SLAC–PUB–4463
IASSNS-HEP-87/61
November 1987
(T)

SOME GLOBAL ISSUES IN STRING PERTURBATION THEORY*

JOSEPH J. ATICK AND GREGORY MOORE

Institute for Advanced Study Princeton, New Jersey O8540, USA

and

ASHOKE SEN†

Stanford Linear Accelerator Center,
Stanford University, Stanford, California 94305, USA

ABSTRACT

Calculations of the type II string vacuum amplitude using the picture changing prescription have been shown to lead, in general, to a positive cosmological constant. We show that there is a global obstruction to the choices of gauge slice for superteichmuller space that lead to such measures. We discuss the general restrictions on gauge slices appropriate for use in explicit fermionic string calculations. We also discuss the relation of the functional determinant and conformal field theory versions of the path integral measure, and show that, at arbitrary genus and in arbitrary backgrounds preserving tree level $N = 1$ supersymmetry, the measure is an exact differential. We evaluate the boundary integrals of this total derivative at genus two in two ways for target space R^{10} to show that the integrals are zero. Finally, we use the factorization hypothesis to show that in appropriate compactified spacetimes the boundary integrals continue to vanish.

Submitted to Nuclear Physics B

* Work supported by the Department of Energy, Contract Nos. DE–AC02–76ER–02220 (IAS) and DE–AC03–76SF00515 (SLAC).

† Address after March 1, 1988: Theoretical Physics Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India.

1. Introduction

There has recently been some confusion in the subject of superstring perturbation theory. Three issues in particular have caused trouble. First, there are several approaches to writing down the supermeasure on supermoduli space. There is the approach through gauge fixed path integrals, the approach through conformal field theory and picture changing operators, and the approach through algebraic supergeometry [1][2][3][4][5][6][7][8][9]. While very *similar*, the equivalence of the resulting measures is not obvious. Second, the correct statement of holomorphic factorization has been in question. Third, recent calculations [10][11] have indicated that there are many subltleties involved in applying the picture changing formalism of[1][12][6] to the calculation of higher loop superstring amplitudes. Ambiguities involved in the choice of location for picture changing operators were discovered in [6] and interpreted in [10] to be due to an intrinsic ambiguity in defining integration over the variables of the grassman algebra [13]. A prescription for handling the ambiguities inherent in the choice of location of picture changing operators was given in [10] for the genus two case. In [11] it was shown that for a (standard) choice of gauge slice the measure for the two loop vacuum amplitude of the type II string is positive definite and fails to be modular invariant. The relevant choices of gauge slice are those which lead to a measure factorized in the contributions of left- and right- movers where the superconformal ghost correlator is computed only in terms of picture changing operators. One of the purposes of this paper is to resolve some of the problems pointed out in [11].

The heart of the matter is the validity of the choice of gauge slice for supermoduli space. In section two we discuss the issues involved in choosing a gauge slice. In particular, we note (in subsections C and F) that certain standard results from teichmuller theory imply that it is impossible to choose a gauge slice for which gravitinos have δ-function support and the measure is manifestly positive definite. Furthermore we analyze the specific choice made in calculations on hyperelliptic curves and show that the gauge slices used in the past cannot be everywhere transverse to the gauge action. Finally, we note that, in contrast to the bosonic string, in superstring theory modular invariance puts constraints on allowed gauge slices. The choice made in hyperelliptic calculations [14][15][16][17][11] does not satisfy these conditions.

In the remaining sections we show that a good choice of gauge in fact leads to a vanishing two-loop cosmological constant for the type II string (in appropriate backgrounds) as is physically reasonable. In section three we discuss the appropriate formula for the measure, and in particular the relation between the formulae derived from the functional determinant and conformal field theory approaches. In the course of this treatment we give a component version proof of the holomorphic factorization of ghost determinants, confirming the calculations of [7][5] for this case. We do not address the more subtle question of the matter superdeterminants in any detail. In section four we show, using the formalism of conformal field theory, that the type II measure is a total derivative on moduli space, to all orders of perturbation theory and in arbitrary backgrounds preserving tree level $N = 1$ supersymmetry. In section five we evaluate the boundary integrals for the case of genus two. In the first two subsections we consider the case of flat space. We first use the factorization hypothesis and show that no operators of the relevant dimension and ghost charge can be exchanged. We then confirm the arguments with an explicit calculation in terms of theta functions. In subsection C we again use the factorization hypothesis to show that the boundary integrals vanish for spacetimes preserving tree level $N = 1$ supersymmetry[1] except for theories which could develop Fayet-Iliopoulos D-terms at one loop [29][30]. For these backgrounds we find that the two loop cosmological constant is proportional to the square of the Fayet-Iliopoulos D-terms (if any) induced at one loop. Several technical points are treated in the appendices. Appendix A reviews elements of teichmuller theory relevant to the choice of slice and appendix B contains proofs of some assertions needed in section two. Appendix C gives an understanding of why concentrating

1 Examples of vacuua preserving $N = 1$ tree level supersymmetry in type II string theory can be found in [18][19][20][21][22][23][24][25][26][27][28]

gravitino support at the Weierstrass points leads to a point-by-point vanishing measure. The remaining appendices contain technical details on manipulation of the formula for the superstring measure.

2. Global issues in the choice of slice

A. *Supermoduli space*

We begin with a brief review of the differential-geometric approach to supermoduli space [31][32][2][3][4][5][33][34][35]. Supermoduli space can be thought of in algebro-geometric terms as the moduli space of super riemann surfaces [36][37][38] or in terms of teichmuller space for deformations of superfuchsian groups [39][40]. We mostly use the superdifferential geometry approach, which emulates the fiber bundle approach to teichmuller theory [41] since this is most closely connected with the gauge-fixed path integral. Thus, instead of constructing a universal family of superriemann surfaces, from which all other families are obtained by pullback, we consider a fixed supermanifold and the space of certain structures on that supermanifold. We then define an equivalence relation on these structures and define supermoduli space to be the set of resulting equivalence classes. The appropriate structures in this case are frames ${E_M}^A$ in WZ gauge satisfying the torsion constraints of two-dimensional supergravity [31] and the equivalence relation is just equivalence by superlorentz superweyl and superdiffeomorphism transformations.

More precisely, consider a C^∞ real 2-surface Σ. We choose

a.) an open covering $\{U_\alpha\}$,

b.) a complex structure with holomorphic cotangent bundle K_0

c.) Two spin bundles, $K_0^{\frac{1}{2}}$ and $\hat{\overline{K}}_0^{\frac{1}{2}}$

d.) Nonvanishing C^∞ sections

$$\theta_\alpha \in \Gamma(U_\alpha, K_0^{\frac{1}{2}})$$
$$\overline{\theta}_\alpha \in \Gamma(U_\alpha, \hat{\overline{K}}_0^{\frac{1}{2}})$$

such that $(\theta_\alpha^2)^* = \overline{\theta}_\alpha^2$.

Note that $e_\alpha = \theta_\alpha^2$ and its conjugate defines a frame, which specifies a metric, which induces the complex structure K_0. Of course, there are no global sections of $K_0^{\frac{1}{2}}$, but we may choose θ_α so that

$$\begin{aligned} \theta_\alpha(p) &= e^{i\varphi_{\alpha\beta}(p)}\theta_\beta(p) \\ \overline{\theta}_\alpha(p) &= e^{-i\psi_{\alpha\beta}(p)}\overline{\theta}_\beta(p) \end{aligned} \tag{2.1}$$

for $p \in U_\alpha \cap U_\beta$ with $e^{2i\varphi} = e^{2i\psi}$ defining the transition functions of the complexification of the tangent space $T\Sigma \otimes C$. Having made these choices we form the supermanifold $\hat{\Sigma} = (\Sigma, \mathcal{A})$ where $\mathcal{A}$ is the sheaf of algebras $\mathcal{A}(U_\alpha) = \mathcal{O}(U_\alpha) \otimes \Lambda^*(\theta_\alpha, \overline{\theta}_\alpha)$ [42]. Here $\mathcal{O}$ is the structure sheaf of the reduced manifold (i.e. to each open set it assigns the abelian group of C^∞ functions on that set) and Λ^* is the exterior algebra with the indicated generators. A frame is a basis for the sheaf of derivations of $\mathcal{A}$ as an $\mathcal{A}$-module. We will consider frames which transform diagonally across patches, and hence may be specified in terms of tangent space indices $A = z, \bar{z}, +, -$. Objects with an upper $+$ index transform as $e^{i\varphi}$ etc. A frame is denoted by E_A and its dual by E^A. To obtain an interesting geometry one imposes the torsion constraints [31]. As explained in [33] some of the torsion constraints are conveniences, some ensure that the frame defines a complex structure and some ensure that the frame defines a superconformal structure. Thus a frame satisfying the torsion constraints defines a superriemann surface.

One can remove certain trivial degrees of freedom in the superdiffeomorphism and superweyl groups to specify that the frame be in WZ gauge [2]. WZ gauge is really a partial fixing of allowed coordinate systems. Howe shows that a frame in WZ gauge is uniquely specified by the θ-independent parts, $e, \overline{\chi}$, of E^z, E^+. Since we will consider families of structures on a fixed supermanifold, and not families of supermanifolds this prescription is unambiguous. It follows that there is a one-one correspondence between frames in WZ

[2] Our definition of WZ gauge is related to Howe's definition of WZ gauge by using superweyl transformations to set his auxiliary field A and the spin-$\frac{1}{2}$ components of the gravitino to zero.

gauge satisfying the torsion constraints and pairs $(e, \overline{\chi})$, where $e = e^z$ is a C^∞ section of K_0 and $\overline{\chi}$ is a C^∞ section of the C^* bundle $K_0^{-\frac{1}{2}} \otimes \overline{K_0}$. Specifically, referring $\overline{\chi}$ to the orthonormal frame e, θ, so that $\overline{\chi} = \overline{\chi}_{\bar{z}}^+$ we have [31][34]

$$\begin{aligned} E^z &= e - d\theta \cdot \theta + \bar{e}\theta\overline{\chi} \\ E^+ &= d\theta + \tfrac{1}{2}e(i\theta\omega + \overline{\theta}\theta\overline{D}\overline{\chi}) + \tfrac{1}{2}\bar{e}(i\theta\overline{\omega} + \overline{\chi}) \end{aligned} \tag{2.2}$$

where ω is the usual spin connection. $E^{\bar{z}}, E^-$ are obtained formally by conjugation. Since E^- has a different spin structure from E^+ this conjugation is only formal. Note that we obtain a family of frames by varying $e, \overline{\chi}$, holding the transition functions implied by the frame indices A, and the meaning of θ fixed. We could further specify complex coordinates for the reduced manifold $u, \bar{u}$, which are compatible with the complex structure at the basepoint K_0. Hence, (2.2) is just the supersymmetric analog of the representation of all frames on a manifold by $e^z = du e_u{}^z + d\bar{u} e_{\bar{u}}{}^z = e^\phi(du + \mu d\bar{u})$, where the frame indices $z, \bar{z}$ and coordinates $u, \bar{u}$ are held fixed and the complex structure varies with μ, the beltrami differential.

The set of pairs $\mathcal{F} = \{(e, \overline{\chi})\}$ is an infinite dimensional manifold. The action of local weyl and lorentz symmetries can be combined into a set of maps $\{f_\alpha : U_\alpha \to C^*\}$ which agree on overlaps, and form a group $\mathcal{C}$, acting by

$$\begin{aligned} e &\to fe \\ \overline{\chi} &\to \overline{f}^{-1} f^{\frac{1}{2}} \overline{\chi} \end{aligned} \tag{2.3}$$

In passing to WZ gauge, the θ-dependent part of superdiffeomorphism symmetries have been fixed [31]. What remains are ordinary diffeomorphisms acting by pullback and supergravity transformations, specified by an anticommuting $(-\frac{1}{2}, 0)$ form ϵ, and acting by

$$\begin{aligned} \delta\partial &= 0 \\ \delta\overline{\partial} &= \epsilon\overline{\chi}\partial \\ \delta\overline{\chi} &= 2\overline{\partial}\epsilon \\ \delta\chi &= 0 \end{aligned} \tag{2.4}$$

Denoting by $\mathcal{D}$ the set of superdiffeomorphisms, we find that the commutators of the infinitesimal actions of $\mathcal{C} \times \mathcal{D}$ close and exponentiate to form an infinite-dimensional supergroup.

We define supermoduli space $\mathcal{M}$ in terms of frames in WZ gauge as the set of equivalence classes of frames $(e, \overline{\chi}) \in \mathcal{F}$ under the action of the symmetries $\mathcal{C} \times \mathcal{D}$. This is the space of interest for computation of the Polyakov path integral. It has not been completely proven that it is the moduli space of super riemann surfaces in the sense of algebraic geometry, although some steps in this direction have been taken in [33].

Similarly, superteichmuller space $\mathcal{T}$ is obtained by dividing just by the diffeomorphisms connected to the identity $\mathcal{D}_0$. The set $\mathcal{F}/\mathcal{C}$ can be thought of as the space of sections of a homogeneous manifold, analogous to $GL(2,R)/C^*$ in the bosonic case, and can probably be topologized in such a way that the theory of [41] can be repeated in this setting. We have not done this, but will assume it can be done and proceed. From the index theorem applied to the operator $\overline{\partial}$ acting on vector fields and $(-1/2,0)$ forms we see that supermoduli space is a real superspace of dimension $(6g-6|4g-4)$. In fact, it is a complex superspace [40][5][34] of dimension $(3g-3|2g-2)$. This completes our review.

B. *Slices and total derivative ambiguities*

Amplitudes for the superstring are integrals of volume forms over supermoduli spaces (perhaps with punctures). In contrast to the bosonic string, superstring densities are made from a cotangent space which has an even and an odd part. Existing computations of superstring amplitudes typically begin by integrating out first the odd moduli to obtain a density on ordinary moduli space. The final integrand suffers from an ambiguity because it changes by a total derivative in the moduli under a change in the choice of slice [6]. The ambiguity has its origin [10] in an intrinsic ambiguity in defining integration over elements of a grassman algebra [13] and may be illustrated by the following simple example. Let us consider the integral

$$\int dx d\theta d\phi F(x,\theta,\phi) \tag{2.5}$$

where x is an even element of the grassman algebra, and θ, ϕ are the odd elements. In order to define the integral, we need to express x as,

$$x = y + z\theta\phi \tag{2.6}$$

where y and z are real numbers, and choose a contour in the (y,z) plane:

$$z = h(y) \tag{2.7}$$

where h is a single-valued function of y. The integral (2.5) may then be written as,

$$\int d(y + h(y)\theta\phi) d\theta d\phi F(y + h(y)\theta\phi, \theta, \phi) \tag{2.8}$$

It is not difficult to see that if we take a different function $h'(y)$, the difference is a total derivative in y. The point to note here is that although $y + z\theta\phi$ and $y + z'\theta\phi$ are two different points in the x plane (for $z \neq z'$), we do not include both these points in the domain of integration. Instead, for each y, we choose one and only one value of z given by $z = h(y)$.

Let us now consider the case where the x space is taken to be compact. More specifically, let us take two points x and $\tilde{x}$ to be equivalent if $\tilde{x} = x + 1$. This, of course, makes sense if the function F to be integrated is invariant under such a transformation. At this point, note that in order for the integral (2.8) to be well defined (i.e., in order that the value of the integral does not depend on which fundamental domain in y space we choose), the function $h(y)$ must satisfy,

$$h(y+1) = h(y) \tag{2.9}$$

Furthermore, notice that given two such functions $h(y)$ and $h'(y)$, both satisfying (2.9), the difference in contribution to (2.8) will be a total derivative, and after integration will give equal and opposite boundary contribution at the two ends of the y integral. Thus the value of the integral is independent of the choice of $h(y)$ as long as it satisfies (2.9).

Let us now return to the case where the variables of integration are the graviton and the gravitino, defining some even and odd variables of a grassman algebra. There are two different approaches that can be pursued for computing the superstring vacuum diagram. In the first approach, one treats the even and odd coordinates of the supermoduli space on an equal footing, and expresses the superstring partition function as an integral over the supermoduli space with a certain measure. The computation of the measure, in turn, requires us to choose a gauge fixing slice. In this scheme, a *slice* is just a map from a region $R \subset C^{3g-3|2g-2}$ to frame space: $f : R \to \mathcal{F}$. It is useful to keep the following picture in mind:

$$\begin{array}{ccc} & & \mathcal{F} \\ & \nearrow & \\ R & & \downarrow \pi \\ & \searrow & \\ & & \mathcal{T} \end{array} \tag{2.10}$$

A *good slice* is one for which $\pi \circ f : R \to \mathcal{T}$ is one to one and onto a region in $\mathcal{T}$. It is often useful to distinguish between local and global properties on $\mathcal{T}$, so we reserve the term global slice for $g = \pi \circ f$ which are *onto* $\mathcal{T}$. For a good slice, the image of f is transverse to the action of the gauge group and hence defines a local cross-section of the fibration $\mathcal{F} \to \mathcal{T}$. Conversely, if we are given a one-one map $g : R \to \mathcal{T}$ and if s is a cross-section, then $s \circ g$ is a good slice. Finally, the space $\mathcal{F}/C$ can be identified with the space of pairs $\{(\mu, \chi)\}$, which has a natural complex structure, so we can speak of a ***holomorphic slice***. This is a holomorphic map $C^{3g-3|2g-2} \to \mathcal{F}$. Superteichmuller space also has a complex structure, and the map π is holomorphic, so if f is a holomorphic slice then $g = \pi \circ f : C^{3g-3|2g-2} \to \mathcal{T}$ is holomorphic. Holomorphic slices are useful in the computation of the string amplitudes because for such slices the Fadeev-Popov determinant factorizes into a contribution for the left and right movers, which is important to holomorphic factorization. From the well-known theorems that T is Stein and topologically trivial, [43] together with Grauert's theorem that holomorphic bundles on topologically trivial Stein manifolds are holomorphically trivial [44] , it follows from the Bers embedding that $\mathcal{T}$ is isomorphic to a

region $R \subset C^{3g-3|2g-2}$. This does *not* mean that there exists a good global holomorphic slice. In fact, already in the case of ordinary teichmuller space, a theorem of Earle [45] asserts that there is no global holomorphic section of the space of conformal structures over teichmuller space [3]. Thus we cannot work with a global holomorphic slice. This, however, does not prevent us from choosing slices which are holomorphic on local coordinate patches in R.

In this way of doing the computation, we have not introduced any ambiguity in defining the integration measure so far. But at this point, in carrying out the integration over the elements of the grassman algebra, we have to invoke eq.(2.8) for defining integration over the even elements. As a result we get a total derivative ambiguity, due to the ambiguity in choosing the function $h(y)$.

In the second way (which has proved to be of more practical use so far, and which is the method we shall use in later sections of this paper) one proceeds as before, but carries out integration over the odd variables of the grassman algebra before attempting to calculate an expression for the measure in the supermoduli space. As should be clear from the example quoted above, one must choose the analog of the section $h(y)$ in eq.(2.7) before carrying out the integration over the odd variables. In this case the role of the variable y is played by the $6g-6$ real ($3g-3$ complex) coordinates t^r of the moduli space of an ordinary genus g surface, whereas the role of θ, ϕ is played by $4g-4$ odd elements of the grassman algebra $C^{0|2g-2}$. Locally, the space spanned by (t^r, ς^a) is isomorphic to a subset of a ***vector space***: $\mathcal{R} \subset C^{3g-3|0} \oplus C^{0|2g-2}$. In order to carry out the integral, we need a map to a supermanifold $h : \mathcal{R} \to R \subset C^{3g-3|2g-2}$. The map h is analogous to the map $(y, \theta, \phi) \in R^{1|0} \oplus R^{0|2} \to (y + h(y)\theta\phi, \theta, \phi) \in R^{1|2}$ that we needed in the previous example. The situation is best explained by extending the diagram (2.10) as follows:

$$\begin{array}{ccccc} & & & & \mathcal{F} \\ & h & & \nearrow & \\ \mathcal{R} & \to & R & & \downarrow \pi \\ & & & \searrow & \\ & & & & \mathcal{T} \end{array} \tag{2.11}$$

[3] For completeness we sketch the idea of the proof of Earle's theorem in appendix A

Note that we must distinguish between the spaces $\mathcal{R}$ and R. The even elements of $\mathcal{R}$ have no even nilpotent part, and are analogs of the variable (y, θ, ϕ) in the previous example. On the other hand elements of R have even nilpotent parts, and are the analogs of the variables (x, θ, ϕ) in the previous example. If $f : R \to \mathcal{F}$ is the map defined previously, then we get a new map $\tau : \mathcal{R} \to \mathcal{F}$ from (2.11) as $\tau = h \circ f$. Thus given any point $(t^r, \varsigma^a) \in \mathcal{R}$, the mapping τ gives a specific configuration of the zweibein $e(\vec{u}, t^r, \varsigma^a)$ and gravitino $\chi(\vec{u}, t^r, \varsigma^a)$, $\vec{u}$ denoting coordinates on the genus g surface in some fixed coordinate system. In the present formalism, it is the map τ (i.e. specifying $e(\vec{u}, t^r, \varsigma^a)$, $\chi(\vec{u}, t^r, \varsigma^a)$ for every point on $\mathcal{R}$) that is defined to be a *gauge slice*. τ thus contains information about h as well as f, in fact for a given τ, both f and h are uniquely determined using the local isomorphism g between R and $\mathcal{T}$.

It is then clear that choosing the slice (e, χ) not only amounts to choosing a specific point on a fiber containing gauge equivalent configurations, but also a specific choice of contour out of many gauge non-equivalent contours, through the map h. It is thus hardly surprising that when we make a different choice of slice τ, in general we have made a different choice of the map h, and the final integrand changes by a total derivative as a consequence. By the same token, since the moduli space is obtained from the teichmuller space by identifying various points related to each other by the action of the modular group, the choice of the slice τ must satisfy constraints analogous to the constraint (2.9). In particular, if t and $\tilde{t}$ are two points in the teichmuller space related to each other by a modular transformation, and (e, χ), $(\tilde{e}, \tilde{\chi})$ are the corresponding points on the slice above $(t, \tilde{t})$, then $(\tilde{e}, \tilde{\chi})$ must be related to (e, χ) by a combination of (global) diffeomorphism and supergravity transformations. For given (e, χ) this means that if we fix $\tilde{\chi}$ and the reduced part of $\tilde{e}$, the even nilpotent part of $\tilde{e}$ is completely determined. (In this form the constraint is manifestly of the form of eq.(2.9) for when the reduced part of x is one then the even nilpotent part of x is determined in terms of θ, ϕ, and $h(1)$, which, in turn, is determined by $h(0)$, i.e. the the value of x at $y = 0$.) On the other hand, if we fix both,

the reduced and the nilpotent parts of $\tilde{e}$, then $\tilde{\chi}$ is determined by this constraint in terms of $\tilde{e}$, e and χ. We shall discuss the implication of these constraints for a special class of slices in subsection D.

Finally, note that even if the map f gives a holomorphic slice on local coordinate patches in R, the map $\tau = h \circ f$ may not be holomorphic if h is not holomorphic. In fact, as we shall show in subsection G, we cannot find a map τ which is holomorphic in each local coordinate patch in $\mathcal{R}$, and can be patched together globally in a way that the string integrand is well-defined (i.e. takes the same value in the two coordinate patches) in all the overlap regions. Even if we do not work with a holomorphic slice, it is still useful to use complex coordinates on supermoduli space since these help distinguish the contributions of left and right- movers to the measure.

