, \mathbb{Z}_2)$
- $Z^{\xi}_{KM} = \sum \frac{p^l s^r}{l!\, r!} P_D(\rho^l \Sigma^r)$

$$= \sinh\left(\tfrac{1}{2}s^2 + 2p\right) \qquad X = K3.$$

$\boxed{10:38:50 \quad 27M}$

Generalizations: Other $N=2$ Field Theories

~~[illegible]~~ SYM w/ G VM

HM: $\quad \mathbb{W}$ $\mathbb{H}$-repn of G.

$$\mathbb{W} = R \oplus \bar{R} \qquad R \text{ cplx of } G.$$

$$q \oplus \tilde{q}$$

$$(q, \tilde{q}^*) \qquad \underline{SU(2)_R \text{ doublet}}$$

Top Tw: $\quad M = q \oplus \tilde{q}^* \in \Gamma(S^+ \otimes \mathcal{R})$

assoc. to P via R

<u>Exple:</u> $G = U(1)$ $\quad R \cong \mathbb{C}$ charge 1 : Spin-c.

<u>Before twisting</u> Eliminate auxiliary $D \Longrightarrow$

$$D(M) = \mathbb{H}\text{-momentmap} \in \mathfrak{g}^\vee \otimes \mathbb{R}^3 \cong \mathfrak{g} \otimes \mathbb{R}^3$$

$T \in \mathfrak{g}$

$$T \cdot \mu_r = \langle q, Tq \rangle - \langle \tilde{q}, T\tilde{q} \rangle$$

$$T \cdot \mu_c = \tilde{q} \cdot Tq \qquad \bar{R} \cong R^\vee$$

top tw: $\quad \mu : \Gamma(S^+ \otimes \mathcal{R}) \longrightarrow \Gamma(\Lambda^+ \otimes \operatorname{ad} P)$

$$QX = 0 \iff F^+ = \mu(M)$$
$$Q(\text{gaugino}) = 0 \iff \not{D} M = 0.$$

gen. monopole eqs.

$G = U(1)$, $R \cong \mathbb{C}$ S-W eqs.

$$S_{VM+HM} = Q(\Psi_S) + \text{cnst} \int \mathrm{Tr}\, F \wedge F$$

Generalization? Yes, if QFT exists.

G must be semi-simple

G $\beta = -2h^\vee + C_2(R) \leq 0.$

$R = \mathfrak{g} \otimes \mathbb{C}$ $\beta = 0$

✓ $SU(N)$ Marcos Mariño Kronheimer

✓ $SU(2)$ $R = \underbrace{2 \oplus \cdots \oplus 2}_{N_f \leq 4}$

Feehan + Leness $N_f = 1$

- What good is it?

$t g_{\mu\nu}$ $t \to \infty$

$$Z_{DW}^{\xi}(p, S) = Z_{\cancel{LEEF}\ IR}^{\xi}(p, s) = \left\langle e^{p\, \mathcal{O}_{IR}(p) + s^\alpha \mathcal{O}_{IR}(\Sigma_\alpha) + \cdots} \right\rangle_{\cancel{LEEF}\ IR}$$

$\boxed{10:56}$ $\boxed{20M}$

Vacuum structure of

LEET: $N=2$ QFT on $\mathbb{R}^3 \times \mathbb{R}^1$

Now on $\mathbb{R}^4$ Vacua?