C. *Derivatives of slices*

In order to carry out computation with a given gauge slice, or to determine if a given slice is a good slice, we need to define tangents to the space of slices. Choose coordinates $(\hat{t}^r, \varsigma^a)$ with $r = 1, \ldots, 6g-6$, $a = 1, \ldots, 4g-4$ for R and a slice $f : R \to \mathcal{F}$ with $f(\hat{t}, \varsigma) = (e(\hat{t}, \varsigma), \overline{\chi}(\hat{t}, \varsigma))$. (Note that we have used the variables $\hat{t}^r$ to indicate that they denote even elements of the grassman algebra, rather than real numbers.) Then the tangents are defined as,

$$\begin{aligned}
\hat{\eta}_{r\bar{z}}{}^{z} &= (e^m_{\bar{z}} \frac{\partial}{\partial \hat{t}^r} e^z_m) \\
\hat{\eta}_{a\bar{z}}{}^{z} &= (e^m_{\bar{z}} \frac{\partial}{\partial \varsigma^a} e^z_m) \\
\hat{\eta}_{r\bar{z}}{}^{+} &= \frac{\partial}{\partial \hat{t}^r} \overline{\chi} \\
\hat{\eta}_{a\bar{z}}{}^{+} &= \frac{\partial}{\partial \varsigma^a} \overline{\chi} \\
\hat{\eta}_{rz}{}^{-} &= \frac{\partial}{\partial \hat{t}^r} \chi \\
\hat{\eta}_{az}{}^{-} &= \frac{\partial}{\partial \varsigma^a} \chi
\end{aligned} \tag{2.12}$$

where $e^m \equiv (e^m_z, e^m_{\bar z})$ is the inverse of the matrix $(e^z_m, e^{\bar z}_m)$. If (t^r, ς^a) denote coordinates for $\mathcal{R}$, then given the map $\tau : \mathcal{R} \to \mathcal{F}$ we may define the tangents to $\mathcal{R}$ as

$$\begin{aligned}
\eta_{r\bar z}{}^{z} &= (e^m_{\bar z} \frac{\partial}{\partial t^r} e^z_m) \\
\eta_{a\bar z}{}^{z} &= (e^m_{\bar z} \frac{\partial}{\partial \varsigma^a} e^z_m) \\
\eta_{r\bar z}{}^{+} &= \frac{\partial}{\partial t^r} \overline{\chi} \\
\eta_{a\bar z}{}^{+} &= \frac{\partial}{\partial \varsigma^a} \overline{\chi} \\
\eta_{rz}{}^{-} &= \frac{\partial}{\partial t^r} \chi \\
\eta_{az}{}^{-} &= \frac{\partial}{\partial \varsigma^a} \chi
\end{aligned} \tag{2.13}$$

which may be obtained from the $\hat\eta$'s using the chain rule of differentiation using the map $h : \mathcal{R} \to R$.

In order to give proper meaning to eq.(2.13) we must first define what is meant by differentiation of a slice with respect to moduli. If we are given a family of frames with fixed transition functions, as in the family (2.2) then we simply differentiate patchwise. However, sometimes we may be given a family of frames in which the transition functions implied by the tangent space indices also vary. We will need to discuss how to define the derivative of frames in that case.

The definition of differentiation relies on the fact that differentiable C^* bundles and $U(1)$ bundles are classified up to isomorphism by their first Chern class. Using the isomorphisms between $K^{1/2}$ and $K_0^{1/2}$ we may consider e, χ to be families of sections of a fixed bundle, which may be differentiated in an obvious way. We now recall the proof of some standard facts which make this possible [46]. We will denote by $\mathcal{A}_{C^*}, \mathcal{A}_R, \ldots$ the sheaf of differentiable $C^*, R\ldots$-valued functions. Isomorphism classes of differentiable C^* bundles may be thought of as elements of $H^1(\mathcal{A}_{C^*})$. From the exponential sequence

$$0 \to Z \to \mathcal{A}_C \to \mathcal{A}_{C^*} \to 0 \tag{2.14}$$

and the fact that $H^q(\mathcal{A}_C) = 0$ for $q > 0$, [4]we learn from the long exact sequence that

$$H^1(\mathcal{A}_{C^*}) \cong H^2(Z) \cong Z \tag{2.15}$$

where the connecting homomorphism is given by the Chern class. Similarly, from the map $f \to f/|f|$ we obtain

$$0 \to \mathcal{A}_R \to \mathcal{A}_{C^*} \to \mathcal{A}_{U(1)} \to 0 \tag{2.16}$$

giving $H^1(\mathcal{A}_{C^*}) \cong H^1(\mathcal{A}_{U(1)})$. Thus, we may speak interchangeably of C^* and $U(1)$ bundles.

In concrete terms what we have shown is the following. Suppose we have a family of riemann surfaces over a small open set $S \subset T$. We can choose a fixed covering U_α of the *topological* surface, together with transition functions $\{g_{\alpha\beta}(t)\}$ for K_t, $t \in S$. From the above considerations we see that for $t \in S$ near some point $t_0 \in S$, there exist C^∞ C^*-valued functions $f_\alpha(t)$ defined on each patch such that

$$f_\alpha(t)/f_\beta(t) = g_{\alpha\beta}(t)/g_{\alpha\beta}(t_0) \tag{2.17}$$

Using the trivialization $\{f_\alpha(t)\}$ we may consider a section e_α of K_t as a section $\{f_\alpha^{-1} e_\alpha\}$ of the fixed bundle K_{t_0}. Thus, to differentiate a family of frames, we refer them to a standard bundle and differentiate patchwise.

Since we must also differentiate the gravitino with respect to the moduli we must actually find a set of C^∞ trivializations $\{\kappa_\alpha(t)\}$ of $K_t^{1/2}$. [5]If we adopt the above definition

[4] One may prove this using a partition of unity. Technically, $\mathcal{A}_C$ is a fine sheaf [46].
[5] The existence of such trivializations does not contradict the fact that there are 2^{2g} inequivalent spin bundles. The transition functions of a spin structure $k_{\alpha\beta}$ are $U(1)$-valued functions such that $k^2_{\alpha\beta}$ are the transition functions of the canonical bundle. A trivialization of the difference k_1/k_2 of two spin structures requires $U(1)$- valued functions f_α such that $f_\alpha/f_\beta = \pm 1$ on the overlaps. By contrast, if t_1 and t_0 are related by modular transformations then $\kappa_\alpha(t_1)/\kappa_\alpha(t_0)$ needn't be constant. On the contrary, if the spin structures are inequivalent, they cannot be constant.

of differentiation with respect to the moduli, then we must write the gravitino in terms of these trivializations; the section $\{\bar\chi_\alpha(p,t)\}$ at t of $K_t^{-1/2}\otimes\bar K_t$ is written as the section

$$\{\kappa^{-1}(t,p)\bar\kappa^2(t,p)\bar\chi(p,t)\} \tag{2.18}$$

of $K_{t_0}^{-1/2}\otimes\bar K_{t_0}$ This choice of slice has the difficulty that the whole construction is extremely noncanonical. There is a further apparent difficulty because there are points in T fixed by symplectic modular transformations $Sp(2g,Z)\subset\Gamma$. At such a point τ^* the squareroot $K^{1/2}$ is ambiguous. Although two choices of $K_{\tau^*}^{1/2}$ will be isomorphic as C^* bundles, the isomorphism must be nontrivial since they are inequivalent spin structures. Thus $\{\kappa(\tau^*,p)\}_\alpha$ are ill-defined, and this ambiguity will lead to discontinuities in the section (2.18). However, these discontinuities occur on manifolds of high codimension so they shouldn't matter [6]. In any case, as we shall see in subsection D, the $\kappa(t,p)$ appearing in eq.(2.18) drop out completely from our calculation, and hence the final result is free from any ambiguity that occurs in choosing $\kappa(t,p)$.

One fairly concrete realization of the above procedure is provided by the uniformization theorem. Recall that in the bosonic case every riemann surface Σ may be regarded as the quotient of the upper half plane U by a fuchsian group Γ. Furthermore, U has a unique complex structure. Thus, if two riemann surfaces Σ_0, Σ are represented by $\pi_0: U\to\Sigma_0$ and $\pi: U\to\Sigma$ we know that there is a quasiconformal map w such that

$$\begin{array}{ccc} U & \xrightarrow{w} & U \\ \pi_0\downarrow & & \pi\downarrow \\ \Sigma_0 & \rightarrow & \Sigma \end{array} \tag{2.19}$$

commutes. In terms of metrics g_0, g inducing the complex structures, if we arrange that $\pi_0^* g_0 = e^{\phi}|du|^2$ then $w^*\pi^* g = e^{\tilde\phi}|du+\mu d\bar u|^2$ where μ is a beltrami differential for Γ. In

[6] It is only for the hyperelliptic locus at $g=3$ that the discontinuities occur on a codimension one subvariety. In this case the nontrivial diffeomorphism is the hyperelliptic involution which does not change the homology basis and therefore fixes the spin structure.

this way we can map a family of riemann surfaces to a family of metrics on the fixed coordinate system provided by a fundamental domain for Γ in U. Also, in this way we see the correspondence between beltrami differentials and tangents to the gauge-fixing slice in the Polyakov path integral. Note that the choice of w is not unique since a metric and its diffeomorphic image define the same riemann surface. Hence, even in this more restricted and concrete setting provided by the uniformization theorem, the meaning of differentiation is highly noncanonical.

In the super case we will use the *old* uniformization theorem to pull back frames $\mathcal{F}$ to frames on the upper half plane: $e^{-\phi/2}\theta$ pulls back to a (not necessarily holomorphic) automorphic form for Γ of weight -1 and multiplier system specified by the spin structure. Thus, transition functions identifying the boundary of a fundamental region are of the form $z\to\frac{az+b}{bz+d}$ and $e^{-\phi/2}\theta\to(cz+d)^{-1}e^{-\phi/2}\theta$. This should not be confused with superfuchsian uniformization.

One advantage of this viewpoint is that we can use it to give a simple criterion for a good slice. This is done with the help of the simple and well-known

Lemma: *A necessary and sufficient condition for the slice $f(\hat t,\varsigma)$ to be a good slice is that for all $(\hat t,\varsigma)\in R$ the matrices:*

$$\begin{gathered}(\hat\eta_a(\hat t),\nu^b(\hat t)) \\ (\hat\eta_i(\hat t),\psi^j(\hat t))\end{gathered} \tag{2.20}$$

are invertible, where ψ^j,ν^b are nondegenerate bases for $H^0(K^2)$ and $H^0(K^{3/2})$, respectively.

Proof: We obtain a good slice iff for all $(\hat t,\hat\varsigma)$ the map $d(\pi\circ f): TR_{(\hat t,\hat\varsigma)}\to T\mathcal{T}_{g(\hat t,\hat\varsigma)}$ is invertible. Splitting tangent vectors into even and odd components we have

$$\begin{aligned} df\left(\frac{\partial}{\partial\hat t^r}\right) &= \hat\eta_{r\bar z}{}^{z}\oplus\hat\eta_{r\bar z}{}^{+} \\ df\left(\frac{\partial}{\partial\varsigma^a}\right) &= \hat\eta_{a\bar z}{}^{+}\oplus\hat\eta_{a\bar z}{}^{z}\end{aligned} \tag{2.21}$$

Furthermore, $(T\mathcal{T})^* \cong H^0(K^2) \oplus H^0(K^{3/2})$, and, given $v = v_0 \oplus v_1 \in \Gamma(K^{-1} \otimes \bar{K}) \oplus \Gamma(K^{-1/2} \otimes \bar{K})$ and $\psi = \psi_0 \oplus \psi_1 \in H^0(K^2) \oplus H^0(K^{\frac{3}{2}})$ we have the dual pairing

$$(v, \psi) = \int v_0 \psi_0 + \int v_1 \psi_1 \tag{2.22}$$

Thus, $g = \pi \circ f$ is invertible iff

$$Ber \begin{pmatrix} (\hat{\eta}_r, \psi^j) & (\hat{\eta}_a, \psi^j) \\ (\hat{\eta}_r, \nu^b) & (\hat{\eta}_a, \nu^b) \end{pmatrix} \neq 0 \tag{2.23}$$

which is true iff the matrices

$$\begin{aligned} &(\hat{\eta}_a(\hat{t}), \nu^b(\hat{t})) \\ &(\hat{\eta}_i(\hat{t}), \psi^j(\hat{t})) \end{aligned} \tag{2.24}$$

are invertible. ∎

Note that the matrices (2.24) are invertible if and only if they are invertible after reduction by nilpotents. Furthermore, the η's differ from the $\hat{\eta}$'s by nilpotent terms. Hence the criteria for τ to be a good slice is that the matrices

$$\begin{aligned} &(\eta_a(t), \nu^b(t)) \\ &(\eta_i(t), \psi^j(t)) \end{aligned} \tag{2.25}$$

are invertible.

The condition for a holomorphic slice is also conveniently phrased in this language. We use a set of holomorphic coordinates for $\mathcal{R}$, $(t^\imath, t^{\bar\imath}, \varsigma^\ell, \varsigma^{\bar\ell})$ where $\imath, \bar\imath$ runs from 1 to $3g-3$ and $\ell, \bar\ell$ runs from 1 to $2g-2$. When there is no need for distinction we write these indices as (r, a) to stand for $r = (\imath, \bar\imath)$ and $a = (\ell, \bar\ell)$. The condition for a holomorphic slice is

$$\eta_{\bar\imath \bar z}{}^z = \eta_{\bar\imath \bar z}{}^+ = \eta_{\bar\ell \bar z}{}^z = \eta_{\bar\ell \bar z}{}^+ = 0 \tag{2.26}$$

One of the most important consequences of the integration ambiguity is that the positivity properties of the superstring measure depend on the holomorphy of the slice. In the bosonic string the measure is positive definite. If we choose a holomorpic slice the measure is in fact a square-modulus of a holomorphic form (up to factors of $det Im\tau$) [47].

If we do not choose a holomorphic slice the measure is not naturally an absolute square, but the change of variables (on the cotangent space)

$$ds^\jmath = dt^\imath(\eta_{\imath \bar z}{}^z, \psi^\jmath_{zz}) + dt^{\bar\imath}(\eta_{\bar\imath \bar z}{}^z, \psi^\jmath_{zz}) \tag{2.27}$$

on the (analog of) $\mathcal{R}$ shows that the measure

$$\prod_\imath dt^\imath \wedge dt^{\bar\imath} det \begin{pmatrix} (\eta_{\imath \bar z}{}^z, \psi^\jmath_{zz}) & (\eta_{\bar\imath \bar z}{}^z, \psi^\jmath_{zz}) \\ (\eta_{\imath z}{}^{\bar z}, \psi^{\bar\jmath}_{\bar z \bar z}) & (\eta_{\bar\imath z}{}^{\bar z}, \psi^{\bar\jmath}_{\bar z \bar z}) \end{pmatrix} \tag{2.28}$$

which is not manifestly positive, is in fact just $\prod ds^\imath \wedge ds^{\bar\imath}$, which is manifestly positive.

In the superstring this argument fails because of the integration ambiguity. For a holomorphic slice the measure is still positive semidefinite in the supersymmetric case, as we will see in section III. However we cannot relax the holomorphy condition, for even if we choose a slice with

$$\eta_{\bar\imath \bar z}{}^+ = \eta_{\bar\ell \bar z}{}^z = \eta_{\bar\ell \bar z}{}^+ = 0 \tag{2.29}$$

if $\eta_{\bar\imath \bar z}{}^z \neq 0$ the superanalog of ψ contains even nilpotent terms and the superanalog of the change of variables (2.27) alters the measure by the addition of a total derivative. Thus, in superstring theory Earle's theorem has the important consequence that we cannot choose a good global slice for which the measure is manifestly positive semidefinite This, by itself, does not mean that the measure is not positive semidefinite, since one could try to choose holomorphic slices on local coordinate patches, in a way that on the region of overlap the two slices give the same answer for the string integrand. As we shall see in subsection G, in some cases it is even impossible to choose such a set of local holomorphic slices.

D. *Special slices and δ-function gravitini*

We now discuss a class of slices that we shall be using in our computation. These slices are characterized by two conditions, the first of which is,

$$\frac{\delta g^{\alpha\beta}}{\delta \varsigma^a} = 0 \tag{2.30}$$

i.e. the even nilpotent part of the metric is taken to be zero. This, in particular means that $\eta_{az}{}^{\bar z} = \eta_{a\bar z}{}^{z} = 0$. The second condition is that $\bar\chi$ have δ-function support. We shall discuss this is some detail now. First some notation: In earlier subsections we have introduced a fixed coordinate system $(u,\bar u) \equiv \vec u$ on the riemann surface. In particular $\vec u$ could be taken to be the coordinate system such that at a specific point t_0 in teichmuller space $g^{uu} = g^{\bar u\bar u} = 0$, the corresponding frame indices were denoted by $z, \bar z, +, -$. It is convenient at this stage and for later analysis to define a new family of coordinate systems. This is done as follows: Let $v^\mu \equiv (v_t, \bar v_t)$ be the coordinate system such that the metric components $g^{vv} = g^{\bar v\bar v}$ at the point t in teichmuller space vanish. In other words $(v_t, \bar v_t)$ for a given t is defined to be the coordinate system that diagonilizes the metric on the riemann surface at the point t in teichmuller space. By definition the $(u,\bar u)$ coordinate system is just $(v_{t_0}, \bar v_{t_0})$. We should also associate with the $(v_t, \bar v_t)$ coordinate system the compatible frame indices $(z_t, \bar z_t, +_t, -_t)$. However in order to avoid cluttering our formulae below we shall suppress the subscript t on all frame indices and ocassionally we do the same thing with the subscript on the coordinates $(v_t, \bar v_t)$. No confusion should arise, the precise meaning should be clear from the context.

A convenient choice of slice for the gravitino is given by:

$$\bar\chi_{\bar z}{}^{+}(v_t, t) = \sum_a \varsigma^a \delta^{(2)}(v_t - v_t(q_a(t))) \tag{2.31}$$

for every t, where $v_t(q_a(t))$ are the coordinates in the v_t system of some set of points $\{q_a(t) : a = 1, \cdots, 2g-2\}$ where the gravitino has its support on the riemann surface. A similar expression can be written for $\chi_z{}^{-}$ with a support at $\{q_a(t) : a = 2g-1, \cdots, 4g-4\}$. These points in general will be allowed to move as we change t. We now apply the remarks of subsections B and C to this specific class of slices.

We first discuss the differentiation of δ-function slices. As explained in subsection C one way to give a meaning to slice differentiation when the slice is defined in a "moving coordinate system" (as is the case in (2.31)) is to pull back $\bar\chi_{\bar z}{}^{+}(v_t, t)$ for every t to a fixed

coordinate system, say the u system (this gives a family of sections $\bar\chi_{\bar z}{}^{+}(u,t)$ parametrized by t, of one fixed bundle) and then differentiate the resulting family patchwise in the obvious way. More concretely to compute $\frac{\partial\bar\chi}{\partial t}$ at some point t we pull back both $\bar\chi(v_t, t)$ and $\bar\chi(v_{t+\delta t}, t+\delta t)$ to the u coordinates and then take the obvious difference. However for computational purposes, it may be more convenient to pull back only $\bar\chi(v_{t+\delta t}, t+\delta t)$ to the v_t coordinate system and then take the infinitesimal difference with $\bar\chi(v_t, t)$. (This is of course equivalent to computing the difference in the u coordinate system and then pulling the result back again to the v_t coordinates). $\bar\chi(v_{t+\delta t}, t+\delta t)$ expressed in the v_t coordinates takes the following form:

$$\bar\chi(v_t, t = t+\delta t) = \sum_a \varsigma^a \delta^{(2)}(v_t - v_t(q_a(t+\delta t))) \left[\frac{\kappa^{-1}(t+\delta t)\bar\kappa^2(t+\delta t)}{\kappa^{-1}(t)\bar\kappa^2(t)} \frac{\partial(v_t, \bar v_t)}{\partial(v_{t+\delta t}, \bar v_{t+\delta t})}\right]_{v_t = v_t(q_a(t))} \tag{2.32}$$

where the jacobian factor comes from the transformation of the δ function. For a delta function slice all the manipulations that we shall carry out below can be shown to be manifestly invariant under a change $\delta^{(2)}(v - v_a) \to f(t)\delta^{(2)}(v - v_a)$ of the basis, where $f(t)$ is an arbitrary function of t. In other words, in calculating $\frac{\partial\bar\chi}{\partial t}$ we may ignore $\frac{\partial f(t)}{\partial t}$ term. [7]Therefore we may ignore the factors of κ in the transformation law, and the basis given in eq.(2.32)is equivalent to a basis where,

$$\bar\chi(v_t, t+\delta t) = \sum_a \varsigma^a \delta^{(2)}(v_t - v_t(q_a(t+\delta t))) \tag{2.33}$$

and so,

$$\begin{aligned}\frac{\partial\bar\chi}{\partial t} &= \lim_{\delta t \to 0} \frac{(\bar\chi(v_t, t+\delta t) - \bar\chi(v_t, t))}{\delta t} \\ &= \sum_a \varsigma^a \frac{\partial}{\partial v_t^\mu} \delta^{(2)}(v_t - v_t(q_a(t))) \frac{\partial v_t^\mu(q_a(t))}{\partial t}\end{aligned} \tag{2.34}$$

---

[7] This may be seen as follows. The only place where $\frac{\partial\bar\chi}{\partial t}$ appear in our formulae, is in $\frac{\partial}{\partial t}\int d^2v f^a(t)\delta^{(2)}(v - v_a)\beta(v)$, where β is a commuting ghost. Each such term is also multiplied by a factor of $\delta(\beta)$. Thus all terms where the $\frac{\partial}{\partial t}$ operator acts on $f(t)$ vanish identically, being proportional to $\beta(v_a)\delta(\beta(v_a))$.

where,

$$\frac{\partial v_t^\mu(q_a(t))}{\partial t} = \lim_{\delta t\to 0}\left[\frac{v_t^\mu(q_a(t+\delta t)) - v_t^\mu(q_a(t))}{\delta t}\right] \tag{2.35}$$

In particular, the choice $\frac{\partial \bar\chi}{\partial t}=0$ corresponds to $\frac{\partial q_a(t)}{\partial t}=0$, i.e. the points do not move on the riemann surface for this choice. As mentioned in subsection C, a useful way to compute $\partial v/\partial t$ throughout teichmuller space is provided by the uniformization theorem. We will use that point of view in subsection F. In a similar way we we can work out differentiation with respect to odd moduli.

It is not obvious that the gravitino slice (2.31) (which partially specifies the map f in (2.10)) defines a good slice. By the above lemma, a necessary condition for a slice with $\bar\chi_{\bar z}^{+} = \sum_a \varsigma^a \delta^{(2)}(v-v_a)$ to be transverse to the gauge directions is simply

$$det\nu^a(t,v_b)\neq 0 \tag{2.36}$$

for all t, where $\nu^a(t,p)$ is a basis of holomorphic 3/2 differentials. As explained below eq. (2.10) we know that we can find a globally defined basis $\nu^a(t,p)$ of holomorphic 3/2 differentials. For such a basis the condition that (2.36) hold for all $t\in T$ is the condition for a good slice.

When the condition (2.36) is fulfilled then, given any $\bar\chi_{\bar z}^{+}$, we may find a gauge transformation parameter ϵ such that,

$$\bar\chi_{\bar z}^{+} = \sum_{a,b}\varsigma^a R_{ab}\delta^{(2)}(v-v_b) + \partial_{\bar z}\epsilon \tag{2.37}$$

where,

$$\epsilon = \int d^2v' G(v,v')\bar\chi_{\bar z}^{+}(v') \tag{2.38}$$

and,

$$\varsigma^b R_{ba} = (\nu^a(z_c))^{-1}\int d^2v'\nu^c(v')\bar\chi_{b\bar z}{}^{+}(v') \tag{2.39}$$

$G(v,v')\in K^{-1/2}|_v\otimes K^{3/2}|_{v'}$ is the parametrix for the operator $\bar\partial$ acting on $(-1/2,0)$ forms, satisfying,

$$\partial_{\bar v}G(v,v') = \delta^{(2)}(v-v') - \sum_{a,b}\delta^{(2)}(v-v_b)\nu^a(v')(\nu^a(v_b))^{-1}. \tag{2.40}$$

Thus we can always pass to a δ-function supported gravitino slice as long as (2.36) is satisfied.

E. *Constraints of Modular Invariance: Implications of the Integration Ambiguity*

We now attempt to clarify the role of modular invariance in the choice of slice. Again, we shall confine our discussion to the specific class of slices defined by eqs.(2.30) and (2.31). We have been describing a slice for superteichmuller space, but string amplitudes are obtained by integration over supermoduli space, obtained by dividing $\mathcal{T}$ by the action of the modular group $\mathcal{D}/\mathcal{D}_0$. Recall that if $[(e,\overline{\chi})]\in\mathcal{T}$ and $\gamma=[\phi]\in\mathcal{D}/\mathcal{D}_0$ then this action is defined by $\gamma\cdot[(e,\overline{\chi})]=[\phi\cdot(e,\overline{\chi})]$ where ϕ acts by pullback. In the case of the bosonic string, where the measure is completely slice-independent, modular invariance provides *no* additional restrictions on a slice. Recall that if t, $\tilde t\in R$ map to points $g(t),g(\tilde t)\in\mathcal{T}$ related by a modular transformation: $\gamma\cdot g(t)=g(\tilde t)$ for $\gamma\in\mathcal{D}/\mathcal{D}_0$ then, if we represent γ by a particular global diffeomorphism $\phi\in\mathcal{D}$, the action of ϕ will take the frame $f(t)$ to another frame $\phi\cdot f(t)$ which needn't be equal to $f(\tilde t)$. By the definition of a slice, the frames $\phi\cdot f(t)$ and $f(\tilde t)$ are related to one another by the action of gauge transformations connected to the identity. Thus, an anomaly free measure μ computed in two ways with the aide of one slice passing through $\phi\cdot f(t)$ and another passing through $f(\tilde t)$ will yield the same volume form at $\tilde t\in\mathcal{T}$; indeed this is what is meant by "anomaly free." In the bosonic string the question of modular invariance is the question of whether $\gamma^*(\mu_{g(\tilde t)})=\mu_{g(t)}$ for all $\gamma\in\mathcal{D}/\mathcal{D}_0$. This is a nontrivial condition on the measure, and not on the slice. Since $\mu_{g(t)}$ is completely specified in a given theory, this, in turn, is a constraint on the theory.

In the superstring the above argument is not valid because of the integration ambiguity, and modular invariance gives a constraint on the slice, as discussed in subsection B.

Given two points $t^r, \tilde{t}^r$ in teichmuller space which are related by a modular transformation, we know from the previous discussion that the slice (e, χ) at (t, ς) should be related to the slice $(\tilde{e}, \tilde{\chi})$ at $(\tilde{t}, \varsigma)$ by a gauge transformation. Gauge transformations in general include diffeomorphisms and supergravity transformations. But the supergravity transformations (2.4) lead to even nilpotent terms in the metric. Thus if we want to work in the gauge (2.30) where the even nilpotent part of the metric is zero at every point t in the teichmuller space, the slice (e, χ) at t must be related to $(\tilde{e}, \tilde{\chi})$ at $\tilde{t}$ purely through a diffeomorphism. Since the global diffeomorphism ϕ representing γ is not unique one might ask if there are further conditions on the choice of slice. The answer is no. As we show in appendix E the measure is invariant under change of slice by diffeomorphisms connected to the identity. Thus the above condition on the slice is both necessary and sufficient. The condition of modular invariance is the analog of the condition (2.9). We will call a slice $\tau = h \circ f$ a *modular invariant* slice it it satisfies this criterion. With such a choice of slice, the condition $\gamma^*(\mu_{g(\tilde{t})}) = \mu_{g(t)}$ on the measure is automatically satisfied in a modular invariant theory, as long as γ is a pure diffeomorphism.

We now analyze how the condition of modular invariance constrains the δ-function slices satisfying (2.30). Although any two δ-function supported gravitinos are related by a supergravity transformation, the condition of modular invariance requires that χ and $\tilde{\chi}$ at $t, \tilde{t}$ must be related by a diffeomorphism. In addition e and $\tilde{e}$ must also be related by the *same* diffeomorphism. If we consider diffeomorphisms as active transformations then we consider a family of frames $e_\alpha^z(t, \vec{u})$ in a fixed coordinate system $\vec{u}$, as in subsection C. In the coordinate system $\vec{u}$ the gravitino support is located at $\vec{u}_a(t) \equiv \vec{u}(q_a(t))$. If $\tilde{t} = \gamma \cdot t$ then, having chosen $e(t)$ and $e(\tilde{t})$ there is a unique [8] ϕ, with $[\phi] = \gamma$, and $\phi^*(e(t)) = e(\tilde{t})$. The condition of modular invariance then includes

$$\phi^{-1}\vec{u}_a(t) = \vec{u}_a(\tilde{t}) \tag{2.41}$$

[8] This is true except at the orbifold points of the moduli space where $g^{\alpha\beta}(t)$ has isometries. In this case the points $\vec{u}_a(t)$ must be taken to be at the fixed points of the isometry of $g^{\alpha\beta}(t)$[6].

The physical outcome of imposing the constraint of modular invariance is the following. After integration over the odd variables, we may express the partition function as an integral over the teichmuller parameters with a certain measure. In order for the answer to make sense, the measure must be modular invariant. However, as we discussed earlier, under a shift of the points $\vec{u}_a$, the integrand changes by a total derivative in the moduli. Hence if t and $\tilde{t}$ are two points in the teichmuller space related to each other by a modular transformation, then, for arbitrary choice of the $\vec{u}_a(t)$ and $\vec{u}_a(\tilde{t})$ the integrands cannot be related to each other by modular transformation. What the constraint (2.41) does is to determine $\vec{u}_a(\tilde{t})$ for given $\vec{u}_a(t)$ in a way that ensures that the integrand at $\tilde{t}$ is indeed the modular transform of the integrand at t.

In fact, the constraint that the $\vec{u}_a$'s are chosen so as to satisfy the requirement (2.41) will be crucial for our analysis. In sec.IV we shall show that the superstring partition function may be expressed as $\int \prod_r dt^r \frac{\partial M^r}{\partial t^r}$, where M^r is some known correlation function of operators inserted at the points $\vec{u}_a$. If we represent the moduli space as a fundamental domain in the teichmuller space, then the domain, in general, has many boundaries which are identified with each other by modular transformation. In order that the integral over t^r does not receive contribution from these boundaries, M^r must transform like a vector density under modular transformation. Again, since M^r depends on the choice of the points $\vec{u}_a$, this is not going to happen if $\vec{u}_a(t)$ and $\vec{u}_a(\tilde{t})$ are chosen arbitrarily. Again, the constraint (2.41) chooses $\vec{u}_a(\tilde{t}^r)$ for a given $\vec{u}_a(t^r)$ in such a way as to ensure the M^r transforms like a vector density in the moduli space. In this case $\int \prod_r dt^r \frac{\partial M^r}{\partial t^r}$ receives contribution only from true boundaries of the moduli space, namely, when the surface degenerates into two lower genus surfaces. In appendix G we show that when (2.41) is satisfied M^r transforms correctly.

Even after a good global modular invariant slice is chosen, the ambiguity may not be completely resolved. The ambiguity is a total derivative and the constraint of modular invariance only ensures that this total derivative yields mutually cancelling contributions at the boundaries of the fundamental domain in teichmuller space. If moduli space were truly compact that would be the whole story. However moduli space has a boundary describing riemann surfaces that degenerate into two lower genus surfaces. Compactifying moduli space does not help since the integrand could develop a pole on that hypersurface in moduli space. Consequently, in general, as was shown in [10] for heterotic string theories (equally valid conclusion holds for the type II string) the ambiguity is present, even after integrating over the moduli because of non-vanishing contibutions from the total derivative in the integrand at that boundary. In this case it seems that one has to invoke the constraint of $BRST$ invariance in order to determine the correct choice of slice. At genus $g = 2$ this was enough to eliminate this ambiguity [10] for the vacuum amplitude (see sec. 5). It is not known at this moment whether $BRST$ invariance at higher genera (or for higher point functions) will be equally powerful, or whether one needs to invoke additional principles. We again emphasize that the ambiguity is not related to the fact that we have carried out the integration over the odd coordinates before the even ones. The ambiguity is present even in the formalism where we treat the even and the odd coordinates on equal footing, only it appears at a later stage. In short, the ambiguity comes from an ambiguity in choosing the domain of integration, not in choosing the measure[10].

F. *Applications to hyperelliptic calculations*

The discussion in the previous section may be made more concrete by working in the hyperelliptic representation of $g = 2$ curves.[9]In this context we shall also critically examine a further condition on the choice of slice that has been used in the past, besides those given in eqs.(2.30) and (2.31). This constraint is,

$$\frac{\partial}{\partial t}\bar{\chi} = 0 \tag{2.42}$$

[9] Calculation of the vacuum amplitude in hyperelliptic coordinates for flat space string theories has been carried out by many authors. See for example [48][14][15][16][17][49][11].

As we will see in section III it is this constraint which leads to the insertion only of picture changing operators.

Let us begin our discussion by noting that any (affine) $g = 2$ curve can be written in hyperelliptic coordinates: [50][14][15][16]

$$S_{\vec{e}} = \{(y,x) \in C^2 | y^2 = \prod_{i=1}^{6}(x - e_i)\} \tag{2.43}$$

where x is the projection to C and the hyperelliptic involution is $j(y,x) = (-y,x)$. We may fix three branch points using projective transformations, say e_4, e_5, e_6, at $0, 1, \infty$. The remaining branch points live in the space

$$\mathcal{E} = \{\vec{e} \in C^3 | e_i \neq 0, 1, e_i \neq e_j\} \tag{2.44}$$

At $g = 2$ this is a finite (720-fold) covering of moduli space, and teichmuller space is its universal cover. For $t \in T$ we also denote the curve with branch points $e_i(t)$ by S_t. Because $\mathcal{F}/C \to T$ is topologically trivial we can find a (real-analytic) family of quasiconformal maps $w_t : S_0 \to S_t$ for all $t \in T$ which commute with the hyperelliptic involution, $w_t j = j w_t$ and thus induce a family of maps $\tilde{w}_t : C \to C$ such that the diagram

$$\begin{array}{ccc} S_0 & \xrightarrow{w_t} & S_t \\ \pi_0 \downarrow & & \pi_t \downarrow \\ C & \xrightarrow{\tilde{w}_t} & C \end{array} \tag{2.45}$$

commutes. The $\tilde{w}_t$ are not holomorphic, for a metric inducing the holomorphic structure on S_t will be of the form $\pi_t^*(e^{\phi}|dx|^2)$ and will pull back under $\tilde{w}_t$ to a metric $\pi_0^*(e^{\tilde{\phi}}|dx + \mu_t d\bar{x}|^2)$ which is not diagonal. Notice further that since $w_t j = j w_t$, w_t takes branch points to branch points, since these are the fixed points of j. In particular $\tilde{w}_t(e_i(0)) = e_i(t)$. If we are given a family of gravitinos on S_t

$$\bar{\chi}_{\bar{z}}{}^{+}(z) = \pi_t^*\Big(\sum \varsigma^a \delta^{(2)}(x - x_a(t))\Big) \tag{2.46}$$

where $x_a(t) \in C$, we can pull them back and differentiate. Dropping multiplicative factors as usual we have

$$\left(\frac{\partial}{\partial t^i}\tilde{\chi}\right)(z) = \pi_0^*\left(\sum \varsigma^a \frac{\partial}{\partial t^i}\delta^{(2)}(x - \tilde{w}_t^{-1}(x_a(t)))\right) \tag{2.47}$$

which is to be considered as a gravitino field on S_0.

We may now discuss the meaning of the constraint (2.42) in this coordinate system. Because of the natural projection $\pi_t : S_t \to C$ a choice of gravitino slice which might seem natural is to consider (2.46) with $x_a \in C$ held constant. Although it might seem natural to say this family has constant support, recall that the family w_t is undetermined up to the small diffeomorphisms $Diff_0(S_0)$ of S_0, and for generic choices of w_t we see that in fact $\frac{\partial}{\partial t}\tilde{\chi} \sim \frac{\partial}{\partial t}(\tilde{w}_t^{-1}x_a(t)) \neq 0$ even if $\frac{\partial x_a}{\partial t} = 0$, in the sense in which we have defined it. Thus there are two notions of constant gravitino support. On the one hand we could demand $x_a(t) = x_a(0) = constant$, on the other hand we could demand that the RHS of (2.47) vanish. That there is a choice of family w_t for which the second notion coincides with the first is a consequence of the following [10]

Theorem 1: *Given a family of points $x_a(t) \in C$, $t \in T$, $a = 1, \dots n$ with $x_a(t) \neq x_b(t)$ for $a \neq b$, define*

$$T' = T - \{t | x_a(t) = e_i(t)\}$$

If $W \subset T'$ is connected and simply connected then there exists a family of quasiconformal maps $w_t : S_0 \to S_t$, $t \in W$, commuting with the hyperelliptic involution, such that $\tilde{w}_t(x_a(0)) = x_a(t)$.

We give the proof of this theorem in appendix B [11]. In plain english this simply says (with $n = 2$) that if the δ-function support generically avoids branch points then we can

[10] For the present discussion we only need this theorem for the case where the family is $x_a(t) = x_a(0)$. We will need the more general statement below.

[11] We thank C. Earle for his very generous assistence in proving this theorem.

use small diffeomorphisms to gauge transform the gravitino to have constant support, as viewed from the fixed coordinate system of S_0.

We now apply the ideas of subsections C, D and E to determine the constraints on $x_a(t)$ from transversality condition (2.36) and the modular invariance constraint (2.41). In this context we shall show that the choice of slice with $x_a(t)$ held constant cannot be a good gauge slice. [12]On hyperelliptic curves we can write a basis of holomorphic 3/2-differentials quite explicitly. Choosing an even spin structure amounts to choosing three special branch points, which we may call A_i, the remaining three will be called B_i. The degree $g-1=1$ divisor of the even spin structure will be $A_1 + A_2 - A_3$. [13]Holding $e_6 = B_3$ fixed, permutation of the remaining 5 points yields the $\binom{5}{2} = 10$ even spin structures [14][15][16][17]. We may then take

$$\begin{aligned} \nu^1 &= \Pi_A(x)\frac{dx^{3/2}}{y^{3/2}(x)} \\ \nu^2 &= \Pi_B(x)\frac{dx^{3/2}}{y^{3/2}(x)} \end{aligned} \tag{2.48}$$

where we define

$$\begin{aligned} \Pi_A(x) &\equiv \sqrt{\prod_i (x - A_i)} \\ \Pi_B(x) &\equiv \sqrt{\prod_i (x - B_i)} \end{aligned} \tag{2.49}$$

Therefore, if the determinant (2.36) vanishes then

$$\Pi_A^2(x_1)\Pi_B^2(x_2) - \Pi_A^2(x_2)\Pi_B^2(x_1) = 0 \tag{2.50}$$

[12] The constraints on $x_a(t)$ that we shall derive, are independent of the choice of w_t, since at no stage w_t will enter the discussion. It is only when we try to see the implication of these constraints on $\frac{\partial \chi}{\partial t}$, that the choice of w_t becomes relevant.

[13] Recall that the divisor of a meromorphic function has class zero and that that of an abelian differential is the canonical class K. Examining the meromorphic functions $(x - e_i)^{-1}$ and y we learn that $\infty_1 + \infty_2 - 2e_i \sim 0$ and $e_1 + \cdots + e_6 \sim 3(\infty_1 + \infty_2)$, while $div(y^{-1}dx) = \infty_1 + \infty_2$ is in the class K. Thus $A_1 + A_2 + A_3 \sim B_1 + B_2 + B_3$ and $A_1 + A_2 + A_3 = A_1 + A_2 - A_3 + 2A_3$ is in the class $K^{1/2} \otimes K$.

Considered as an equation in the branch points for fixed x_i this has the trivial solutions when two branch points coincide with the x_a. These solutions describe a subvariety of complex codimension two, whereas (2.50) has solutions on a subvariety of codimension one. Actually (2.49) are not well-defined on the space of branch points but on its universal cover. The existence of solutions of (2.50) implies that there are solutions of $det\nu^a(x_b) = 0$ on T other than the trivial ones. Thus, the constant support, δ-function basis, in the sense that we keep $x_a(t)$ fixed, is not a good slice. These considerations may be generalized to the hyperelliptic locus at any genus. If the x_a are fixed the determinant condition reduces to an algebraic equation on the branch points which in general has solutions other than the trivial ones.

Since the slice (2.46) with x_a constant becomes singular in moduli space the matrix R_{ab} in (2.39) becomes singular and it is possible that taking $\frac{\partial}{\partial t}\bar{\chi}$ can induce δ-function singularities. However, in the measure such singularities always multiply $\delta(\beta)\beta$ which is formally zero (see section III), so an unambiguous and correct answer must be obtained by a limiting procedure. We will avoid this issue by choosing a slice for which (2.50) is never satisfied by choosing x_a to be appropriate functions of the branch points. We now show that such a choice is always possible. Recall we have used $SL(2,C)$ to set $e_6 = B_3$ (which is the same for all 10 even spin structures) to ∞. Then we may let, for example,

$$x_1 = 1 + \sum_{i=1}^{5} |e_i|$$

which never coincides with a branch point. Plugging in this equation for x_1, (2.50) can be viewed as a set of 10 third order polynomial equations in x_2 which have at most 30 distinct solutions. Clearly we can choose x_2 to depend continuously on e_i so that it avoids e_i, x_1 and these solutions. Combining such a gravitino slice with a good global slice $e(t)$ for bosonic moduli space we see that the conditions of the above lemma are satisfied and we therefore have a good global slice for superteichmuller space in which g is independent of ς, and $\bar{\chi}$ has (moving) delta function support.

In the above argument we have excluded the possibility that the delta function support of the gravitino is located at the branch points. In fact, with the above (noncanonical!) choice of w_t, if we take $x_1 = e_1$, for some branch point e_1, then, strange as it may seem: $\frac{\partial}{\partial t^I}\delta(x - \tilde{w}_t^{-1}e_1(t)) = \frac{\partial}{\partial t^I}\delta(x - e_1(0)) = 0$, since $\tilde{w}_t^{-1}(e_i(t)) = e_i(0)$. If we take $x_1 \to e_1$ and $x_2 \to e_1$ in such a way that our gravitino satisfies $\bar{\chi} = \pi_t^*(\varsigma^1\delta(x - e_1) + \varsigma^2\delta'(x - e_1))$ then $\frac{\partial}{\partial t}\bar{\chi} = 0$ everywhere in moduli space. The condition for transversality is then satisfied since

$$det(\eta_a, \nu^b) = \frac{1}{\hat{y}^2(e_1)} = \frac{1}{\prod_{i=2}^{6}(e_1 - e_i)} \tag{2.51}$$

never vanishes in moduli space. Note that this is the choice of gauge which lead to the pointwise vanishing vacuum amplitudes in [15][16][17][11][51]. Still, this slice is not without other difficulties, as we will see soon.

Let us now consider the implications of modular invariance for hyperelliptic calculations. The difficulties with modular invariance caused by gravitino slices with $x_a(t)$ =constant have been pointed out in [11]. We can now use the above considerations to find the origin of the problem. To begin we must understand how the modular group acts on the space of branch points $\mathcal{E}$. An isomorphism $(y, x) \to (\tilde{y}, \tilde{x})$ of the curves

$$\begin{aligned} S_e : y^2 &= \prod_{i=1}^{6}(x - e_i) \\ \tilde{S}_{\tilde{e}} : \tilde{y}^2 &= \prod_{i=1}^{6}(\tilde{x} - \tilde{e}_i) \end{aligned} \tag{2.52}$$

will project to an isomorphism of $\hat{C}$, which must therefore be $\tilde{x} = \frac{\alpha x+\beta}{\gamma x+\delta}$. Thus we see that we must have

$$\{e_i\} = \{\frac{\delta\tilde{e}_i - \beta}{-\gamma\tilde{e}_i + \alpha}\} \tag{2.53}$$

as unordered sets. As a special case let us fix e_4, e_5, e_6 to $0, 1, \infty$ and take $\tilde{x} = \frac{1}{e_1}x$, together with a permutation of points one and five. This induces the modular transformation

$$\tilde{e}_1 = \frac{1}{e_1}, \quad \tilde{e}_2 = \frac{e_2}{e_1} \quad \tilde{e}_3 = \frac{e_3}{e_1} \tag{2.54}$$

In terms of the coordinates on C, the constraint of modular invariance on the gravitino supports is just that $x_a(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ must be the image of $x_a(e_1, e_2, e_3)$ under the isomorphism $\tilde{x} = \frac{x}{e_1}$, i.e.

$$x_a(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3) = \frac{1}{e_1} x_a(e_1, e_2, e_3) \tag{2.55}$$

It follows that if we choose x_a to be independent of the branch points then we will have a conflict with (2.55) and hence with modular invariance. Also, in passing to the gravitino slice $\chi = \pi_t^*(\varsigma^1 \delta(x - A_1) + \varsigma^2 \delta'(x - A_1))$, the limit $x_a \to A_1$ is not modular invariant. For example, if we choose $x_1(\{e_i\}) \to e_1$ then we also require $x_1(\{\tilde{e}_i\}) \to \tilde{e}_1 = \frac{1}{e_1}$. However, modular invariance (2.55) requires $x_1(\{\tilde{e}_i\}) \to \frac{1}{e_1} e_1 = 1$. This is another reflection of the loss of modular invariance inherent in the choice of special point R_* in [15][16][17][11]. As noted in [11] it is difficult to see how the prescription can lead to modular invariant nonvanishing string integrands. In the case of uncompactified superstring theory in ten dimensions, the integrand of the vacuum amplitude calculated with this choice of basis turns out to vanish identically [15][16][17][11] [51] and hence we do not have any problem in defining the integral. But in other cases, (where the cosmological constant is expected to be non-zero) this prescription probably will not lead to a sensible answer.

Our discussion so far has shown that the image under π_t of the points q_a on C (i.e. x_a) cannot be held fixed as a function of t. One might ask if, when these images $x_a(t)$ move so as to satisfy the conditions of transversality and modular invariance, a family w_t can be chosen so that $\frac{\partial \chi}{\partial t}$ in (2.47) vanishes. We can apply the above theorem once more to answer this in the affirmative.

We have seen that the choices of (e, χ) are in one-one correspondence with choice of families $(x_a(t), w_t)$ $a = 1, 2$ $t \in T$. The theorem states that for generic choices of $x_a(t)$ we can choose a family w_t so that for almost all $t \in T$ (i.e. everywhere in T except possibly

on a subvariety of real codimension one. [14])

$$\begin{aligned} \tilde{w}_t^{-1}(x_a(t)) &= x_a(0) \\ e(t) \sim \tilde{w}_t^*(dx) &\sim dx + \mu_t d\bar{x} \end{aligned} \tag{2.56}$$

In this way a transverse and modular invariant slice can be put into a form which satisfies (2.42). Combining this discussion with our previous remarks we see that there exists an almost global good slice satisfying (2.42).

For the reader who, perchance, has lost his way in the thicket of mathematical terminology we summarize the situation. Let us work in a fixed coordinate system on the riemann surface, and assume that we have found one consistent choice of the slice $(e(t), \chi(t))$ which is consistent with the the requirements of modular invariance, as well as transversality to the gauge directions. Let $q_a(t)$ denote the trajectory of the points q_a in this coordinate system. Using the freedom of local diffeomorphism, we can, for every value of t, bring the point $q_a(t)$ back to a fixed point $q_a(0)$, as long as we change the metric accordingly. With this choice of slice, $\frac{\partial \chi}{\partial t^r} = 0$ everywhere in the moduli space. The section $e(t)$ obtained in the above way will not be a holomorphic section. That $e(t)$ is not a global holomorphic section is a consequence of Earle's theorem [15].

G. *Local Slices*

So far we have been discussing global slices, and we have used theorems on global slices, like Earle's theorem. In the bosonic string theory, the same theorem prevents us

[14] This is not completely satisfactory, since a subvariety of real codimension one, even though naively a set of measure zero on the teichmuller space, may give a finite contribution to the superstring functional integral, if, for example, the metric changes discontinuously across the surface so that the tangents $\eta_{iz}{}^{\bar{z}}$, $\eta_{i\bar{z}}{}^{z}$ have delta function singularities. We shall rectify this state of affairs in subsection G by choosing local slices, and patching them together.

[15] More precisely, by Earle's theorem any global slice cannot be holomorphic and the modifications of the slice involved in the proof of theorem theorem 1 cannot restore holomorphy. We have no proof of this intuitively obvious statement. Note however that if w_t is in fact defined on the complement of a complex codimension one, (= real codimension two), subvariety then by Earle's theorem and Hartog's theorem w_t cannot be holomorphic.

from choosing a global holomorphic section as well. However, in the bosonic string theory we can choose a holomorphic slice in each local coordinate patch in the teichmuller space and compute the partition function. In the region of overlap of two patches the slices in the two patches will be related by a gauge transformation, and the partition function computed with the two slices will be identical, since it is gauge invariant. In other words, the partition function may be expressed as a holomorphic square (up to factors of $det Im\tau$) at each point in the teichmuller space.