Classical Vacua $H = \sum$ squares

$\Rightarrow$ ϕ, M constants

$\phi \in \mathfrak{g}\otimes\mathbb{C}$, $M \in R\otimes 2$

$$\mathcal{M}^{class} = \left\{ (\phi, M) \;\middle|\; \begin{array}{c} [\phi,\phi^*] = 0 \\ \mu(M) = 0 \\ \phi\cdot q = 0 \;\&\; \phi^*\cdot\tilde{q}^* = 0 \end{array} \right\} / G$$

ϕ is s.s. $\Rightarrow$ $\phi \in \mathfrak{t}\otimes\mathbb{C}$

$$G \xrightarrow[SSB]{} \mathrm{Stab}(\phi)$$

$$\underbrace{(\mathfrak{t}\otimes\mathbb{C} - \Delta)/W}_{\mathcal{M}^{class.}_{Coul.}} \amalg \underbrace{\cdots\cdots}_{\text{Hybrid}} \amalg \underbrace{W /\!/\!/ G}_{\mathcal{M}^{class}_{Higgs}}$$

$\mathcal{M}^{class.}_{Coul.}$: $M=0$.

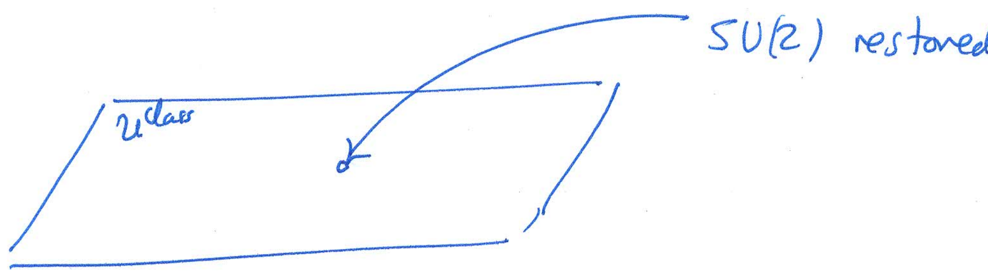
$\mathcal{M}^{class}_{Higgs}$: $\phi=0$, $M\neq 0$.

LEET $\underline{\underline{T-VM}}$

explain "Coulomb" & "Higgs"

$G = SU(2)$ no HM.

$$\phi = \begin{pmatrix} a & \\ & -a \end{pmatrix} \qquad u^{class} \sim \mathrm{Tr}\, \phi^2 = 2a^2$$



$$m(W^\pm) \sim |a|$$

Quantum Vacua: G simple

Expand around classical vacuum at Weak coupling

$$\tau_0 = \frac{4\pi i}{g_0^2} + \frac{\theta_0}{2\pi}$$

$$\cancel{\Lambda^{2h^\vee} = \Lambda_0 \, e^{2\pi i \tau_0}}$$

$$T(\Lambda) \qquad \Lambda^{2h^\vee} = \Lambda_0^{2h^\vee} e^{2\pi i \tau_0} \qquad \begin{array}{l} \Lambda_0 \to \infty \\ g_0 \to 0 \end{array}$$

$N=2$ susy @ weak coupling Vac. Deg. Not Lifted!

LEET VM's in T

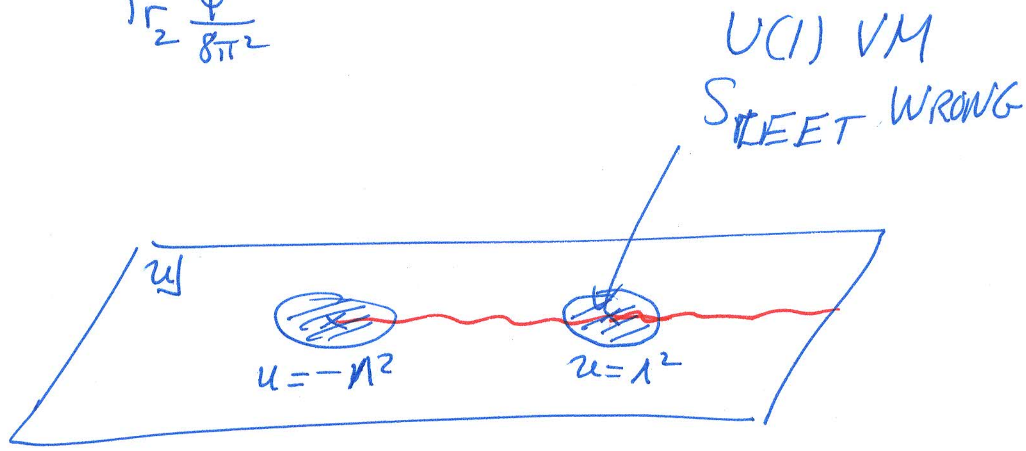
Classify $N=2$ susy actions: No Spont $\cancel{SUSY}$