Once again, the above argument fails in fermionic string theory because of the integration ambiguity, since if we choose local slices then on the overlaps they will be related by gauge transformations which, in general, include supergravity transformations. This does not prevent us from choosing local slices related purely by small diffeomorphisms. Thus we may choose a set of slices $\tau_\alpha(t,\varsigma) = (e^\alpha(t,\varsigma), \bar{\chi}^\alpha(t,\varsigma))$ for $t \in U_\alpha$, relative to an open covering $\{U_\alpha\}$ of T such that for $t \in U_\alpha \cap U_\beta$, τ_α is related to τ_β by small diffeomorphisms. There is no local obstruction to choosing holomorphic families w_t^α of quasiconformal maps [43]. Thus, it would appear to be possible to choose in this way a set of good local slices satisfying conditions (2.30) and (2.42) and differing on overlaps by small diffeomorphisms. For such a set of slices the remarks of [11] would apply and we would have a positive semidefinite measure, which, for generic choices of gravitino support, would be positive definite. We now show that there is a global obstruction to choosing a set of such holomorphic slices τ_α for an entire covering U_α of teichmuller space. We begin by quoting a powerful theorem from teichmuller theory: [52][53]

Hubbard's theorem: *There is no global holomorphic section of the universal teichmuller curve except for the Weierstrass points at $g = 2$.*

The universal teichmuller curve is simply the fiber space over teichmuller space where the fiber over $t \in T$ is just a copy of the riemann surface defining the point t. A slightly more formal description can be found in appendix A and in [43]. Hubbard's theorem

implies that exactly those slices which could lead to positive definite measures in fact do not exist. This is a consequence of the following

Theorem 2: *A set of holomorphic slices τ_α with δ-function supported gravitinos, defined for an open covering U_α of T, differing on overlaps by diffeomorphisms and satisfying (2.42) defines a holomorphic section of the universal teichmuller curve.*

The reason for this is simple. We may describe the local slices by the pairs $(x_a^\alpha(t), w_t^\alpha)$ where w_t^α is quasiconformal, and $t \in U_\alpha$. The condition that the slices be related on overlaps purely by diffeomorphisms is the condition that for $t \in U_\alpha \cap U_\beta$, $((\tilde{w}_t^\alpha)^{-1}(x_a^\alpha(t)), dx + \mu_t^\alpha d\bar{x})$ is related to $((\tilde{w}_t^\beta)^{-1}(x_a^\beta(t)), dx + \mu_t^\beta d\bar{x})$ through a diffeomorphism. Furthermore, the condition (2.42) means that $x_a(t) = \tilde{w}_t(x_a(0))$ in both patches. Thus the $x_a(t)$ vary holomorphically with t. The unique choice of diffeomorphism relating the two slices is $(w_t^\alpha)^{-1}(w_t^\beta)$ as can be seen by comparing the metrics. By assumption this diffeomorphism must also relate gravitino supports. Thus we must have $x_a^\alpha(0) = (\tilde{w}_t^\alpha)^{-1}(\tilde{w}_t^\beta)x_a^\beta(0)$, i.e. $x_a^\alpha(t) = x_a^\beta(t)$. In other words, the points $x_a(t)$ are *globally* defined, and depend holomorphically on t. Such a set of points defines a global holomorphic section of the universal teichmuller curve. A more formal discussion of these matters is given in appendix B.

Note that the crucial difference from the bosonic string theory is that in that case w_t^α and w_t^β are completely free, whereas, in the present case, they must satisfy the constraint $(\tilde{w}_t^\alpha)^{-1}\tilde{w}_t^\beta x_a^\beta(0) = x_a^\alpha(0)$ at every point t in the overlap region. This is the constraint that prevents w_t from being holomorphic. We may, instead, relax this constraint on w_t, and choose $x_a(t)$ in a way that the two slices are related by appropriate diffeomorphism in the overlap region. In this case, there is no obstruction to choosing local holomorphic slice for $e(t)$, but $\frac{\partial \chi}{\partial \bar{t}}$ will not vanish any more. In general both the holomorphic and the antiholomorphic derivatives of $\tilde{w}_t^{-1}x_a(t)$ will be non-vanishing. This can be interpreted as a further (but more specialized) obstruction to holomorphic factorization above and beyond the coupling of the zero modes of the scalar fields. In the computation in the following sections, we shall work with a general choice of slice, and assume neither holomorphy, nor $\frac{\partial \chi}{\partial t} = 0$. We can use the above observations to explain why the hyperelliptic calculations lead to point-by-point vanishing when the gravitinos are supported on Weierstrass points. This is done in appendix C.

3. Measure for Supermoduli

At present the best way of studying multiloop string amplitudes is through the "conformal field theory" of first order ghost systems [1]. An alternative approach makes use of ζ-function regulated functional determinants. While far more awkward, the latter approach has the advantage of being mathematically well-defined. In this section we will indicate the relation between the two approaches. Perhaps the most fruitful point of view is that the functional determinants give rigorous meaning to the quantities and manipulations of conformal field theory.

We begin by reviewing the gauge fixing approach which was pursued in [2] for the heterotic string and in [4] for the type II string. We will focus on the type II case although the formulae are easily adapted to the heterotic case.[16] In WZ gauge the action for the type II string is:

$$S = \int \left(\partial X \overline{\partial} X + \psi \overline{\partial} \psi + \overline{\psi} \partial \overline{\psi} + \overline{\chi} \psi \partial X + \chi \overline{\psi} \overline{\partial} X + \tfrac{1}{2} \psi \overline{\psi} \chi \overline{\chi} \right) \qquad (3.1)$$

where $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ on a flat world-sheet, and is the appropriate cauchy-riemann operator coupled to 0-forms X, and $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ -forms ψ and $\overline{\psi}$. Superdiffeomorphisms

[16] Both [2] and [4] contain important errors in the formula for the supermoduli correction. The measure in [2] omits the contribution of the matter supercurrent, while that of [4] omits the ghost supercurrent. Neither contribution can be left out. This is clear simply on grounds of superconformal invariance. It is only for the sum of the ghost and matter supercurrents that the conformal anomaly term in the operator product expansion cancels.

defined by an even vector field will be called "ordinary" diffeomorphisms, while odd diffeomorphisms are referred to as supergravity transformations. The action of ordinary diffeomorphisms is [17]

$$\begin{aligned}
\delta \partial &= \partial \overline{\xi} \overline{\partial} \\
\delta \overline{\partial} &= 0 \\
\delta X &= -\overline{\xi} \overline{\partial} X \\
\delta \psi &= -\overline{\xi} \overline{\partial} \psi \\
\delta \overline{\psi} &= -(\overline{\xi} \overline{\partial} \overline{\psi} + \tfrac{1}{2} \overline{\partial} \overline{\xi} \overline{\psi}) \\
\delta \overline{\chi} &= -\overline{\partial}(\overline{\xi} \overline{\chi}) \\
\delta \chi &= -\overline{\xi} \overline{\partial} \chi + \tfrac{1}{2} \overline{\partial} \overline{\xi} \chi
\end{aligned} \qquad (3.2)$$

where $\overline{\xi}$ is an even vector field (i.e. a $(0,-1)$ form).

The supergravity transformations are:

$$\begin{aligned}
\delta \partial &= 0 \\
\delta \overline{\partial} &= \epsilon \overline{\chi} \partial \\
\delta \overline{\chi} &= 2 \overline{\partial} \epsilon \\
\delta \chi &= 0 \\
\delta X &= -\epsilon \psi \\
\delta \psi &= \tfrac{1}{2} \epsilon \chi \overline{\psi} + \epsilon \partial X \\
\delta \overline{\psi} &= 0
\end{aligned} \qquad (3.3)$$

for a supersymmetry transformation by an anticommuting parameter ϵ (a $(-1/2, 0)$ form). In (3.2) and (3.3) we have only displayed the $(\overline{\xi}, \epsilon)$ dependence. The analogous expressions with $(\xi, \overline{\epsilon})$ can be obtained from (3.2) and (3.3) by interchange of barred and unbarred quantities.

[17] We have actually taken a linear combination of a diffeomorphism, and appropriate lorentz and weyl transformations for convenience.

From the transformation laws (3.2) and (3.3) we see that if we define $h_a^b = e_a^m(\delta e_m^b)$ then the change of variables from the original fields to the gauge transformation parameters and the (super-)moduli is,[18]

$$\begin{pmatrix} \delta h_{\bar z}{}^{z} \\ \delta \bar\chi_{\bar z}{}^{+} \\ \delta h_{z}{}^{\bar z} \\ \delta \chi_{z}{}^{-} \end{pmatrix} = \begin{pmatrix} \bar\partial & -\frac{1}{2}\bar\chi & & \\ -(\partial\bar\chi) + \frac{1}{2}\bar\chi\partial & \bar\partial & & \\ & & \partial & -\frac{1}{2}\chi \\ & & -(\bar\partial\chi) + \frac{1}{2}\chi\bar\partial & \partial \end{pmatrix} \begin{pmatrix} \xi \\ \epsilon \\ \bar\xi \\ \bar\epsilon \end{pmatrix}$$
$$+ \begin{pmatrix} \hat\eta_{\imath\bar z}{}^{z} & \hat\eta_{\ell\bar z}{}^{z} & \hat\eta_{\bar\imath\bar z}{}^{z} & \hat\eta_{\bar\ell\bar z}{}^{z} \\ \hat\eta_{\imath\bar z}{}^{+} & \hat\eta_{\ell\bar z}{}^{+} & \hat\eta_{\bar\imath\bar z}{}^{+} & \hat\eta_{\bar\ell\bar z}{}^{+} \\ \hat\eta_{\imath z}{}^{\bar z} & \hat\eta_{\ell z}{}^{\bar z} & \hat\eta_{\bar\imath z}{}^{\bar z} & \hat\eta_{\bar\ell z}{}^{\bar z} \\ \hat\eta_{\imath z}{}^{-} & \hat\eta_{\ell z}{}^{-} & \hat\eta_{\bar\imath z}{}^{-} & \hat\eta_{\bar\ell z}{}^{-} \end{pmatrix} \begin{pmatrix} \delta\hat t^{\imath} \\ \delta\varsigma^{\ell} \\ \delta\hat t^{\bar\imath} \\ \delta\varsigma^{\bar\ell} \end{pmatrix} \tag{3.4}$$

where the various tangents $\hat\eta$ have been defined in sec. II. Here $(\hat t^{\imath}, \hat t^{\bar\imath}, \varsigma^{\ell}, \varsigma^{\bar\ell})$ where $\imath, \bar\imath$ runs from 1 to $3g-3$ and $\ell, \bar\ell$ runs from 1 to $2g-2$ are complex coordinates on superteichmuller space. When there is no need for distinction we write these indices as (r, a) to stand for $r = (\imath, \bar\imath)$ and $a = (\ell, \bar\ell)$.

At this point we can choose the map h discussed in section 2. This allows us to express the last term in (3.4) in terms of the variables $(t^{\imath}, t^{\bar\imath}, \varsigma^{\ell}, \varsigma^{\bar\ell})$, where $t^{\imath}$ and $t^{\bar\imath}$ are complex coordinates without nilpotent parts. Equation (3.4) then takes the form:

$$\begin{pmatrix} \delta h_{\bar z}{}^{z} \\ \delta \bar\chi_{\bar z}{}^{+} \\ \delta h_{z}{}^{\bar z} \\ \delta \chi_{z}{}^{-} \end{pmatrix} = \begin{pmatrix} \bar\partial & -\frac{1}{2}\bar\chi & & \\ -(\partial\bar\chi) + \frac{1}{2}\bar\chi\partial & \bar\partial & & \\ & & \partial & -\frac{1}{2}\chi \\ & & -(\bar\partial\chi) + \frac{1}{2}\chi\bar\partial & \partial \end{pmatrix} \begin{pmatrix} \xi \\ \epsilon \\ \bar\xi \\ \bar\epsilon \end{pmatrix}$$
$$+ \begin{pmatrix} \eta_{\imath\bar z}{}^{z} & \eta_{\ell\bar z}{}^{z} & \eta_{\bar\imath\bar z}{}^{z} & \eta_{\bar\ell\bar z}{}^{z} \\ \eta_{\imath\bar z}{}^{+} & \eta_{\ell\bar z}{}^{+} & \eta_{\bar\imath\bar z}{}^{+} & \eta_{\bar\ell\bar z}{}^{+} \\ \eta_{\imath z}{}^{\bar z} & \eta_{\ell z}{}^{\bar z} & \eta_{\bar\imath z}{}^{\bar z} & \eta_{\bar\ell z}{}^{\bar z} \\ \eta_{\imath z}{}^{-} & \eta_{\ell z}{}^{-} & \eta_{\bar\imath z}{}^{-} & \eta_{\bar\ell z}{}^{-} \end{pmatrix} \begin{pmatrix} \delta t^{\imath} \\ \delta\varsigma^{\ell} \\ \delta t^{\bar\imath} \\ \delta\varsigma^{\bar\ell} \end{pmatrix} \tag{3.5}$$

[18] In writing down (3.4) we have actually redefined ϵ by $\epsilon \to \epsilon - \bar\xi\bar\chi$ to remove off diagonal terms in the fluctuation matrix. Furthermore we have rescaled ϵ by $\frac{1}{2}$.

As mentioned in section two a holomorphic slice satisfies

$$\eta_{\bar\imath\bar z}{}^{z} = \eta_{\bar\imath\bar z}{}^{+} = \eta_{\bar\ell\bar z}{}^{z} = \eta_{\bar\ell\bar z}{}^{+} = 0$$

For a holomorphic slice, the transformation (3.5) becomes block diagonal, corresponding to the decoupling of left and right-moving modes of the ghosts. For the reasons explained earlier we will not, in general, choose a holomorphic slice. Even if we do not use a holomorphic slice it is still convenient to use complex coordinates $(t^{\imath}, t^{\bar\imath}, \varsigma^{\ell}, \varsigma^{\bar\ell})$.

In changing variables from $h_{\bar z}{}^{z}$, $h_{z}{}^{\bar z}$, $\bar\chi_{\bar z}{}^{+}$ and $\chi_{z}{}^{-}$ to ξ, ϵ, $\bar\xi$, $\bar\epsilon$, $(t^{\imath}, t^{\bar\imath})$ and $(\varsigma^{\ell}, \varsigma^{\bar\ell})$ we pick up a jacobian factor. Let ϕ^{i} $(i = 1, \cdots, N)$ denote the original field variables, x^{α} $(\alpha = 1, \cdots, n)$ denote the gauge transformation parameters, and y^{m} $(m = n+1, \cdots, N)$ the coordinates $(t^{\imath}, t^{\bar\imath})$ and $(\varsigma^{\ell}, \varsigma^{\bar\ell})$. (Thus here N and n are infinite, although $N - n$ is finite). Let us write,

$$\delta\phi^{i} = A_{i\alpha}\delta x^{\alpha} + A_{im}\delta y^{m} \tag{3.6}$$

where A is the matrix displayed in eq. (3.5). Then the jacobian for the change of variables is simply given by $sdet(A)$. There are two ways of interpreting this quantity as we now describe.

A. *Gauge fixing with ghosts*

We shall now express the jacobian factor in terms of a functional integral over the ghost fields. Let us introduce variables B^{i}, C^{α} with the property that B^{i} and C^{α} have exactly the opposite statistics of the variables ϕ^{i} and x^{α} respectively. Thus, for example, B^{i} is anti-commuting if ϕ^{i} is commuting. Then it can be shown that,

$$sdet(A) = \int \prod_{i=1}^{N} dB^{i} \prod_{\alpha=1}^{n} dC^{\alpha} exp(B^{i}A_{i\alpha}C^{\alpha}) \prod_{m=n+1}^{N} \delta(B_{i}A_{im}) \tag{3.7}$$

One way to prove this formula is to define auxiliary variables,

$$\tilde C_{\alpha} = B^{i}A_{i\alpha}$$
$$\tilde D_{m} = B^{i}A_{im} \tag{3.8}$$

With the above change of variables, we may express the right hand side of eq. (3.7) as,

$$sdet(A)\int\prod_{\alpha} dC^{\alpha}d\tilde{C}_{\alpha}\prod_{m} d\tilde{D}_{m}exp(\tilde{C}_{\alpha}C^{\alpha})\prod_{m}\delta(\tilde{D}_{m}) = sdet(A) \tag{3.9}$$

thus proving eq.(3.7). Note that the reason that what appears in eq. (3.9) is $sdet(A)$, and not $(sdet(A))^{-1}$ is that $\tilde{C}_{\alpha}$, $\tilde{D}_m$ and B_i have exactly the opposite statistics of x^{α}, y^m and ϕ^i respectively. Since $sdet(A)$ was defined for the change of variables given in eq. (3.6), the jacobian factor appearing in the change of variables given in eq. (3.8) is $sdet(A)^{-1}$.

The relevant matrix $A_{i\alpha}$, A_{im} in our case may be read directly from eq. (3.5). Introducing the B ghosts b_{zz}, $\bar{b}_{\bar{z}\bar{z}}$, β_{z+} and $\bar{\beta}_{\bar{z}-}$, and the C ghosts c^z, $\bar{c}^{\bar{z}}$, γ^+ and $\bar{\gamma}^-$ we may express the jacobian factor as,

$$sdet(A) = \int dbdcd\bar{b}d\bar{c}d\beta d\gamma d\bar{\beta}d\bar{\gamma}exp(-S_{gh})\prod_{r=1}^{6g-6}(\eta_r, B)\prod_{a=1}^{4g-4}\delta((\eta_a, B)) \tag{3.10}$$

where,

$$S_{gh} = \int(b\bar{\partial}c+\beta\bar{\partial}\gamma+\overline{\chi}(\tfrac{1}{2}b\gamma+\tfrac{1}{2}\beta\partial c+\partial(\beta c))+\bar{b}\partial\bar{c}+\bar{\beta}\partial\bar{\gamma}+\chi(\tfrac{1}{2}\bar{b}\bar{\gamma}+\tfrac{1}{2}\bar{\beta}\bar{\partial}\bar{c}+\bar{\partial}(\bar{\beta}\bar{c})) \tag{3.11}$$

and we define scale-factor-independent pairings by,

$$\begin{aligned}(\eta_r, B) &= \int \eta_{r\bar{z}}{}^{z}b_{zz} + \int \eta_{r\bar{z}}{}^{+}\beta_{z+} + \int \eta_{rz}{}^{\bar{z}}b_{\bar{z}\bar{z}} + \int \eta_{rz}{}^{-}\beta_{\bar{z}-}\\ (\eta_a, B) &= \int \eta_{a\bar{z}}{}^{z}b_{zz} + \int \eta_{a\bar{z}}{}^{+}\beta_{z+} + \int \eta_{az}{}^{\bar{z}}b_{\bar{z}\bar{z}} + \int \eta_{az}{}^{-}\beta_{\bar{z}-}\end{aligned} \tag{3.12}$$

In writing down eq. (3.10) we have replaced $\delta((\eta_i, B))$ by (η_i, B), since this is an anti-commuting object. Adding the matter action to the ghost action we see that the gravitino field $\bar{\chi}$ couples to the full supercurrent,

$$\psi\partial X + \tfrac{1}{2}b\gamma + \tfrac{1}{2}\beta\partial c + \partial(\beta c) \tag{3.13}$$

The above ghost supercurrent differs slightly from that in [1]. The difference is entirely due to a difference in convention for the kinetic terms for ψ. We now proceed to the second way to make sense of $sdet A$.

B. *Gauge fixing with functional determinants*

We leave conformal field theory briefly and describe now the approach that uses ζ-function regulated functional determinants. Returning to (3.4) we introduce operators P and $\hat{\eta}$ so that (3.4) becomes

$$\begin{pmatrix}\delta h_{\bar{z}}{}^{z}\\ \delta\chi_{\bar{z}}{}^{+}\\ \delta h_{z}{}^{\bar{z}}\\ \delta\chi_{z}{}^{-}\end{pmatrix} = P\begin{pmatrix}\xi\\ \epsilon\\ \bar{\xi}\\ \bar{\epsilon}\end{pmatrix} + \hat{\eta}\begin{pmatrix}\delta\hat{t}^{i}\\ \delta\varsigma^{\ell}\\ \delta\hat{t}^{\bar{\imath}}\\ \delta\varsigma^{\bar{\ell}}\end{pmatrix} \tag{3.14}$$

Introducing the obvious [19] L^2 inner products on each component we can define adjoint operators. (This involves a choice of metric.) For $g > 1$ the operator P has no zero modes, but the operator $P^{\dagger}$ has zero modes. When expanding in nilpotents $3g-3$ zero modes begin with holomorphic quadratic differentials ψ^{J}_{zz}, $3g-3$ with antiholomorphic quadratic differentials $\psi^{\bar{J}}_{\bar{z}\bar{z}}$, while $2g-2$ zero modes begin with holomorphic 3/2-differentials ψ^{p}_{z+}, and $2g-2$ with anti-holomorphic 3/2 differentials $\psi^{\bar{p}}_{\bar{z}-}$. We may arrange these zero modes into a 4×4 matrix Ψ, defined by

$$\Psi = \begin{pmatrix}\Psi^{\bar{J}}_{\bar{z}\bar{z}} & \Psi^{\bar{p}}_{\bar{z}\bar{z}} & & \\ \Psi^{\bar{J}}_{\bar{z}-} & \Psi^{\bar{p}}_{\bar{z}-} & & \\ & & \Psi^{J}_{zz} & \Psi^{p}_{zz}\\ & & \Psi^{J}_{z+} & \Psi^{p}_{z+}\end{pmatrix} \tag{3.15}$$

In general we cannot take the tangents to be identical with these zero modes. Accounting for this in the usual way [55][56]we obtain for the jacobian

$$sdet(\Psi^{\alpha}, \hat{\eta}_I)\left(\frac{sdet P^{\dagger}P}{sdet((\Psi^{\dagger})_I, \Psi^{\beta})}\right)^{1/2} \tag{3.16}$$

The inner product matrix $(\Psi^{\alpha}, \hat{\eta}_I)$ is defined as,

$$(\Psi^{\alpha}, \hat{\eta}_I) = \int(\Psi^{\alpha}_{zz}\hat{\eta}_{I\bar{z}}{}^{z} + \Psi^{\alpha}_{\bar{z}\bar{z}}\hat{\eta}_{Iz}{}^{\bar{z}} + \Psi^{\alpha}_{\bar{z}-}\hat{\eta}_{Iz}{}^{-} + \Psi^{\alpha}_{z+}\hat{\eta}_{I\bar{z}}{}^{+}) \tag{3.17}$$

[19] Actually, it is not so obvious. The chief difference between the component formalism and the superfield formalism enters at this stage. In the superfield formalism one uses the metric introduced in [54]for fluctuations of superfields. These are not the same metrics on field space.

where α stands for the indices $\jmath, \bar{\jmath}(=1,\dots 3g-3)$, $p, \bar{p}(=1,\dots 2g-2)$, and I stands for the indices $\imath, \bar{\imath}(=1,\dots 3g-3)$, $\ell, \bar{\ell}(=1,\dots 2g-2)$. The quantity $(\Psi^\dagger)_I$ involves a metric, turning $(\Psi^\alpha)^*$ into a beltrami differential.

Again at this point, by a choice of the map h of section 2 we can express the jacobian as

$$sdet(\Psi^\alpha, \eta_I)\Big(\frac{sdet P^\dagger P}{sdet((\Psi^\dagger)_I, \Psi^\beta)}\Big)^{1/2} \tag{3.18}$$

where we now integrate over $(t^\imath, t^{\bar{\imath}}, \varsigma^\ell, \varsigma^{\bar{\ell}})$. Note that the matrix P, as well as Ψ has block diagonal form, reflecting decomposition into holomorphic and anti-holomorphic parts. This is, however, not manifestly true for the matrix (Ψ^α, η^I), due to the fact that we have not chosen the tangents η compatible with the complex structure in moduli space. Any further manipulations in this subsection can be carried out in either representation (one formally can go from one to the other by replacing $\hat{t}, \hat{\eta}$ by t, η and vice versa). For practical purposes we shall only exhibit our fomulae from now on in the second representation.