$G = SU(2)$ $\quad |u| \gg |\Lambda^2|$

$$2u = \langle \Omega(u) | \underset{\underset{\mathrm{Tr}_2 \frac{\phi^2}{8\pi^2}}{\shortparallel}}{\mathcal{O}} | \Omega(u) \rangle$$

SW 1994

$\mathcal{M}^{\text{Quant}}_{\text{Coul.}}$



LEET breaks down at $u = \pm\Lambda^2$

<u>U(1) VM</u> : $a, A, D, \eta, \chi, \psi$ $\mathbb{R}^4$

$\uparrow$

Scalar: $\mathbb{R}^4 \longrightarrow \mathcal{M}^{\text{Quant}}_{\text{Coul.}}$

keep!

$$S_{LEET} \overset{\text{Gen Thms of SUSY}}{=} \int i\left(\overline{\tau(a)}(F^+)^2 + \tau(a)(F^-)^2\right)$$

$$+ \mathrm{Im}\,\tau \; da * da + \mathrm{Im}\,\tau \; D * D$$

$$+ \tau \psi * d\eta + \bar{\tau}\, \eta\, d * \psi + \tau \psi d \chi - \bar{\tau} \chi d \psi$$

$$+ i \frac{d\bar{\tau}}{d\bar{a}} \eta \chi (F^+ + D) + \cdots$$

$\tau(a)$ arbitrary holomorphic.

Gen. Thm of $\overset{N=2}{\text{SUSY}}$ VM's Γ

$$\mathcal{U} \longrightarrow \mathcal{M}^{qu}_{Coul.}$$

$$\Gamma \longrightarrow \mathcal{M}_{Coul} \qquad \Gamma_u = H_1(\mathcal{U}_u; \mathbb{Z})$$

lattice of elect + mag. charges

N=2 central charge $\quad Z : \Gamma \longrightarrow \mathbb{C}$

<u>SW94</u> $\quad G = SU(2)$

$$E_u : \qquad y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2)$$

$$\Gamma_u = H_1(E_u; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\gamma \longmapsto \oint_\gamma \lambda_{SW} \; ; \; \lambda_{SW} = \frac{dx}{y}(x-u)$$

@ $u=\infty$ $\quad M_\infty \qquad \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} \overset{\in \Gamma^0(4)}{Z}$

$\Longrightarrow$ almost monodromy inv't cycle A-cycle.

$$a = \oint_A \lambda_{SW}$$

$$\Gamma_u = \Gamma_u^{el} \oplus \Gamma_u^{mag} \qquad \text{"duality frame"}$$

Abelian S-duality

$\exists$ B cycle $\quad a_D = \oint_B \lambda_{SW} \longrightarrow 0 \quad u = \Lambda^2$

$\quad a + a_D \longrightarrow 0 \quad u = -\Lambda^2$

$$u = \frac{1}{2}\frac{\vartheta_2^4 + \vartheta_3^4}{(\vartheta_2\vartheta_3)^2} = \frac{1}{8q^{1/4}}\left(1 + 20q^{1/2} + \cdots\right)$$

$$a(u) = \frac{1}{6}\frac{2E_2(\tau) + \vartheta_2^4 + \vartheta_3^4}{\vartheta_2\vartheta_3}$$

$\boxed{11{:}19 \mid 25M}$

$E_u \quad y^2 = x^2(x-u) + \frac{\Lambda^4}{4}x \quad M \in \Gamma^0(4)$

$u = \pm\Lambda^2$

u-plane $\sim$ modular curve

τ