The above expressions are still formal. For example, it is not obvious how to take the squareroot in (3.18). The proper definition will emerge when we expand (3.18) in nilpotents. This exercise is also useful because it allows us to define all quantities rigorously. Write the operator P as the sum of its zeroth order and nilpotent part: $P = \overline{D} + \not{A}$, where

$$\not{A} = \begin{pmatrix} 0 & -\frac{1}{2}\overline{\chi} & & \\ -(\partial\overline{\chi}) + \frac{1}{2}\overline{\chi}\partial & 0 & & \\ & & 0 & -\frac{1}{2}\chi \\ & & -(\overline{\partial}\chi) + \frac{1}{2}\chi\overline{\partial} & 0 \end{pmatrix} \tag{3.19}$$

Making this separation for the superdeterminant we find

$$sdet P^\dagger P = sdet \overline{D}^\dagger \overline{D} sdet\big(1 + \frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger \overline{D} + \frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger \not{A} + \frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger \not{A}\big) \tag{3.20}$$

With a little algebra one may write the second superdeterminant on the RHS as

$$\big|sdet(1 + \frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger \not{A})\big|^2 sdet\big(1 + \frac{1}{(1 + \frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger \overline{D})(1 + \frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger \not{A})} \frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger \Pi_0 \not{A}\big) \tag{3.21}$$

where

$$\Pi_0 = 1 - \overline{D} \frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger \tag{3.22}$$

is the projector on the zero modes of $\overline{D}^\dagger$. We compare this with the square norm of the zero modes of $P^\dagger$. Since

$$(\overline{D}^\dagger + \not{A}^\dagger)\Psi = 0 \tag{3.23}$$

we can write $\Psi = \psi + \delta\psi$ where ψ is zeroth order in $\overline{\chi}$ and can be written in terms of "ordinary" zero modes as:

$$\psi = \begin{pmatrix} \psi^{\bar{\jmath}}_{\bar{z}\bar{z}} & 0 & & \\ 0 & \psi^{\bar{p}}_{\bar{z}-} & & \\ & & \psi^{\jmath}_{zz} & 0 \\ & & 0 & \psi^{p}_{z+} \end{pmatrix} \tag{3.24}$$

we can further split $\delta\psi = \sum \delta\psi^{(i)}$ according to the number of odd moduli it contains. Writing the zero mode equation order by order in χ, $\overline{\chi}$ and solving recursively gives

$$\Psi = \frac{1}{(1 + \overline{D}\frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger)} \psi \tag{3.25}$$

Thus

$$sdet(\Psi^\dagger, \Psi) = sdet\Big(\psi^\dagger, \Big[1 - \frac{1}{(1 + \not{A}\frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger)} \frac{1}{(1 + \overline{D}\frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger)} \overline{D} \frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger\Big] \psi\Big) \tag{3.26}$$

Using the identities

$$\begin{gathered} \frac{1}{(1 + \overline{D}\frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger)} \overline{D} = \overline{D} \frac{1}{(1 + \frac{1}{\overline{D}^\dagger \overline{D}} \not{A}^\dagger \overline{D})} \\ \Big(\psi^\dagger, \Big[\frac{1}{(1 + \not{A}\frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger)} \overline{D}\Big] \phi\Big) = -\Big(\psi^\dagger, \Big[\not{A} \frac{1}{(1 + \frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger \not{A})}\Big] \phi\Big) \end{gathered} \tag{3.27}$$

(for any matrix ϕ), we see that the ratio

$$\frac{sdet P^\dagger P}{sdet(\Psi^\dagger, \Psi)} = \frac{sdet \overline{D}^\dagger \overline{D}}{sdet(\psi^\dagger, \psi)} \big|sdet(1 + \frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger \not{A})\big|^2 \tag{3.28}$$

is manifestly factorized in the supermoduli. Therefore, the jacobian will be

$$sdet(\Psi,\eta)\Big(\frac{sdet\overline{D}^\dagger\overline{D}}{sdet(\psi^\dagger,\psi)}\Big)^{1/2} sdet\big(1+\frac{1}{\overline{D}^\dagger\overline{D}}\overline{D}^\dagger A\big) \tag{3.29}$$

The ratio $\frac{sdet\overline{D}^\dagger\overline{D}}{sdet(\psi^\dagger,\psi)}$ in (3.29) may be defined using ζ-function regularization. As is well-known, if the ψ vary holomorphically, then the ratio of these superdeterminants is a holomorphic square up to the usual liouville factor [47][57][58][59][60][61]. When the anomalies cancel we may ignore this factor and take the holomorphic squareroot. The last factor may be written as

$$exp(-str\sum\frac{1}{2n}(\mathcal{B}^n+\overline{\mathcal{B}}^n)) \tag{3.30}$$

where

$$\mathcal{B}=\begin{pmatrix} -\frac{1}{2}\frac{1}{\overline{\partial}_2}\overline{\chi}\frac{1}{\overline{\partial}_{3/2}}(-\partial\overline{\chi}+\frac{1}{2}\overline{\chi}\partial) & 0 \\ 0 & -\frac{1}{\overline{\partial}_{3/2}}(-\partial\overline{\chi}+\frac{1}{2}\overline{\chi}\partial)\frac{1}{2}\frac{1}{\overline{\partial}_2}\overline{\chi} \end{pmatrix} \tag{3.31}$$

is an operator on C^∞ sections $\mathcal{B}:\Gamma(K^{-1}\oplus K^{-1/2})\to\Gamma(K^{-1}\oplus K^{-1/2})$ and $\overline{\partial}_2^{-1}$ etc. are defined by

$$\begin{aligned}\frac{1}{\overline{\partial}_2}&\equiv\frac{1}{\overline{\partial}_2^\dagger\overline{\partial}_2}\overline{\partial}_2^\dagger\\ \frac{1}{\overline{\partial}_{3/2}}&\equiv\frac{1}{\overline{\partial}_{3/2}^\dagger\overline{\partial}_{3/2}}\overline{\partial}_{3/2}^\dagger\end{aligned} \tag{3.32}$$

Since $\mathcal{B}$ is nilpotent the sum in (3.30) terminates after a finite number of terms. For smooth χ the traces are well-defined and give a rigorous definition of the last factor in (3.29). Similarly, we can define (3.25) and hence the first factor in (3.29) rigorously. In this way we can define carefully the jacobian (3.29).

We note here that a corollary of the above derivation is the superholomorphic factorization of ghost superdeterminants which has been addressed in different ways in [62][7][5]. We have explicitly shown factorization in the supermoduli in (3.29). As we will see in the next subsection, the first and third factors are not holomorphic in the moduli, but their product is, provided the slice is holomorphic. The corresponding definition of the matter

superdeterminant is straightforward, although the corresponding discussion of holomorphic factorization is not, and must be left to future work. (See, however, [5][32].)

C. *Defining ghost correlators*

We can now define the quantities in (3.10). We expand the ghost quantum fields in terms of the zero modes Ψ and the orthogonal eigenmodes of $PP^\dagger$:

$$\begin{aligned}b_{zz}&=\Psi^r_{zz}b^r_0+\Psi^a_{z+}\beta^a_0+\cdots\\ \beta_{z+}&=\Psi^r_{z+}b^r_0+\Psi^a_{zz}\beta^a_0+\cdots\end{aligned} \tag{3.33}$$

where the ellipsis refers to orthogonal modes. Similarily for the antiholomorphic ghosts. Then the basis-independent measure for the ghost zero-modes is

$$(sdet(\Psi^\dagger,\Psi))^{-1/2}\prod_{r,a}db^r_0 d\beta^a_0 \tag{3.34}$$

and one may evaluate

$$\begin{aligned}&\int\prod_{r,a}db^r_0 d\beta^a_0\prod_i(\eta_i,B)\prod_a\delta((\eta_a,B))\\ &=\int\prod_{r,a}db^r_0 d\beta^a_0\prod_i(\eta_i,B)\prod_a\delta(((\eta_a,\Psi^b)\beta^b_0+(\eta_a,\Psi^r)b^r_0+\cdots))\\ &=\int\prod_r db^r_0\frac{1}{det(\eta_a,\Psi^b)}\prod_i(((\eta_i,\Psi^r)-(\eta_i,\Psi^a)(\eta_a,\Psi^b)^{-1}(\eta_b,\Psi^r))b^r_0+\cdots)\\ &=sdet(\Psi,\eta)\end{aligned} \tag{3.35}$$

using the definition in (3.12) and properties of the Dirac delta function. The orthogonal modes lead to $(sdet'P^\dagger P)^{\frac{1}{2}}$, thus reproducing (3.18).

While this discussion suffices to define the ghost path integral, the expansion (3.33) is difficult to work with because of the dependence of Ψ on $\overline{\chi}$. From the point of view of conformal field theory it is more natural to expand in modes of $\overline{D}^\dagger\overline{D}$ and $\overline{D}\overline{D}^\dagger$. For example, separating B into its zeromode and nonzero mode pieces

$$\begin{aligned}B&=B_0+B_\perp\\ &\equiv\psi_\alpha b^\alpha_0+B_\perp\end{aligned} \tag{3.36}$$

we have

$$dB_0 = (sdet(\psi^\dagger, \psi))^{-1/2} \prod db_0^i d\beta_0^p d\bar{b}_0^{\bar{i}} d\bar{\beta}_0^{\bar{p}} \tag{3.37}$$

so it is clear that we will obtain the middle factor in (3.29) after integration over $B_\perp$. The last factor, written as (3.30) may be interpreted as correlation functions of the ghost supercurrent where we identify the parametrices with the ghost correlators through:

$$\begin{aligned}
\langle c(x)b(y)\rangle_\perp &\equiv \frac{1}{\bar{\partial}_2^\dagger \bar{\partial}_2} \bar{\partial}_2^\dagger(x,y) \in K^{-1}|_x \otimes K^2|_y \\
&\sim \frac{1}{\pi}\frac{1}{x-y} \\
\langle \gamma(x)\beta(y)\rangle_\perp &\equiv \frac{1}{\bar{\partial}_{3/2}^\dagger \bar{\partial}_{3/2}} \bar{\partial}_{3/2}^\dagger(x,y) \in K^{-1/2}|_x \otimes K^{3/2}|_y \\
&\sim \frac{1}{\pi}\frac{1}{x-y}
\end{aligned} \tag{3.38}$$

with similar relations involving the antiholomorphic ghosts. We see that we just obtain the correlators of the ghost supercurrent.

These correlators are *not* convenient for use in conformal field theory. For example, they are not meromorphic on the riemann surface. In conformal field theory we must compute correlators in the presence of insertions of operators soaking up background charge. These latter correlators are meromorphic. We now show that the factor $sdet(\Psi, \eta)$ can be interpreted as a correction changing the correlators (3.38) into those of conformal field theory. To do this we return to the ghost expression (3.10). Defining an index $\{I\} = \{i, a\}$ we may rewrite (3.10) as

$$\int dBdC \prod_I \delta((\eta_I, B)) e^{(B,PC)} \tag{3.39}$$

since $\delta(b) = b$ for an anticommuting object. Using the integral representation for the delta function we get

$$\int d\lambda^I dBdC exp\Big(i \sum \lambda^I(\eta_I, b) + (B, PC)\Big) \tag{3.40}$$

We again expand the quantum fields in a basis of eigenmodes. However this time not in eigenmodes of $(P^\dagger P)$ but of $\overline{D}^\dagger D$.

We may then shift $B_\perp, C$ in the action to obtain

$$\int d\lambda^I dBdC exp\Big(i b_0^\alpha(\psi_\alpha, (1 - \not{A} P^{-1})\eta_I)\lambda^I + (B_\perp, PC)\Big) \tag{3.41}$$

where $P^{-1} : (ker \overline{D}^\dagger)_\perp \to \Gamma(K^{-1} \oplus K^{-1/2})$ is defined by

$$P^{-1} = \frac{1}{1 + \frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger \not{A}} \frac{1}{\overline{D}^\dagger \overline{D}} \overline{D}^\dagger \tag{3.42}$$

and satisfies $((1 - \Pi_0)P)P^{-1} = 1 - \Pi_0$. The integral over b_0^J, λ^I exactly reproduces the factor $sdet(\Psi, \eta)$, and we have already obtained the other two factors in (3.29).

The important point here is that all the (valid) manipulations of superconformal ghost systems can in principle be rigorously justified through manipulations of parametrices of differential operators. One trivial example of this is the set of OPE's of b, c, β, γ which follow from (3.38). Similarly, the holomorphy of the correlators of β, γ and b, c in the presence of background charges shows the validity of the use of the equations of motion—and hence of contour deformation—as long as no two operators have coinciding arguments. A slightly less trivial example is the pair of OPE's

$$\begin{aligned}
\beta(z)\delta(\beta(w)) &\sim (z-w)\beta'(w)\delta(\beta(w)) \\
\gamma(z)\delta(\beta(w)) &\sim (z-w)^{-1}\delta'(\beta(w))
\end{aligned} \tag{3.43}$$

These follow from the OPE of β with γ, together with the general properties of the δ-function, which in turn may be derived from the integral representation used in (3.40). Finally, through the manipulations that lead from (3.40) to (3.29) we may justify (3.43) using rigorous operator techniques.

D. *Special Slices*

Having indicated how the methods we will use could be made rigorous we will proceed in the rest of the paper using the techniques of CFT. As mentioned, the traces in, e.g., (3.30) are well-defined for smooth $\bar{\chi}$. If we allow $\bar{\chi}$ to have δ-function support then the individual terms do not behave well, but the expressions in CFT seem to make sense, indicating that the whole expression for the measure has a good limit. Therefore, we choose a good global slice, or set of slices of the kind discussed in section 2.F, assume it can be chosen to be modular invariant, and simply substitute into (3.10). In particular, we take g to be independent of ς, so that $\eta_{a\bar{z}}{}^{z} = \eta_{az}{}^{\bar{z}} = 0$, and we take the gravitinos to be linear in the supermoduli and to have delta function support:

$$\begin{aligned} \bar{\chi}_{\bar{z}}{}^{+} &= \sum_{a=1}^{2g-2} \varsigma^a \delta^{(2)}(v - v_a) \\ \chi_z{}^{-} &= \sum_{a=2g-1}^{4g-4} \varsigma^a \delta^{(2)}(v - v_a) \end{aligned} \tag{3.44}$$

where $v_a = v_t(q_a(t))$. Then $\eta_{a\bar{z}}{}^{+} = \delta^{(2)}(v - v_a)$ $(a = 1, \cdots, 2g-2)$, and $\eta_{az}{}^{-} = \delta^{(2)}(v - v_a)$ $(a = 2g-1, \cdots, 4g-4)$. Taking the support of $\chi, \bar{\chi}$ to be different, the quartic term in the action (3.1) can be ignored.[20] Using (3.10) the partition function becomes after integration over ς^a,

$$\Lambda = \int \prod_{i=1}^{6g-6} dt^i D[XBC]e^{-S_0} \prod_{a=1}^{4g-4} \delta(\check{\beta}(v_a)) \left[\prod_{a=1}^{4g-4} (\check{T}_F + \frac{\partial}{\partial \varsigma^a}) \prod_{r=1}^{6g-6} (\eta_r, B) \right]_{\{\varsigma^a = 0\}} \tag{3.45}$$

where S_0 denotes (3.1) with $\chi = \bar{\chi} = 0$, and $\check{\beta}(v_a)(\check{T}_F(v_a))$ is defined to be $\beta(v_a)(T_F(v_a))$ for $a = 1, \cdots, 2g-2$, and $\bar{\beta}(v_a)(\bar{T}_F(v_a))$ for $a = 2g-1, \cdots, 4g-4$. The formula (3.45) was the starting point in the important paper [6].

At this point we must find the proper CFT interpretation of the operators $\delta(\beta), \delta(\bar{\beta})$ and give a prescription for calculating correlation functions of such operators on any riemann surface [6]. For this we use the bosonization prescription of Friedan, Martinec and

[20] One can take the limit $p_{a+2} \to p_a$ $(a = 1, 2)$ to recover the formulae in [11].

Shenker [1]. It was shown by these authors that on the sphere, the β, γ system may be replaced by a pair of free fermions ξ, η, and a free boson ϕ, with the identification,

$$\begin{aligned} \beta &= \partial \xi e^{-\phi} \\ \gamma &= \eta e^{\phi} \end{aligned} \tag{3.46}$$

From this we may derive the operator products,

$$\begin{aligned} \beta(z) e^{q\phi(w)} &\sim (z-w)^q \partial\xi(w) e^{(q-1)\phi(w)} \\ \gamma(z) e^{q\phi(w)} &\sim (z-w)^{-q} e^{(q+1)\phi(w)} \eta(w) \end{aligned} \tag{3.47}$$

On the other hand, we have seen in the previous subsection that we have the operator products (3.43). Thus we see that $\delta(\beta)$ develops the same singularities near β and γ as $e^{\phi(w)}$.

On higher genus surfaces, the bosonization prescription given in eq.(3.46) is no longer true if we interprete ϕ, ξ, η as independent free fields. However, if we define the operators $e^{q\phi}$ through their operator product expansion(3.47), then the correlators involving products of the operators $e^{q\phi}$, β and γ are completely determined [63][64] via the stress tensor method [65][66][67]. Thus comparing eqs.(3.47)and (3.43)we see that we may identify $\delta(\beta)$ with e^{ϕ} for calculation of any correlation function involving $\delta(\beta)$. A similar identification may be made for the $\bar{\beta}$, $\bar{\gamma}$ system.

We should mention at this stage that the operator $e^{\phi(z)}$ develops a pole near $T_F(w)$ due to the presence of a term proportional to $b(w)\gamma(w)$ in the ghost stress tensor. Thus we must define this operator product through a particular normal ordering prescription. We choose the prescription of [1] to define a BRST invariant normal ordering prescription. Namely, we define,

$$\begin{aligned} : e^{\phi} T_F : &= Y(z) \equiv \{Q_B, \xi(z)\} \\ &= c\partial\xi + e^{\phi} T_F^{matter} - \frac{1}{4}\partial\eta e^{2\phi} b - \frac{1}{4}\partial(\eta e^{2\phi} b) \end{aligned} \tag{3.48}$$

where Q_B is the BRST charge. $Y(z)$ is BRST invariant, since $Q_B^2 = 0$. A justification for choosing this particular normal ordering prescription may be given by noting that, formally,

$$\delta(\beta)T_F = \delta(\beta)[Q_B, \beta] = [Q_B, H(\beta)] \tag{3.49}$$

where H is the Heaviside step function. Thus the operator $\delta(\beta)T_F$ is formally BRST invariant, and we preserve this feature in the regulated theory by defining it as in eq.(3.48). Comparing (3.48)and (3.49)we see that ξ may be identified with $H(\beta)$[6].

Finally, we see what happens when we further specialize the measure to the slice (2.42). This means $\eta_{r\bar{z}}{}^{+} = \eta_{rz}{}^{-} = 0$. Then we have

$$\frac{\partial}{\partial\varsigma^a}(\eta_r, B) = 0 \tag{3.50}$$

and the integrand reduces to $\langle \prod Y(z_a)\rangle$. If we choose complex coordinates to describe the moduli space, and take $\eta_{iz}{}^{\bar{z}} = \eta_{\bar{i}\bar{z}}{}^{z} = 0$, only the zero-modes of the X-field do not split chirally. Since the correlator

$$\langle \partial X_1 \overline{\partial} X_3\rangle\langle \partial X_2 \overline{\partial} X_4\rangle = \omega_1 \cdot (Im\tau)^{-1} \cdot \bar{\omega}_3 \omega_2 \cdot (Im\tau)^{-1} \cdot \bar{\omega}_4 \tag{3.51}$$

(where ω are normalized abelian differentials and τ is the period matrix on the riemann surface) is a positive measure in the limit $p_3 \to p_1, p_4 \to p_2$, we find that in this limit the path integral measure is positive semidefinite. More specifically if we choose a holomorphic slice it is a sum of absolute squares $|A|^2 + |B|^2$. As shown in [11], for generic choices of points p_1, p_2 these semidefinite terms are in fact positive definite and thus appear to lead to a nonvanishing cosmological constant. We are now ready to begin rectifying this alarming state of affairs.

We have argued in the previous section that a holomorphic slice satisfying (2.42) connot even be defined by local patching if we only allow diffeomorphisms across the

patches of moduli space. If we insist upon using δ-function gravitinos and a choice of the metric that gives $\eta_{iz}{}^{\bar{z}} = \eta_{\bar{i}\bar{z}}{}^{z} = 0$, we must let the points q_a move. So:

$$\eta_{i\bar{z}}{}^{+} = \varsigma^a \frac{\partial u_a^\mu}{\partial t^i} \partial_\mu \delta^{(2)}(u - u_a) \tag{3.52}$$

where $u_a^\mu = (u(q_a(t)), \bar{u}(q_a(t)))$ is the image of q_a in the u coordinate system and $\frac{\partial u_a^\mu}{\partial t^i}$ is computed with the prescription of section II. If the points q_a move then (η_i, B) will have terms proportional to $\varsigma^a (a = 1 \dots 4)$, and the measure is no longer positive semidefinite. In fact, it may be shown to have the general form

$$|A|^2 + |B|^2 - |C|^2 - |D|^2 + Re(E) \tag{3.53}$$

where C, D, E are proportional to $\partial u_a^\mu/\partial t^i$. (3.53) is not positive, and, in the following sections we show that it is in fact a total divergence with zero boundary integral for appropriate spacetimes. (These include R^{10}.) On the other hand using reparametrization invariance we can always set $\partial u_a^\mu/\partial t^i = 0$ but this in general will lead to nonvanishing $\eta_{iz}{}^{\bar{z}}$ and $\eta_{\bar{i}\bar{z}}{}^{z}$. As we discussed in section II.C, if the holomorphy constraint is relaxed the integrand is not positive semidefinite, even if $\partial u_a^\mu/\partial t^i = 0$.

4. The Vacuum Amplitude as an Exact Differential

In this section we show that, for any genus and any tree level supersymmetry preserving background the cosmological constant is the integral of a total divergence.

As we have seen in the last section, after integration over the supermoduli our density on moduli space takes the compact form

$$\left[\prod_{a=1}^{4g-4} (\check{Y}_a + e^{\check{\phi}_a} \frac{\partial}{\partial\varsigma^a}) \prod_{i=1}^{6g-6} (\eta_i, B)\right]_{\{\varsigma^a=0\}} \tag{4.1}$$

For the special choice of basis of the super-beltrami differentials given in (3.44) we can make this more explicit by noting that

$$e^{\check{\phi}_a}\frac{\partial}{\partial\varsigma^a}(\eta_i,B)=\frac{\partial z_a^\mu}{\partial t^i}\partial_\mu\check{\xi}(z_a)\equiv\partial_i\check{\xi}_a \tag{4.2}$$

where $\check{\phi}^a(\check{\xi}^a)$ stands for $\phi(z_a)(\xi(z_a))$, $a=1,\cdots,2g-2$, and $\bar{\phi}(z_a)(\bar{\xi}(z_a))$, $a=2g-1,\cdots,4g-4$. In deriving (4.2) we have assumed that $\frac{\partial g}{\partial\varsigma^a}=0$.[21]

A significant simplification of the algebra that follows can be accomplished by defining

$$D_i\equiv\frac{\delta}{\delta(\eta_i,b)} \tag{4.3}$$

which is to be considered as an operator on antisymmetric polynomials in (η_i,b); where $(\eta_i,b)=(\eta_i,B)_{\{\varsigma^a=0\}}$. Note that D_j is purely a book-keeping device, which, acting on some expression, removes a factor of (η_j,b) if it is present, otherwise annihilates it. For manipulations that we shall carry out below it is useful at this stage to note that this operator satisfies the following simple properties

1.) $\{D_i,D_j\}=0$

2.) By definition, $D_j(\frac{\partial\eta_k}{\partial t^i},b)=0$, similarly, $D_j(\eta_k,\{Q_B,b\})=0$, where $Q_B=\oint j_B$ is the BRST operator. (This is true for any other operator as well). Thus, trivially, $[D_i,\partial_j]=\{D_i,Q_B\}=0$, where $\partial_j=\frac{\partial}{\partial t^j}$.

3.) Note also that the operator $\oint j_B$ acting on polynomials in (η_i,b) is equivalent to $\sum_j(\eta_j,T)D_j$, since $\delta_{BRST}(\eta_i,b)=(\eta_i,T)$. Similarily ∂_i acting on polynomials in (η_j,b) is equivalent to $(\frac{\partial\eta_j}{\partial t^i},b)D_j$. (The implicit dependence of b on moduli is accounted for by an insertion of (η_i,T) as we shall see below).

With the above definitions and properties in mind the cosmological constant (3.45) is most conveniently written as

$$\Lambda=\int\prod_{i=1}^{6g-6}dt^i\int D[XBC]e^{-S_0}\prod_{a=1}^{4g-4}(\check{Y}_a+\partial_i z_a^\mu\partial_\mu\check{\xi}^a D_i)\prod_{i=1}^{6g-6}(\eta_i,b) \tag{4.4}$$

[21] Since in the remainder of the paper we do not make essential use of frame indices, we shall change to a more standard notation and adopt z as the coordinate system which we previously defined as v_t. In this notation z_a is nothing other than $v_t(q_a(t))$.

If we explicitly evaluate the above correlator with the help of (5.30) and (5.31) below we shall find that the correlator has poles whenever $z_a=z_b(a,b=1,\cdots,2g-2)$ (or $a,b=2g-1,\cdots,4g-4$) or when $\vartheta[\delta](\sum_{a=1}^{2g-2}\vec{z}_a-2\vec{\Delta})=0$ (similarly when $\vartheta[\delta](\sum_{a=2g-1}^{4g-4}\vec{\bar{z}}_a-2\vec{\bar{\Delta}})=0$). The condition $det\upsilon^a(z_b)\neq 0$ discussed in section II, is precisely the condition that ensures that the trajectory of $q_a(t)$ avoids these singularities. This can be readily seen from the bosonization formulae [68][69][70][71][72].

We are now ready to show that the measure is a total derivative in moduli. The argument has three parts.

A. *The "dilaton trick"*

We use the method introduced in [73][74] to calculate the vacuum amplitude in the heterotic string theory. We shall see here that this same method can be used to show that the measure (4.4) for the superstring is also a total derivative on the moduli space.

Consider the following amplitude[22]

$$\int_{\Sigma-\{z_a\}}d^2y\langle V\rangle\equiv\int_{\Sigma-\{z_a\}}d^2y\langle\partial X^\mu(y)\bar{\partial}X^\mu(y)\rangle \tag{4.5}$$

where $\langle\ \rangle$ stands for the functional integral as defined in (4.4). In this notation the cosmological constant is just given by $\langle I\rangle$, with I the identity operator. As indicated the y integration runs over the riemann surface Σ excluding the points $\{z_a : a=1,\cdots 4g-4\}$ where the gravitino has support.

The self contraction of V above gives $\pi\omega(y)\cdot(Im\tau)^{-1}\cdot\bar{\omega}(y)d\langle I\rangle$, which upon integration over y is just $gd\langle I\rangle$ on a genus g surface, where d is the dimension of uncompactified space-time. If this were the only contribution to (4.5) then up to an overall numerical constant one would write the cosmological constant as

$$\Lambda=\int_{\Sigma-\{z_a\}}d^2y\int[\prod_{i=1}^{6g-6}dt^i](\partial X(y)\bar{\partial}X(y))\prod_{a=1}^{4g-4}(\check{Y}_a+\partial_i z_a^\mu\partial_\mu\check{\xi}^a D_i)\prod_{i=1}^{6g-6}(\eta_i,B) \tag{4.6}$$

[22] In our conventions throughout the rest of the paper, $d^2z=\frac{i}{2\pi}dz\wedge d\bar{z}\equiv\frac{1}{\pi}dxdy$.

To ensure that (4.6) is true we must therefore verify that the contractions of V with the other fields in the measure (e.g. $\check{Y}(z_a)$) yield no contribution. Consider first the contractions of V with Y_a, Y_b for a, b holomorphic indices (i.e. $a, b = 1, \cdots, 2g-2$). Then the y-dependent terms are of the form

$$(\underbrace{\partial X(y)\partial X(z_a)})\,(\underbrace{\bar\partial X(y)\partial X(z_b)}) = \partial_y[\,(\underbrace{X(y)\partial X(z_a)})\,(\underbrace{\bar\partial X(y)\partial X(z_a)})] \tag{4.7}$$

The total derivative in (4.7) could contribute at the boundary of the y integration only if the correlator within the square brackets develops a singularity of the form $(\bar y - \bar z_a)^{-1}$. It is not difficult to verify that no such singularity exists. Similarly, no contribution arises when a, b are both antiholomorphic indices. When a is holomorphic and b is antiholomorphic we find two contractions:

$$(\underbrace{\partial X(y)\partial X(z_a)})\,(\underbrace{\bar\partial X(y)\bar\partial X(z_b)}) + (\underbrace{\partial X(y)\bar\partial X(z_b)})\,(\underbrace{\bar\partial X(y)\partial X(z_a)}) \tag{4.8}$$

The second term is nonsingular, and the integral over y gives

$$\pi\omega(z_a)\cdot(Im\tau)^{-1}\cdot\bar\omega(z_b) \tag{4.9}$$

The first term may be written as

$$\partial_y[\,(\underbrace{X(y)\partial X(z_a)})\,\underbrace{\bar\partial X(y)\bar\partial X(z_b)})] \tag{4.10}$$

In this case the correlator involved possesses a simple pole in $\bar y$ at $\bar z_b$. The boundary integral consequently has a contribution near $y \cong z_b$, given by the residue of the simple pole. More explicitly it is given by

$$-\pi\underbrace{\bar\partial X(z_b)\partial X(z_a)} \quad \sim -\pi\omega(z_a)\cdot(Im\tau)^{-1}\cdot\bar\omega(z_b) \tag{4.11}$$

which exactly cancels the contribution (4.9). This establishes the validity of (4.6).

B. *SUSY contour deformation*

The advantage of the representation (4.6) for the cosmological constant is that it enables us to express Λ as a contour integral of the right-handed space-time supersymmetry current J_α around some vertex operator: (A set up which, as we shall see, is very useful for calculating Λ). More specifically, we shall proceed by rewriting (4.6) at this stage as

$$\delta_\alpha^\beta\Lambda = \int_{\Sigma-\{z_a\}} d^2y \oint_y \frac{dx}{2\pi i}\langle J_\alpha(x)V^\beta(y)\rangle \tag{4.12}$$

where

$$J_\alpha(x) = e^{-\frac{1}{2}\phi}\hat S^+ S_\alpha \tag{4.13}$$

and

$$\begin{aligned} V^\alpha(y) =&\bar\partial X^\mu(\gamma_\mu)^{\alpha\dot\beta}[e^{\frac{1}{2}\phi}\partial X^\nu(\gamma_\nu)_{\dot\beta\gamma}\hat S^- S^\gamma \\ &+\frac{1}{2}e^{\frac{3}{2}\phi}\eta b\hat S^- S_{\dot\beta} + e^{\frac{1}{2}\phi}S_{\dot\beta}\lim_{w\to z}(w-z)^{1/2}T_F^{int}(w)\hat S^-(z)] \\ \equiv&(\gamma^\mu)^{\alpha\dot\beta}\bar\partial X^\mu\tilde V_{\dot\beta} \end{aligned} \tag{4.14}$$

In this expression, $S_\alpha, S_{\dot\beta}$ are four dimensional spin fields of positive and negative chirality and $\hat S^\pm$ are the internal spin fields which exist if space-time supersymmetry in 4-d exists. (In the free case—uncompactified internal space— $\hat S^\pm$ reduce to $e^{\pm i(\phi^1+\phi^2+\phi^3)/2}$ where ϕ^i are related to the internal fermions through standard bosonization $\psi^i \sim e^{i\phi^i}$). Finally $T_F^{int}(w)$ is the super partner of the world-sheet stress tensor of the internal theory. (In flat space $T_F \sim \psi^i\partial X^i$). The contour integral of $J_\alpha(x)$ around the first term in (4.14) gives the required $\partial X^\mu\bar\partial X^\mu$ term after using the fact that $\langle\partial X^\mu\bar\partial X^\nu\rangle \sim \delta^{\mu\nu}\langle\ \rangle$. The contour integral around the second and the third term in (4.14) on the other hand vanishes since $J_\alpha(x)$ is non-singular around these operators. These have been included in $V^\alpha(y)$ in order to make it a $BRST$ invariant (up to total derivatives) vertex operator. It is important to emphasize that in writing (4.14) we took into consideration the fact that we shall be working on general string vacuua which possess at least the space-time supercurrents $J_\alpha = e^{-\frac{\phi}{2}}\hat S^+ S_\alpha$

and $J_{\dot{\alpha}} = e^{-\frac{\phi}{2}\hat{S} - S_{\dot{\alpha}}}$ of positive and negative 4-dimensional chirality respectively. [23]Ten dimensional flat space time is then viewed as a special case of this general vacuum setting.

Substituting the expressions of $J_\alpha(x)$ and $V^\alpha(y)$ in (4.6) we may explicitly evaluate the relevant correlators as functions of x, the argument of $J_\alpha(x)$. After summing over spin structures, the correlators may be shown to be periodic on the riemann surface as functions of x, and have a pole at $x = y$, as expected from the operator product expansion. If this were the only pole, then in (4.12) we would deform the x contour and shrink it to a point, thereby showing that the right hand side of (4.12) vanishes. From the operator product expansion of $J_\alpha(x)$ with $Y(z_a)$ no singularities are expected at $\{z_a\}$ and none are found by explicit calculation. However, as was first pointed out in ref. [6], the same kind of explicit calculations reveal that the supercurrent $J_\alpha(x)$ has in general spurious poles, i.e. poles not dictated by the operator product expansion.[24]In the present case these poles occur at the zeros of the function $f(x) = \prod_\delta \vartheta[\delta](\frac{1}{2}\vec{y} - \frac{1}{2}\vec{x} + \sum_{a=1}^{2g-2} \vec{z}_a - 2\vec{\Delta})$ (see eq. (5.30)). Let us call this set of points $\{r_l\}$. On a genus $g \geq 2$ riemann surface there are $2^{2g-2}g$ such points [73]. A consequence of the presence of these spurious poles is that (4.12) can now be written as a sum of residues:

$$\Lambda = -\sum_{l=1}^{2^{2g-2}g} \int d^2y \oint_{r_l} \frac{dx}{2\pi i} \langle J_\alpha(x) V^\alpha(y) \rangle \tag{4.15}$$

where no sum over α is implied. (We adopt this convention throughout the paper.)

What we shall show next is that the residues in (4.15) are total derivatives on the moduli space, following a treatment similar to the one used in [6]. To do that let us first consider a new correlator $\langle\ \rangle'$ which is defined in the same way as $\langle\ \rangle$, except that in eqn. (4.6) z_1 is replaced by some other point $\tilde{z}_1$. Then $\langle J_\alpha(x) V^\alpha(y) \rangle'$ as a function of x will have spurious poles at the zeros of $f(x) = \prod_\delta \vartheta[\delta](\frac{1}{2}\vec{y} - \frac{1}{2}\vec{x} + \tilde{z}_1 + \sum_{a\neq 1}^{2g-2} \vec{z}_a - 2\vec{\Delta})$. Let

[23] Note that we are assuming the existence of only holomorphic supersymmetry current in the compactified theory. In other words, we only need the existence of $N = 1$ supersymmetry.

[24] An interpretation of the origin of these poles has been given in ref. [73].

us call these points $\{r_l'\}$. For a general $\tilde{z}_1$, $\{r_l\} \cap \{r_{l'}\} = \emptyset$ and $\langle J_\alpha(x) V^\alpha(y) \rangle'$ does not possess any poles near $\{r_l\}$. Thus we may express the cosmological constant (4.15) equally well as

$$\Lambda = \sum_{l=1}^{2^{2g-2}g} \int d^2y \oint_{r_l} \frac{dx}{2\pi i} \{ \langle J_\alpha(x) V^\beta(y) \rangle - \langle J_\alpha(x) V^\beta(y) \rangle' \} \tag{4.16}$$

C. *Demonstration of total derivative*

Consider the path integral in (4.16). This may be written as,

$$\Lambda = -\sum_{l=1}^{2^{2g-2}g} \int \prod_{i=1}^{6g-6} dt^i \int D[XBC] \int d^2y \oint_{r_l} \frac{dx}{2\pi i} e^{-S_0} J_\alpha(x) V^\alpha(y) \xi(z_0) \bar{\xi}(\bar{z}_0)$$
$$(Y_1 - Y_{\tilde{1}} + \partial_i(\xi_1 - \xi_{\tilde{1}}) D_i) \prod_{a\neq 1}^{4g-4} (\check{Y}_a + \partial_i \check{\xi}_a D_i) \prod_{i=1}^{6g-6} (\eta_i, b) \tag{4.17}$$

In (4.17) we have explicitly exhibited $\xi(z_0), \bar{\xi}(\bar{z}_0)$ needed to soak up the $\xi, \bar{\xi}$-zero modes in the reducible algebra. To calculate (4.17) we first notice that, by definition,

$$Y(z_1) - Y(\tilde{z}_1) = \{Q_B, \xi(z_1) - \xi(\tilde{z}_1)\} \tag{4.18}$$

Although at this stage we can choose Q_B to be the $BRST$ charge associated with the right handed (holomorphic) sector, for later convenience we shall take it to be the sum of the $BRST$ charges associated with the right and left handed sectors. Expressing $\{Q_B, \xi(z_1) - \xi(\tilde{z}_1)\}$ as a contour integral of the $BRST$ current around the points z_1 and $\tilde{z}_1$, we may deform the $BRST$ contour and express the partition function as a sum of residues at the various poles of the $BRST$ current. The poles of j_B can be inferred from the following commutation relations

$$\begin{aligned}
[Q_B, Y(z_a)] &= 0 \\
\{Q_B, \partial_i \xi(z_a)\} &= \partial_i Y(z_a) \\
\{Q_B, (\eta_i, b)\} &= (\eta_i | T) \\
&= -\frac{\partial S_0}{\partial t^i} \\
\{Q_B, J_\alpha(x)\} &= \partial_x (c(x) J_\alpha(x)) \\
\{Q_B, V^\alpha\} &= \partial_{\bar y} V_1^\alpha(y) + \partial_y V_2^\alpha(y)
\end{aligned} \tag{4.19}$$

where

$$V_1^\alpha(y) = (\bar c \bar\partial X^\mu + \bar\gamma \bar\psi^\mu)(\gamma_\mu)^{\alpha\dot\beta} \tilde V_{\dot\beta} \tag{4.20}$$

with $\tilde V_{\dot\beta}$ defined in eq. (4.14) and

$$V_2^\alpha(y) = (\gamma_\mu)^{\alpha\dot\beta} \bar\partial X^\mu(y) \lim_{w\to y} \{Y(w) c(y) e^{-\frac{1}{2}\phi(y)} \hat S^-(y) S_{\dot\beta}(y)\} \tag{4.21}$$

After deforming the $BRST$ contour and using the above relations for j_B, we find that the expression for cosmological constant (4.17) takes the following form:

$$\begin{aligned}
\Lambda = &- \sum_{l=1}^{2^{2g-2}g} \int \prod_{i=1}^{6g-6} dt^i \int D[XBC] \int d^2y \oint_{\tau_l} \frac{dx}{2\pi i} e^{-S_0} J_\alpha(x) V^\alpha(y) \xi(z_0) \bar\xi(\bar z_0) \\
&\Big[(\xi_1 - \xi_{\bar 1}) \sum_{b\neq 1} \partial_j \check Y_b D_j \prod_{\substack{a=2\\ a\neq b}}^{4g-4} (\check Y_a + \partial_i \check\xi_a D_i) \prod_{k=1}^{6g-6} (\eta_k, b) \\
&+ \partial_j (\xi_1 - \xi_{\bar 1}) D_j \prod_{a=2}^{4g-4} (\check Y_a + \partial_i \check\xi_a D_i) \prod_{k=1}^{6g-6} (\eta_k, b) \\
&+ (\xi_1 - \xi_{\bar 1}) (\sum_j (\eta_j, T) D_j) \prod_{a=2}^{4g-4} (\check Y_a + \partial_i \check\xi_a D_i) \prod_{k\neq j}^{6g-6} (\eta_k, b) \Big] \\
&+ W_2
\end{aligned} \tag{4.22}$$

where W_2 is the residue of the $BRST$ current at $V^\alpha(y)$. From (4.19) this is a total derivative in y and $\bar y$. In appendix C the explicit expression for W_2 is exhibited. There we

also prove that W_2 is a total derivative in the moduli. The residue of the $BRST$ current at $\xi(z_0)$ and $\bar\xi(\bar z_0)$ vanishes. This is due to the fact that nowhere in the resulting expression does there exist any ξ factor that could be used to soak up the ξ zero mode and hence the residue vanishes identically by the ξ zero mode. An insertion of (η_j, T) in the correlator generates a factor of $-\frac{\partial S_0}{\partial t^j}$. Turning our attention to the rest of the terms in (4.22) we can use properties (1-3) of the D_i listed above, and the identity [25] $(\partial_i \eta_k, b) = (\partial_k \eta_i, b)$, to see that they all combine to form a total derivative on moduli space. More specifically the partition function in (4.22) can finally be written as:

$$\Lambda = \int \prod_{i=1}^{6g-6} dt^i \sum_j \Big(\frac{\partial}{\partial t^j} M_j + \frac{\partial}{\partial t^j} F_j \Big) \tag{4.23}$$

where M_j is the density

$$\begin{aligned}
M_j = - \sum_{l=1}^{2^{2g-2}g} \int D[XBC] e^{-S_0} \int d^2y \oint_{\tau_l} \frac{dx}{2\pi i} \bar\xi(\bar z_0) \xi(z_1) \xi(\tilde z_1) J_\alpha(x) V^\alpha(y) \\
\times D_j \prod_{a=2}^{4g-4} (\check Y(z_a) + \partial_i \check\xi(z_a) D_i) \prod_{k=1}^{6g-6} (\eta_k, b)
\end{aligned} \tag{4.24}$$

and F_j is the contribution of the residue of the $BRST$ charge at $V(y)$ and has been calculated explicitly in Appendix C with the result:

$$\begin{aligned}
F_j = \sum_{b=2}^{2g-2} \sum_l \oint_{\tau_l(z_b)} \frac{dx}{2\pi i} \int D[XBC] \bar\xi(\bar z_0) \xi(z_1) \xi(\tilde z_1) J_\alpha(x) \\
\hat V^\alpha(z_b) D_j \prod_{\substack{a=2\\ a\neq b}}^{4g-4} (\check Y(z_a) + \partial_i \check\xi(z_a) D_i) \prod_{k=1}^{6g-6} (\eta_k, b)
\end{aligned} \tag{4.25}$$

[25] This identity is proved in appendix F.

with $\hat{V}^\alpha = \frac{1}{2}(\gamma_\mu)^{\alpha\dot{\beta}}(\bar{c}\bar{\partial}X^\mu + \bar{\gamma}\bar{\psi}^\mu)e^{\frac{3}{2}\phi}b\hat{S}^- S_{\dot{\beta}}$. In writing down (4.24) and (4.25) we have used the independence of the measure on z_0 to set $z_0 = \tilde{z}_1$.

In going from (4.22) to (4.23) we had to pass $\frac{\partial}{\partial t^i}$ through vertex operators which have no explicit dependence on $\{t^i\}$ e.g. $V^\alpha(y), J_\alpha(x)$ and $\xi(z_0)$. Nevertheless these operators have an implicit dependence on $\{t\}$ and hence further explanation is needed before we can arrive at (4.23). To be more precise, the implicit dependence on $\{t\}$ comes from the fact that the coordinate system which diagonalizes the metric (and in which our vertex operators are exhibited) is t dependent. Let z be the coordinate system that diagonalizes the metric at t and w that that diagonalizes it at a nearby point $t+\delta t$ in moduli. Consider then a correlation function of a set of vertices $\langle\prod_k V(z_k)\rangle_t$ (for simplicity here V is a dim (0,0) operator). The net change in the correlator as we go from t to $t+\delta t$ is not $(\langle\prod_k V(z_k)\rangle_{t+\delta t} - \langle\prod_k V(z_k)\rangle_t)$ but is given by:

$$\delta_t\langle\prod_k V(z_k)\rangle_t = \langle\prod_k V(w_k)\rangle_{t+\delta t} - \langle\prod_k V(z_k)\rangle_t \tag{4.26}$$

which involves explicit $(t \to t+\delta t)$ as well as implicit $(z_k \to w_k)$ change. In appendix F we show that the RHS of (4.26) is given by:

$$\sim \langle\delta t^r(\eta_r, T)\prod_k V(z_k)\rangle_t \tag{4.27}$$

where T is the stress tensor on the world-sheet. This means that the insertion of the stress tensor in a correlator not only accounts for the explicit change in the moduli in a correlator but also encompasses the implicit moduli dependence of the vertices. Consequently one can pass the derivative with respect to the moduli through any operator that has only implicit moduli dependence within a given correlator.

The equations (4.23) (4.24) and (4.25) can be given a more invariant form. One regards correlation functions with k insertions of b as k-forms on moduli space. Then (4.23) is the statement that the measure is the exterior derivative of a $6g-7$ form. Notice that no particular metric on moduli space is needed in making this assertion. It would be interesting to see if the differential form ω defined by

$$\mu = d\omega \tag{4.28}$$

where μ is the measure, is a naturally defined form on moduli space. That ω transforms as a $6g-7$-form is proven in appendix G.

We can make the expression for the density M_j more explicit by carrying out the action of D_i using the properties listed earlier. The answer is most conveniently expressed in a holomorphic coordinate system $(t^\imath, t^{\bar\imath})$ for the moduli space with a holomorphic slice for the graviton, thus $\eta_{\imath z}{}^{\bar z} = \eta_{\bar\imath\bar z}{}^{z} = 0$. Here we shall only exhibit the explicit form of the density $M_j = (M_\imath, M_{\bar\imath} : \imath, \bar\imath = 1,2,3)$ at $g=2$. Similar concrete expressions can be worked out readily from (4.24) and (4.25) at arbitrary genus:

$$\begin{aligned} M_\imath = &\sum_{l=1}^{8}\int D[XBC]\int d^2y \oint_{\Gamma_l}\frac{dx}{2\pi i}e^{-S_0}\bar{\xi}(\bar{z}_0)\xi(z_1)\xi(\tilde{z}_1)J_\alpha(x)V^\alpha(y) \\ &(-)^\imath\Big[Y(z_2)\bar{Y}(z_3)\bar{Y}(z_4)\prod_{\substack{\jmath=1\\ \jmath\neq\imath}}^{3}(\eta_\jmath, b)\prod_{\jmath'=1}^{3}(\bar{\eta}_{\jmath'}, \bar{b}) \\ &+Y(z_2)\Big(\bar{Y}(z_3)\prod_{\substack{\imath'=1\\ \imath'\neq\imath}}^{3}(\eta_{\imath'}, b)\sum_{\bar\jmath=1}^{3}(-)^{\bar\jmath+1}\partial_{\bar\jmath}\bar{\xi}(z_4)\prod_{\substack{\jmath'=1\\ \jmath'\neq\jmath}}^{3}(\eta_{\bar\jmath}, \bar{b}) + z_3 \leftrightarrow z_4\Big)\Big] \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} M_{\bar\imath} = -&\sum_{l=1}^{8}\int D[XBC]\int d^2y \oint_{\Gamma_l}\frac{dx}{2\pi i}e^{-S_0}\bar{\xi}(\bar{z}_0)\xi(z_1)\xi(\tilde{z}_1)J_\alpha(x)V^\alpha(y) \\ &\Big[(-)^{\bar\imath}Y(z_2)\Big(\bar{Y}(z_3)\sum_{\jmath=1}^{3}(-)^{\jmath+1}\partial_\jmath\bar{\xi}(z_4)\prod_{\substack{\jmath'=1\\ \jmath'\neq\jmath}}^{3}(\eta_{\jmath'}, b)\prod_{\substack{\imath'=1\\ \imath'\neq\imath}}^{3}(\eta_{\bar\imath'}, \bar{b}) + z_3 \leftrightarrow z_4\Big)\Big] \end{aligned} \tag{4.30}$$

where we have dropped terms that would vanish by $(b,\bar{b},\phi,\bar{\phi})$ ghost charge conservation. At $g=2$ the terms F^j in (4.25) vanish by ghost charge.

To sum up, the above analysis shows that the partition function at arbitrary genus in arbitrary backgrounds preserving at least $N=1$ four dimensional supersymmetry can be exhibited as a total derivative in the moduli of a definite correlation function. It is worth pointing out that a simple consequence of this is that the terms spoiling holomorphic factorization on moduli space due to X^μ zero modes[15][16][17] (which come from the part of the picture changing operator involving the matter supercurrent) are always total derivatives: Those terms can always be written as correlators involving the full picture changing operator minus correlators involving the ghost part of $Y(z)$. The earlier correlators are just total derivatives as we have seen while the latter do not spoil holomorphic factorization on moduli space since the ghost determinants factorize as in section 3. However it is also important to keep in mind that the lack of holomorphic factorization due to the reasons discussed in detail in section 2 (see in particular subsection G) still remains.

In the next section we shall evaluate the total derivative at genus two in flat ten dimensional space as well as in general compactified vacua.

5. Evaluation of Boundary Terms

Of course, it does not suffice simply to show that the measure is a total derivative. The real issue is whether or not the boundary terms contribute. In this section we analyse these boundary terms for the type II string at genus 2. We shall see that in flat space-time the boundary terms are indeed zero while on a compactified background, they yield, as they should, to a nonvanishing cosmological constant induced by Fayet-Iliopoulos D-terms (if any) arising in these compactified type II models. The anologous calculation for the heterotic string was carried out in ref. [73]. This section is divided into three subsections. In the first two we evaluate the boundary integrals by factorization and also by explicit

computation for strings in R^{10} using genus two ϑ-functions. In the last subsection we evaluate the boundary integrals for compactified strings using factorization.

Before proceeding with the calculations we must discuss two preliminaries. First we describe the relevant boundary of moduli space and second we discuss the dependence of the path integral on the location of $\{z_a\}$ where the gravitino has its support.

In our analysis below we shall examine some correlators in the neighborhood of the boundaries of the moduli space. There are two boundaries of moduli space: Δ_0 describes riemann surfaces where a nontrivial homology cycle shrinks to zero, leaving behind a torus with two marked points, and Δ_1, describes riemann surfaces where a trivial homology cycle shrinks to zero, leaving behind two tori T_1 and T_2 joined at a node. A good parametrization of the neighborhood of both boundaries is given by the plumbing fixture variable t, with $t\to 0$ corresponding to the boundary. Here we shall recall some of the main features of this well-known parametrization.

A family of riemann surfaces near the boundary Δ_1 of moduli space of a genus g riemann surface may be modelled by gluing an annulus $\mathcal{A}$ with modulus t onto two surfaces Σ^I of genus g_1 and Σ^{II} of genus $g-g_1$ at points p_1,p_2. For a genus two surface, Σ_I denotes a torus T_1 with modular parameter τ_1 and Σ_{II} a torus T_2 with parameter τ_2. More precisely, we may choose coordinate patches and local uniformizers:

$$\begin{aligned} U_I &: \{u \,|\, |u|<R\} \quad in \quad \Sigma^I \\ U_{II} &: \{v \,|\, |v|<R\} \quad in \quad \Sigma^{II} \end{aligned} \tag{5.1}$$

for some finite R. Then we may identify the regions

$$\begin{aligned} &\{|t|^{1/2}<|u-p_1|<\epsilon^{\frac{1}{2}}\}\subset U_I \\ &\{|t|^{1/2}<|v-p_1|<\epsilon^{\frac{1}{2}}\}\subset U_{II} \end{aligned} \tag{5.2}$$

with the annulus $\mathcal{A}$ $(|\frac{t}{\epsilon}|^{1/2}<|w|<|\frac{t}{\epsilon}|^{-1/2})$ via

$$\begin{aligned} w &= \frac{t^{1/2}}{u-p_1} \qquad |\frac{t}{\epsilon}|^{1/2}<|w|<1 \\ w &= \frac{v-p_2}{t^{1/2}} \qquad 1<|w|<|\frac{t}{\epsilon}|^{-1/2} \end{aligned} \tag{5.3}$$

where ϵ is an arbitrarily small but fixed real number. The final result is independent of ϵ. In what follows we shall denote by T_1' the surface T_1 with the region $|u - p_1| < \epsilon^{\frac{1}{2}}$ removed. Similarily T_2' will denote the surface T_2 with the region $|v - p_2| < \epsilon^{\frac{1}{2}}$ removed.

If we parametrize our moduli space by the genus two period matrix τ_{ij}. Then in the neighborhood of Δ_1 we have [75]:

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} + t \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} + O(t^2) \tag{5.4}$$

(where α is some known constant). So in a sufficiently small neighborhood around Δ_1 we may choose as our moduli $\frac{\tau_{12}}{\alpha} = t, \tau_{11} = \tau_1, \tau_{22} = \tau_2$.

The abelian differentials on the genus 2 surface to leading order in t can be taken to be

$$\begin{aligned} \omega_1 &\sim du \\ \omega_2 &\sim t \frac{du}{(u - p_1)^2} \end{aligned} \tag{5.5}$$

on U_I and

$$\begin{aligned} \omega_1 &\sim t \frac{dv}{(v - p_2)^2} \\ \omega_2 &\sim dv \end{aligned} \tag{5.6}$$

on U_{II}. In terms of the w-coordinates we have

$$\begin{aligned} \omega_1 &\sim t^{1/2} \frac{dw}{w^2} \\ \omega_2 &\sim t^{1/2} dw \end{aligned} \tag{5.7}$$

Finally we shall construct a set of beltrami differentials $(\eta_t, \eta_{\tau_1}, \eta_{\tau_2})$ dual to the moduli $(dt, d\tau_1, d\tau_2)$ in leading order as $t \to 0$. One easy way to do that is to start from the following standard result [75]:

$$\frac{\partial \tau_{ij}}{\partial t^\rho} = \int \eta_\rho \omega_i \omega_j \tag{5.8}$$

and apply it to $t^\rho = (\tau_1, \tau_2, \tau_{12} = \alpha t)$ with the period matrix as given in (5.4) and the abelian differentials listed above. This implies a set of equations on the dual beltrami differentials. Those equations can be solved by taking $(\eta_{\tau_1}, \eta_{\tau_2}, \eta_t)$ of the form:

$$\begin{aligned} \eta_{\tau_1} &= \frac{1}{(Im\tau_1)} d\bar{z} \otimes \frac{\partial}{\partial z} \quad z \in T_1' \\ \eta_{\tau_2} &= \frac{1}{(Im\tau_2)} d\bar{z} \otimes \frac{\partial}{\partial z} \quad z \in T_2' \\ \eta_t &= -\frac{1}{t} \frac{w}{\bar{w}} \frac{1}{log|\frac{t}{\epsilon}|} d\bar{w} \otimes \frac{\partial}{\partial w} \quad w \in A \end{aligned} \tag{5.9}$$

To leading order in t, the support of $\eta_{\tau_1}, \eta_{\tau_2}$ t is on T_1' and T_2' respectively while that of η_t is on the annulus A.

Now let us briefly consider Δ_0. If we shrink the nontrivial homology cycle $a_1 \to 0$ then the abelian differentials ω_1, ω_2 are, to leading order in t[75]:

$$\begin{aligned} \omega_1 &\to \frac{1}{2\pi i} dz \partial_z log \frac{\vartheta_1(z - p_1)}{\vartheta_1(z - p_2)} \\ \omega_2 &\to dz \end{aligned} \tag{5.10}$$

for z away from the nodes p_1, p_2. The period matrix τ then becomes

$$\begin{aligned} \tau_{11} &\to \frac{1}{2\pi i} logt \\ \tau_{12} &\to \int_{p_1}^{p_2} dz \end{aligned}$$

while τ_2 is the modular parameter of the remaining torus. Again we can apply (5.8) to find the beltrami differentials corresponding to $\frac{\partial}{\partial t}$ etc. In particular we must have

$$\frac{1}{2\pi i} \frac{1}{t} \sim \frac{\partial}{\partial t} \tau_1 = \int \eta_t \omega_1^2$$

We will take, for a fixed radius r,

$$\eta_t \sim \frac{1}{t} (z - p_1)^2 \delta(|z - p_1| - r) d\bar{z} \otimes \frac{\partial}{\partial z} \tag{5.11}$$

One can write down $\eta_{\tau_{12}}, \eta_{\tau_2}$ so that we satisfy (5.8) to leading order in t. We will not need the explicit expressions for these other beltrami differentials in our analysis.

The second issue we have to clarify before starting is the dependence of the path integral on the location of the insertion points $\{z_a\}$. Using manipulations similar to those in ref. [6][10], it may be shown that under a shift in the points z_a the integrand in (4.29) changes by a total derivative in the moduli space. Such total derivatives are not necessarily vanishing on the boundary Δ_1 [10]. In other words, the final value of the path integral depends on the choice of points $\{z_a\}$ at the boundary of moduli space. This is just a manifestation of the fact that because of the integration ambiguity the path integral is not completely basis independent. We have to determine the correct behaviour of the basis of the superbeltrami differentials at the boundary of moduli space. For that one may invoke unitarity or $BRST$ invariance. At genus two this seems to resolve the ambiguity. By generalizing the analysis given in ref. [10] to the type II string it can be seen that the correct prescription at genus two for the choice of points $\{z_a\}$ is to take the points z_1 and z_3 to coincide with p_1 and z_2 and z_4 to coincide with the point p_2 at the boundary Δ_1 of moduli space.

There are several ways of implementing this constraint on the set $\{z_a : a = 1, \cdots, 4\}$: For example, as the surface degenerates, one could first take z_1 and z_3 $\in T_1'$ while z_2 and $z_4 \in T_2'$. Then after extracting the t behaviour of a given correlator, one would take the limits $z_1, z_3 \to p_1$, and $z_2, z_4 \to p_2$. Alternatively one could take z_1 and z_3 (z_2 and z_4) $\in U_I(U_{II})$ with $|z_3 - p_1| \sim |z_1 - p_1| \sim O(t^{\frac{1}{2}})$ ($|z_4 - p_2| \sim |z_2 - p_2| \sim O(t^{\frac{1}{2}})$). Consequently in the w coordinate we have to keep in mind that $\{w(z_a) : a = 1, \cdots, 4\}$ will have absolute value of order one. It is a matter of convenience which way we choose to implement the constraint in our calculations. In subsections A and C we will work with the first way while in B we adopt the latter.

In order to evaluate the boundary term at $t = 0$, we must carefully determine the relative sign between the contributions from different spin structures. This is most easily done by introducing a pair of fields $P^+(z) = e^{-\frac{\phi}{2}}\hat{S}^+ S^\alpha(z)$ and $P^-(w) = e^{\frac{\phi}{2}}\hat{S}^- S_\alpha(w)$ in a given amplitude, and defining the original amplitude as a residue at the pole at $z = w$. By translating z along various homology cycles, we may interpolate between different spin structures and hence determine the relative phases between the contributions from

different spin structures [76][12]. This prescription can always be implemented in the holomorphic sector which has a conserved supersymmetry current $P^+(z)$. In the case of uncompactified superstring theory, (or compactified theories which have a left handed supersymmetry current as well) this prescription may also be used to determine the relative phases between different spin structures in the anti-holomorphic sector. Note that this procedure may be implemented at any stage during the calculation, before, or after taking the $t \to 0$ limit.

A. *Evaluation of boundary terms through factorization:flat space*

We now evaluate the boundary terms in (4.29) assuming the factorization hypothesis for correlation functions of a conformal field theory near the boundary of moduli space.

First consider the behavior near Δ_0. Near the boundary the measure behaves like

$$\partial_t M_t + \partial_{\bar{t}} M_{\bar{t}} \sim dt \wedge d\bar{t} \frac{1}{(log|t|^{-1})^5} (P(t, \bar{t}) + \cdots) \tag{5.12}$$

where P is a Laurent expansion in $t, \bar{t}$ and the ellipsis indicates terms suppressed by higher powers of $Im\ \tau$. It follows that M_t and $M_{\bar{t}}$ have similar expansions. We can only obtain nonzero boundary contributions from terms of the form

$$\int dt \wedge d\bar{t}\partial_t \left(\frac{1}{(log|t|^{-1})^5} \frac{1}{t^\beta \bar{t}^{\beta+1}} \right) \tag{5.13}$$

and this can only contribute for $\beta > 0$. By the physical factorization hypothesis the measure near the boundary can be expanded as

$$\frac{ds}{s^5} \sum \delta m^2 e^{-\frac{m^2}{2} s} \tag{5.14}$$

where δm^2 are one-loop mass corrections. A term with $\beta > 0$ would correspond to the propagation of a tachyon along the long handle. However as is well known the sum over spin structures on the genus two surface allows only states which survive the GSO projection to propagate along the handle. (An explicit demonstration of how this happens will be given in subsection B.) Thus we conclude that the only terms that occur in an expansion of M_t, $M_{\bar{t}}$ are the ones with $\beta \leq 0$. As a result there is no boundary contribution at Δ_0.

We now concentrate on Δ_1. In the region of that boundary we use the parametrization described in detail above and analyse the correlators in (4.29) and (4.30) as $t \to 0$. Furtheremore only derivatives with respect to $t, \bar{t}$ can contribute, so we need only consider $\frac{\partial}{\partial t}M_t + \frac{\partial}{\partial \bar{t}}M_{\bar{t}}$. There can be boundary contributions only if, in an expansion in $t, \bar{t}$ near $t = 0$, M_t has a term $\sim \frac{1}{t}$ or $M_{\bar{t}}$ has a term $\sim \frac{1}{\bar{t}}$. We shall first examine M_t.

Consider the first term in (4.29) given by

$$M_t^{(1)} = \sum_{l=1}^{8} \int d^2y \oint_{\tau_l} \frac{dx}{2\pi i} D[XBC] e^{-S_0} \xi(z_1)\xi(\tilde{z}_1)\bar{\xi}(\tilde{z}_0) J_\alpha(x) V^\alpha(y)$$
$$Y(z_2)\bar{Y}(z_3)\bar{Y}(z_4) \prod_{i \neq t}^{3} (\eta_i, b) \prod_{\bar{j}=1}^{3} (\eta_{\bar{j}}, \bar{b}) \tag{5.15}$$

We only need to extract the $\frac{1}{t}$ term in the neighborhood of Δ_1. A quick way to analyse the behaviour of the correlator in (5.15) in this neighborhood is to use the factorization hypothesis.

To implement factorization it is more convenient to transform further our coordinate system described by the annular coordinate w in (5.3) into the cylindrical coordinate w':

$$w' = \frac{1}{2\pi i} \ln w \tag{5.16}$$

In these coordinates our genus two surface near Δ_1 is degenerating into two tori T_1 and T_2 connected by a long cylinder C. Writing $t = e^{2\pi i s}$, the length of the cylinder is $Im\ s$ and the twist in C is $Re\ s$. Using this picture and the beltrami differentials (5.9) we can see that in the $t \to 0$ limit the expression

$$\prod_{\bar{i}=1}^{3} (\eta_{\bar{i}}, \bar{b}) \to \frac{1}{\bar{t}} \bar{b}_0 (\eta_{\bar{r}_1}, \bar{b})(\eta_{\bar{r}_2}, \bar{b}) \tag{5.17}$$

where $\bar{b}_0$ denotes the zero mode of the $\bar{b}$ ghost on the cylinder C (i.e. relative to the w'-coordinates where $b = \sum_n b_n e^{-inw'}(dw')^2$). Notice because of the support of η's chosen in (5.9) the ghost factors $(\eta_{\bar{r}_1}, \bar{b})$ and $(\eta_{\bar{r}_2}, \bar{b})$ will lie on T_1' and T_2' respectively.

In applying factorization we introduce a complete set of states at the two boundaries of the cylinder and calculate the various correlators of these states with the operators on tori T_1 and T_2. Since inserting a state Φ at the boundary of the disc defined by $|u - p_1| < \epsilon^{\frac{1}{2}}$ on T_1 is equivalent to inserting an operator $\epsilon^{h_\Phi/2}\epsilon^{\bar{h}_\Phi/2}\Phi(p_1)$ on T_1 and similarily for T_2, and since the propagation of such a state on the cylinder gives a factor of $(\frac{t}{\epsilon})^{h_\Phi}(\frac{\bar{t}}{\epsilon})^{\bar{h}_\Phi}$, we get:[26]

$$\Big\langle \prod_i O_i^{(1)}(z_i^{(1)}) \prod_j O_j^{(2)}(z_j^{(2)}) \prod_{i \neq t}^{3} (\eta_i, b) \prod_{\bar{j}=1}^{3} (\eta_{\bar{j}}, \bar{b}) \Big\rangle_{g=2}$$
$$\sim t^{h_\Phi} \bar{t}^{\bar{h}_\Phi - 1} \big\langle \prod_i O_i^{(1)}(z_i^{(1)})(\eta_1, b)(\bar{\eta}_1, \bar{b})\Phi(p_1) \big\rangle_{T_1} \langle 0|\Phi^\dagger \bar{b}_0 \Psi|0\rangle_C$$
$$\big\langle \Psi^\dagger(p_2) \prod_j O_j^{(2)}(z_j^{(2)})(\eta_2, b)(\bar{\eta}_2, \bar{b}) \big\rangle_{T_2} \tag{5.18}$$

where $O_i^{(1)}(z_i^{(1)}), O_j^{(2)}(z_j^{(2)})$ are any set of operators which go to T_1 and T_2 respectively in the limit $t \to 0$ and $(h_\Phi, \bar{h}_\Phi) = (h_\Psi, \bar{h}_\Psi)$ are the conformal dimensions of (Φ, Ψ). Note that in order for $\langle 0|\Phi^\dagger \bar{b}_0 \Psi|0\rangle_C$ to be nonvanishing Φ and Ψ must have the same conformal dimension. A simple consequence of the explicit t factor in (5.18) is that in order to get a contribution from M_t of order $\frac{1}{t}$, we need to find operators Φ, Ψ of conformal dimension $(0,0)$ that have non-vanishing matrix elements in (5.18).

For the matrix elements to be non-vanishing the operators Φ and Ψ must have appropriate ghost factors so as to conserve the various ghost charges on T_1 and T_2 and on C. We also need one factor of ξ and $\bar{\xi}$ on each T_i and C in order to absorb the $\xi, \bar{\xi}$ zero mode.

[26] In this section $\langle\ \rangle$ denotes an ordinary functional integral; $\langle O(\phi)\rangle = \int [d\phi] O(\phi) e^{-S(\phi)}$

This imposes severe constraints on the operators that could possibly go through. Let us first examine the constraints imposed by ghost charge conservation in the left-handed (anti-holomorphic) sector. Recall, in the process of factorizing (5.15) $\bar{Y}(z_3)$ and $\bar{Y}(z_4)$ will lie on T_1 and T_2 respectively. For definiteness we shall take $\bar{z}_0$ to lie on T_1 although the final result is independent of where it lies. Also since $J_\alpha(x)$ and $V^\alpha(y)$ have no factors involving anti-holomorphic ghosts, they will not affect any of the anti-holomorphic ghost charges regardless of where they lie. Consequently to examine the constraints of the anti-holomorphic ghost charge conservation we need only consider the various factors in $\bar{Y}(z_a)$ and the explicit $\bar{b}$ factors in (5.18). For example, in calculating the matrix element on T_2 we need to consider the contribution from each individual term in $\bar{Y}(z_4)$. An explicit expression for $Y(z)$ was given in (3.48). The term $\bar{c}\bar{\partial}\bar{\xi}$ by ghost charge conservation requires $\Psi^\dagger(p_2)$ to be of the form:

$$\Psi^\dagger(p_2) =: \bar{\xi}(p_2)\bar{\eta}(p_2) : U(p_2) \tag{5.19}$$

where $\bar{\xi}$ is needed to absorb the ξ zero mode. U is an operator of dimension $(0,-1)$ which is neutral under all anti-holomorphic ghost charges. It is not difficult to see that no such operator exists. As a matter of fact this analysis is true for the compactified theory as well, since it is based on the structure of the antiholomorphic ghost sector which is the same in all backgrounds. For the term proportional to $\bar{\partial}(e^{2\bar{\phi}}\bar{b}\bar{\eta})$ or $\bar{\partial}\bar{\eta}e^{2\bar{\phi}}\bar{b}$ in $\bar{Y}(z_4)$ ghost charge conservation constrains $\Psi^\dagger(p_2)$ to

$$\Psi^\dagger(p_2) = \bar{c}(p_2)\bar{\partial}\bar{c}(p_2)e^{-2\bar{\phi}(p_2)}\bar{\xi}(p_2)\bar{\partial}\bar{\xi}(p_2)f(p_2) \tag{5.20}$$

where f is an operator of dimension $(0,0)$ and is neutral under the anti-holomorphic ghost charges. It is therefore independent of the antiholomorphic ghosts. The conjugate of this operator is $\bar{c}\bar{\eta}\tilde{f}$ where again $\tilde{f}$ is ghost independent. However a state of this form cannot propagate on the cylinder because of the $\bar{b}$ zero mode. More precisely

$$\bar{b}_0\bar{c}\bar{\eta}\tilde{f}|0\rangle = 0 \tag{5.21}$$

since $\bar{c}(0)|0\rangle = \bar{c}_1|0\rangle$ and $\bar{b}_0|0\rangle = 0$ (In our convention $\langle 0|\bar{c}_0\bar{c}_1\bar{c}_{-1}|0\rangle_C = 1$). As a result the net contribution from this term also vanishes. Note that this argument is also equally true for the compactified theory.

At this stage we are left with the term in $\bar{Y}(\bar{z}_4)$ proportinal to $e^{\bar{\phi}(z_4)}\bar{T}_F^{matter}(z_4)$. In flat space this is given by $e^{\bar{\phi}}\bar{T}_F^{matter} = e^{\bar{\phi}}\bar{\psi}^\mu\bar{\partial}X^\mu$ and the operator which may contribute to the matrix element on the torus T_2 must have the form:

$$\Psi^\dagger(p_2) = \bar{\xi}(p_2)\bar{c}(p_2)e^{-\bar{\phi}(p_2)}\bar{\psi}^\nu(p_2) \times \mathcal{O} \tag{5.22}$$

where $\mathcal{O}$ is an operator of dimension $(0,0)$ from the holomorphic sector. The relevant correlator involving $\bar{\psi}^\mu$ and $\bar{\beta},\bar{\gamma}$ fields may be calculated and the sum over spin structures of the anti-holomorphic sector performed—we find

$$\sim \left(\sum_{\delta_2}\epsilon[\delta_2]\frac{\vartheta[\delta_2](z_4-p_2)(\vartheta[\delta_2](0))^4}{\vartheta[\delta_2](z_4-p_2)}\right)^* \tag{5.23}$$

(In (5.23) * denotes complex conjugation.) $\epsilon[\delta_2] \equiv \epsilon\left[\begin{smallmatrix}a_2\\ b_2\end{smallmatrix}\right]$ may be determined to be $exp(2\pi i(a_2+b_2))$ using the interpolating spin field method, as explained earlier in this section. As a result (5.23) vanishes by a riemann ϑ-identity (for a reference on ϑ-functions and the riemann identities see [77][75].) This proves that $M_t^{(1)}$ yields no boundary contribution.

Let us next analyse the second term in (4.29). One of the relevant terms is

$$M_t^{(2)} \sim \langle\xi(\bar{z}_1)\bar{\xi}(\bar{z}_0)\xi(z_1)J_\alpha(x)V^\alpha(y)Y(z_2)\bar{Y}(z_3)\partial_{\bar{t}}\bar{\xi}(z_4)(\eta_1,b)(\eta_2,b)(\bar{\eta}_1,\bar{b})(\bar{\eta}_2,\bar{b})\rangle \tag{5.24}$$

with another term where z_3 and z_4 are interchanged. Now factorization of the antiholomorphic sector leads to

$$M_t^{(2)} \sim \bar{t}^{\bar{h}_\Phi}\left\langle\bar{\xi}(z_0)(\bar{\partial}(\bar{\eta}e^{2\bar{\phi}}\bar{b}) + \bar{\partial}\bar{\eta}e^{2\bar{\phi}}\bar{b})(z_3)(\bar{\eta}_1,\bar{b})\Phi(p_1)\right\rangle_{T_1}\left\langle\Phi^\dagger(p_2)(\frac{\partial\bar{z}_4}{\partial\bar{t}})\bar{\partial}\bar{\xi}(z_4)(\eta_2,\bar{b})\right\rangle_{T_2} \tag{5.25}$$

where we have applied charge conservation on the $g=2$ surface to drop irrelevant terms in $\bar{Y}(z_3)$. For convenience we have also dropped operators involving X^μ and all holomorphic fields since they do not affect our discussion. From this we find that

$$\Phi(p_1) = \bar{c}\bar{\partial}\bar{c}e^{-2\bar{\phi}}\bar{\partial}\bar{\xi}U \tag{5.26}$$

where U is a $(0,-1)$ operator which is ghost charge neutral in the antiholomorphic sector. Again no such operator exists. This is also true for arbitrary backgrounds. The same analysis applies to the term with z_3 and z_4 interchanged, leading to the same conclusion.

In analysing the terms in M_t proportional to $\frac{\partial z_a}{\partial \tau_i}$ we should keep in mind that $\frac{\partial z_a}{\partial \tau_i} \to 0$ in the $t \to 0$ limit, since $z_a \to p_a$. However, since we are taking the $z_a \to p_a$ limit after $t \to 0$ limit, one might wonder if there is any subtlety in setting $\frac{\partial z_a}{\partial \tau_i}$ to be zero from the beginning. For example, M_t may give divergent boundary contribution in the $t \to 0$ limit before setting $z_a = p_i$, and then the limit will not be well defined. Here we shall show that such things do not happen. Since it makes sense to take z_1 and z_3 to be independent of τ_2 even before we set $z_1 = z_3 = p_1$, we shall only discuss terms proportional to $\frac{\partial z_a}{\partial \tau_1}$ for $a = 1,3$, and $\frac{\partial z_a}{\partial \tau_2}$ for $a = 2,4$. Then, for example, the term in M_t proportional to $\frac{\partial \bar{z}_4}{\partial \bar{\tau}_2}$ is given by an expression analogous to (5.24) with $\partial_{\bar{t}}$ replaced by $\partial_{\bar{\tau}_2}$, and $(\bar{\eta}_2, \bar{b})$ replaced by $(\bar{\eta}_t, \bar{b})$. Factorization in the antiholomorphic sector gives the following expression,

$$\bar{t}^{h_\Phi - 1}\langle\ldots\rangle_{T_1}\langle\ldots\rangle_C\langle\Psi^\dagger(P_2)\bar{\partial}\bar{\xi}(\bar{z}_4)\rangle_{T_2} \tag{5.27}$$

where $\ldots$ denotes terms not relevent for our analysis. In order to get a non-vanishing boundary contribution $\Psi^\dagger$ must have antiholomorphic conformal dimension zero or less. But in order to soak up all the ghost zero modes on T_2 we need $\Psi^\dagger$ to contain an operator : $bc\eta\xi$:, which already has conformal dimension two, which is too high. Hence the boundary contribution vanishes identically. An identical analysis may be carried out for the terms involving $\frac{\partial \bar{z}_3}{\partial \bar{\tau}_1}$ by looking at the correlator on the torus T_1. This argument also is valid for arbitrary backgrounds.

Finally we can analyse the terms in (4.30) in a similar fashion. This time we have to look for $\frac{1}{\bar{t}}$ singularities. One can either look at the holomorphic sector and demonstrate that such a singularity does not exist, or, equally well, one can analyse the antiholomorphic dependence and try to isolate potential $O((\bar{t})^0)$ terms. It is easier to do the latter. In this case after factorization the antiholomorphic correlator is the same as that in (5.25). We are now interested in operators with $\bar{h}_\Phi = 0$, so the operators of interest have the form $\Phi(p_1) = \bar{c}\bar{\partial}\bar{c}e^{-2\bar{\phi}}\bar{\partial}\bar{\xi}\mathcal{O}$ where $\mathcal{O}$ is an operator of dimension $(0,0)$ and contains no antiholomorphic fields. Thus $\Phi^\dagger = \bar{c} : \bar{\xi}\bar{\eta} : \mathcal{O}^\dagger$,[27] and since $\mathcal{O}$ contains no antiholomorphic fields the left-moving correlator on T_2 is $\langle : \bar{\xi}\bar{\eta} : (p_2)\bar{\partial}\bar{\xi}(z_4)\rangle\langle\bar{c}(p)(\eta_2, \bar{b})\rangle$. The correlator of the anticommuting ghosts is nonsingular and the remaining correlator can be evaluated by standard techniques (see subsection B below for more details on the derivation of this.) We find

$$\frac{1}{\vartheta_1^2(p_2 - z_4)} \sum_\delta \epsilon[\delta] \frac{\vartheta[\delta](2p_2 - 2z_4)}{\vartheta^2[\delta](p_2 - z_4)} \vartheta^5[\delta](0) \tag{5.28}$$

which is certainly not zero. However, implementing the node prescription $z_4 \to p_2$ and using the riemann identities and their derivatives we obtain zero. The term with $z_3 \leftrightarrow z_4$ can be handled similarly.

This completes our proof of the vanishing of the genus two boundary terms for the type II superstring in flat R^{10} and hence establishes the vanishing of the cosmological constant in that background. We now confirm these arguments by explicit calculation. In subsection C we will again use factorization arguments, but in arbitrary backgrounds.

B. *Evaluation of boundary terms by explicit calculation:* R^{10}

We now return to the formulae (4.29) and (4.30) for the total derivative and examine through explicit calculations at genus two the behaviour of the correlators near the boundary. We shall start with Δ_1. Again we need only consider $\frac{\partial}{\partial t}M_t + \frac{\partial}{\partial \bar{t}}M_t$. Consider first

[27] In this case $\mathcal{O}^\dagger$ denotes conjugation in a hilbert space without the antiholomorphic ghost fields. In subsequent analysis $\dagger$ will sometimes stand for conjugation without antiholomorphic and/or holomorphic ghost fields. The precise meaning should be clear from the context.

M_t. The correlator involving the antiholomorphic fields can be written (after applying ghost charge conservation) as a sum of three terms:

$$M_t \sim A\langle\bar\xi(z_0)e^{\bar\phi_3}e^{\bar\phi_4}\rangle\langle\bar\psi^\mu(z_3)\bar\psi^\nu(z_4)\rangle\langle\bar\partial X^\mu(z_3)\bar\partial X^\nu(z_4)\cdots\rangle\langle(\bar\eta_t,\bar b)(\bar\eta_1,\bar b)(\bar\eta_2,\bar b)\rangle$$

$$-\frac{B}{4}\Big[\langle\bar\xi(z_0)\bar c\bar\partial\bar\xi(z_3)(\bar\partial\bar\eta e^{2\bar\phi}\bar b+\bar\partial(\bar\eta e^{2\bar\phi}\bar b))(z_4)(\bar\eta_t,\bar b)(\bar\eta_1,\bar b)(\bar\eta_2,\bar b)\rangle + z_3\leftrightarrow z_4\Big] \tag{5.29}$$

$$-\frac{C}{4}\Big[\langle\bar\xi(z_0)(\bar\partial\bar\eta e^{2\bar\phi}\bar b+\bar\partial(\bar\eta e^{2\bar\phi}\bar b))(z_4)\bar\partial\bar\xi(z_3)\frac{\partial\bar z_3}{\partial\bar t^{\bar\jmath}}D_{\bar\jmath}((\bar\eta_1,\bar b)(\bar\eta_2,\bar b)) + z_3\leftrightarrow z_4\Big]$$

where A, B and C are correlators involving the holomorphic fields. They can be exhibited explicitly from equation (4.29). However since we are here only concerned in isolating the $\frac{1}{t}$ behaviour of M_t we will not need the explicit formulae for A, B and C. Furthermore in (5.29) the ellipsis $\cdots$ signifies insertions of X matter fields from the holomorphic sector. Again we will not need to know explicitly what they are since they will not affect the $\frac{1}{t}$ dependence.

We now examine the terms in (5.29) more closely:

1.) Consider the first term in (5.29). All correlators can be evaluated explicitly. More precisely the relevant correlators can be read off from the following general formulae:

$$\langle\xi(z_0)\prod_{i=1}^{N}e^{q_i\phi(z_i)}(dz_i)^{-q_i(q_i+2)/2}\rangle=\frac{Z_1^{\frac12}}{\vartheta[\alpha](\sum_i q_i\vec z_i-2\vec\Delta)}\frac{1}{\prod_{i<j}(E(z_i,z_j))^{q_iq_j}}\frac{1}{\prod_k(\sigma(z_k))^{2q_k}} \tag{5.30}$$

$$\langle\prod_{i=1}^{N_1}b(x_i)(dx_i)^2\prod_{j=1}^{N_2}c(y_j)(dy_j)^{-1}\rangle=Z_1^{-\frac12}\vartheta(\sum\vec x_i-\sum\vec y_j-3\vec\Delta)$$
$$\frac{\prod_{i<j}E(x_i,x_j)\prod_{i<j}E(y_i,y_j)}{\prod_{i,j}E(x_i,y_j)}\frac{\prod_i\sigma^3(x_i)}{\prod_j\sigma^3(y_j)} \tag{5.31}$$

where $Z_1^{-\frac12}$ is the determinant of a chiral scalar and σ is the $g/2$-differential with no zeros or poles [75] . Because of ghost charge conservation we have $\sum_{i=1}^N q_i = 2g-2$ and

$N_1 - N_2 = 3(g-1)$ for $g \geq 2$. Equation (5.30) can be derived for arbitrary q_i (integral as well as half integral) by applying the stress tensor method. More precisely one first constructs the greens function

$$G(x,y;z_i)\equiv\frac{\langle\beta(x)\gamma(y)\prod_{i=1}^N e^{q_i\phi(z_i)}\rangle}{\langle\prod_{i=1}^N e^{q_i\phi(z_i)}\rangle} \tag{5.32}$$

from the knowledge of its analytic properties and (quasi)-periodicity as a function of x and from the fact that the residue of the simple pole at $x=y$ has to be normalized to one. From G one can construct $T(z;z_i)\equiv\langle T(z)\prod_{i=1}^N e^{q_i\phi(z_i)}\rangle/(\langle\prod e^{q_i\phi(z_i)}\rangle)$, where $T(z)$ is the stress tensor for the superconformal ghosts. First order integrable differential equations for the correlator of interest in all the z_i variables then can be derived by isolating the simple poles in $T(z;z_i)$ as $z\to z_i$. The final result can be integrated to give (5.30). For the special case where all the q_i are integral this procedure is not needed since one can derive (5.30) by requiring the right zeros and poles and the correct (quasi)-periodicities for $-\frac12 q_i(q_i+2)$ differentials. (For half integral q_i the correlator is not even quasi-periodic but transforms into correlators in other spin structures). Equation (5.30) agrees with a special case of a general formula given in ref. [6].

The correlator in (5.31) can be derived similarily through analyticity and periodicity contraints. Alternatively it can be read off from the bosonization formulae of [68][69][70][71][72].

To exhibit the behaviour of the first term in (5.29) near Δ_1 we also need to use the following standard degeneration formulae at genus 2: Let $x\in T_1'$, $y\in T_2'$ and $w\in A$, then: [28]

[28] E is a $(-1/2,-1/2)$ form. When we write $E(x,p)$ we mean the $(-1/2,0)$ form obtained by choosing the coordinate system u near p defined in (5.3) and evaluating the second argument in that coordinate system. The coordinate dependence of t and w guarantees that the whole expression is coordinate independent, as it must be.

$$
\begin{aligned}
Z_1^{-\frac{1}{2}} &\to (Z_1^{-\frac{1}{2}})^I (Z_1^{-\frac{1}{2}})^{II} \\
E(x,w) &\to -E_1(x,p_1) w t^{-\frac{1}{4}} (dw)^{-\frac{1}{2}} \\
E(y,w) &\to E_2(y,p_2) t^{-\frac{1}{4}} (dw)^{-\frac{1}{2}} \\
E(x,y) &\to E_1(x,p_1) E_2(p_2,y) t^{-\frac{1}{2}} \\
E(w_1,w_2) &\to (w_1 - w_2)(dw_1)^{-\frac{1}{2}} (dw_2)^{-\frac{1}{2}} \\
\sigma(x) &\to \frac{(dx)^{\frac{1}{2}}}{(E_1(x,p_1))} \\
\sigma(w) &\to w^{-1}
\end{aligned}
\tag{5.33}
$$

and

$$
\begin{aligned}
&\vartheta[\delta](\sum_i n_i^{(1)}\vec{x}_i + \sum_j n_j^{(2)}\vec{y}_j + \sum_j n_j^{(3)}\vec{w}_j - m\vec{\Delta}) \longrightarrow \\
&\qquad \vartheta[\delta](\sum_i n_i^{(1)} x_i - \sum_i n_i^{(1)} p_1 - \frac{m}{2}(1+\tau))\vartheta[\delta](\sum_i n_i^{(2)} y_i - \sum_i n_i^{(2)} p_2 - \frac{m}{2}(1+\tau))
\end{aligned}
\tag{5.34}
$$

For a derivation of these formulae see [75][70].

Thus up to some irrelevant determinants and numerical factors (independent of $\bar{t}$) it is easy to see that the first term in (5.29) behaves as

$$
\frac{1}{\bar{t}^2}\left(\sum \epsilon[\alpha]\vartheta^4[\alpha](0)\frac{\vartheta[\alpha](\vec{z}_3 - \vec{z}_4)}{\vartheta[\alpha](\vec{z}_3 + \vec{z}_4 - 2\vec{\Delta})}\right)^* \tag{5.35}
$$

where $*$ denotes complex conjugation.

We must estimate the sum over spin-structures. Recall first that $2\Delta \to 2p$ where $p = p_1, p_2$ is the node. (From the point of view of the $g = 2$ surface these are the same point.) Next, from (5.7) and the fact that z_3, z_4 must approach the node we have

$$
\begin{aligned}
\xi_1 &= I(z_3 - z_4) = O(\bar{t}^{1/2}(w_3 - w_4)) \\
\xi_2 &= I(z_3 + z_4 - 2\Delta) = O(\bar{t}^{1/2}(w_3 + w_4))
\end{aligned}
\tag{5.36}
$$

where $I : Pic_0 \to Jac$ is the Abel map. Since we only sum over even spin structures

$$
\vartheta[\alpha](\vec{z}_3 - \vec{z}_4) = \vartheta[\alpha](0) + \tfrac{1}{2}\xi^i\xi^j\vartheta_{ij}[\alpha](0) + O(\bar{t}^2) \tag{5.37}
$$

and similarly for the other argument. Finally, from the riemann identities we learn that

$$
\begin{aligned}
\sum \epsilon[\alpha]\vartheta^4[\alpha](0) &= 0 \\
\sum \epsilon[\alpha]\vartheta_{ij}[\alpha](0)\vartheta^3[\alpha](0) &= 0
\end{aligned}
\tag{5.38}
$$

From these facts it follows that the spin structure sum is $O(\bar{t}^2)$, so there cannot be a pole in $\bar{t}$.

2.) The second term in (5.29) can be written

$$
(\frac{Z_{1/2}^*}{Z_1^*})^5 \lim_{z_6 \to z_4}(2\frac{\partial}{\partial \bar{z}_5} + \frac{\partial}{\partial \bar{z}_4})[\langle(\bar{\eta}_1,\bar{b})(\bar{\eta}_2,\bar{b})\bar{c}(z_3)\bar{b}(z_4)\rangle\langle\bar{\xi}(z_0)\bar{\partial}\bar{\xi}(z_3)\bar{\eta}(z_5)e^{2\bar{\phi}(z_4)}\rangle] \tag{5.39}
$$

plus a term with $z_3 \leftrightarrow z_4$. Here $Z_{1/2}$ is the partition function of a weyl fermion. From (5.31) we learn that the behaviour of the reparametrization ghost correlator, as $t \to 0$, is

$$
\sim Z_2^I Z_2^{II}\frac{1}{\bar{t}^2} \tag{5.40}
$$

up to irrelevant factors. In exhibiting the $\bar{t}$ dependence in (5.40) we have used the w coordinate system for z_3, z_4 and z_5 and the fact that $w_3 \sim w_4 \sim w_5 \sim O(1)$. Note that the final answer does not depend on which coordinate system we use to carry out the calculation since $Y(z)$ is a conformal field of weight zero. The individual terms in $Y(z)$ (In particular $\partial\eta e^{2\phi}b$ and $\partial(\eta e^{2\phi}b)$). are not conformal fields. However their sum, which is what appears in (3.48) , is a conformal field.

The superghost correlator is more conveniently handled in the β,γ system. We first notice that,

$$
\begin{aligned}
&\langle\bar{\xi}(\bar{z}_0)\bar{\partial}\bar{\xi}(z_3)\bar{\eta}(z_5)e^{2\bar{\phi}(z_4)}\rangle \\
&\qquad = \lim_{z_5' \to z_5}\lim_{z_3' \to z_3}(z_5' - z_5)^{-1}(z_3' - z_3)^{-1}\langle\bar{\beta}(z_3)\bar{\gamma}(z_5)e^{\bar{\phi}(z_5')}e^{-\bar{\phi}(z_3')}e^{2\bar{\phi}(z_4)}\rangle
\end{aligned}
\tag{5.41}
$$

Using (quasi)periodicity and analyticity and eq. (5.30) we find

$$\langle\bar\beta(z)\bar\gamma(y)\prod_{i=1}^{N}e^{q_i\bar\phi(z_i)}\rangle$$
$$=\Bigg(\frac{1}{\overline{E(z,w)}}\frac{\vartheta[\alpha](\vec z-\vec y+\sum q_i\vec z_i-2\vec\Delta)}{\vartheta^2[\alpha](\sum q_i\vec z_i-2\vec\Delta)} \tag{5.42}$$
$$\prod_{i=1}^{N}\left(\frac{E(z,z_i)}{E(y,z_i)}\right)^{q_i}\prod_{k<j}\left(\frac{1}{E(z_k,z_j)^{q_kq_j}}\right)\prod_k\frac{1}{(\sigma(z_k))^{2q_k}}\left(\frac{\sigma(z)}{\sigma(y)}\right)^2\Bigg)^*$$

Taking the limit indicated in (5.41) on the relevant correlator in (5.42) we finally arrive at

$$\langle\bar\xi(z_0)\partial\bar\xi(z_3)\bar\eta(z_5)e^{2\bar\phi(z_4)}\rangle=Z_1^{1/2}\left(\frac{\vartheta[\alpha](2\vec z_3-2\vec z_5+2\vec z_4-2\vec\Delta)}{\vartheta^2[\alpha](\vec z_3-\vec z_5+2\vec z_4-2\vec\Delta)}\left(\frac{1}{E(z_3,z_5)}\right)^2\frac{1}{\sigma^4(z_4)}\right)^* \tag{5.43}$$

We can now proceed with an argument similar to that used in the matter case. We must evaluate the right hand side of (5.43) in the w coordinate system since the reparametrization ghost correlator has been calculated in this system. The divisors in (5.43) give vectors of $\mathcal{O}(\bar t^{1/2})$ in the jacobian. Once more (5.38) shows that the spin structure sum is $\mathcal{O}(\bar t^2)$, so there is no pole. These remarks are unchanged if we exchange z_3 for z_4.

3.) The last term in (5.29) can be handled similarly to the second. The ghost correlator is now only $\mathcal{O}(\bar t^{-1})$. The superghost correlator is once more (5.43) and hence $\mathcal{O}(\bar t^2)$, so there is no pole.

Finally, we consider $M_{\bar t}$ and look for a term $\sim\frac{1}{t}$ in an expansion in $t,\bar t$. Rather than showing that no such pole arises from the correlators of the holomorphic fields we once again look at the correlators of the antiholomorphic fields: [29]

$$\langle\bar\xi(z_0)(\bar\eta_1,\bar b)(\bar\eta_2,\bar b)\overline{\partial\xi}(z_3)(\overline{\partial\eta e^{2\phi}b+\partial(\eta e^{2\phi}b(z_4))})\rangle \tag{5.44}$$

[29] Our findings here are independent of the behaviour of $\frac{\partial\bar z_3}{\partial t},\frac{\partial\bar z_2}{\partial\tau_1}$ since the latter can never worsen the power of divengence in $\bar t$.

plus a term with z_3 and z_4 interchanged. The ghost correlator has order $\mathcal{O}(1/\bar t)$ while the superghost correlator is once again (5.43) and hence $\mathcal{O}(\bar t^2)$. Therefore, the expansion in $\bar t$ begins at $\bar t$. Hence there is no term $\sim\frac{1}{t}$.

We next turn our attention to the boundary integral on Δ_0. As we have discussed at the beginning of subsection A the only dangerous terms are of the form (5.13). To see if such terms arise we must evaluate the correlators as $t\to0$.

From the above description of the abelian differentials we see that

$$E(x,y)\to\frac{\vartheta_1(x-y)}{\vartheta_1'(0)}(dx)^{-1/2}(dy)^{-1/2}$$

in the limit $t\to0$, as long as x,y are not near the nodes. In particular the limiting behavior of the prime form involves no powers of t. Similarly, $\sigma(x)$, while complicated, involves no factors of t as long as x is not near the node. On the other hand, in the formula for the vector of riemann constants [77] ,

$$\Delta_k=-\frac{\tau_{kk}}{2}+c_k \tag{5.45}$$

c_k is a vector which has a finite limit as $t\to0$, but $\Delta_1\sim-\frac{\tau_1}{2}\to-i\infty$. Thus, in the bosonization formula for the chiral scalar determinant

$$Z_1^{3/2}=\frac{\vartheta(\sum z_i-w-\Delta)}{det\omega_i(z_j)}\frac{\prod E(z_i,z_j)}{\prod E(z_i,w)}\frac{\prod\sigma(z_i)}{\sigma(w)} \tag{5.46}$$

as long as z_i are not near the nodes we may estimate the t dependence by setting $D=\sum z_i-w$, a fixed divisor, and evaluating

$$\sum_{n_1,n_2}e^{i\pi n_1^2\tau_1+2\pi in_1(D_1-\Delta_1)+\cdots}$$
$$=\sum_{n_1,n_2}exp\left(i\pi n_1^2\tau_1+\pi in_1\tau_1+n_1\sum_i\ln\frac{\vartheta_1(z_i-p_1)\vartheta_1(w-p_2)}{\vartheta_1(z_i-p_2)\vartheta_1(w-p_1)}+\cdots\right) \tag{5.47}$$
$$\sim\vartheta(D_2-\Delta_2|\tau_2)+\vartheta(D_2-\tau_{12}-\Delta_2|\tau_2)\prod_i\left(\frac{\vartheta_1(z_i-p_1)\vartheta_1(w-p_2)}{\vartheta_1(z_i-p_2)\vartheta_1(w-p_1)}\right)^{-1}+\cdots$$

Here $\left[\begin{smallmatrix} D_1 \\ D_2 \end{smallmatrix}\right]$ is the vector in the jacobian for the divisor D. The explicit factors of z_i come from the non-vanishing component of ω_1 on T_2. Thus we see that as $t \to 0$ $Z_1 \sim \mathcal{O}(1)$, i.e. acquires no powers of t. Relating the scalar determinant to the chiral scalar determinant in the usual way leads to a factor of

$$\frac{1}{(det Im\tau)^5} = \frac{1}{(Im\tau_2)^5} \frac{1}{(log|t|^{-1})^5} + \cdots$$

We now turn to the fermionic ghost correlators. The beltrami differentials chosen above have support at a bounded distance from the node. Thus, in computing b, c correlators the insertion points of the operators forms a divisor $D = \sum m_i z_i$ with support a bounded distance from the node, and, by the bosonization formulae the correlator behaves like

$$\begin{aligned} \vartheta(D - 3\Delta) &= \sum_{n_1, n_2} e^{i\pi n_1^2 \tau_1 + 2\pi i n_1 (\frac{3}{2}\tau_1) + n_1 \sum_i m_i \ln \frac{\vartheta_1(z_i - p_1)}{\vartheta_1(z_i - p_2)}} + \cdots \\ &\sim t^{-1} \vartheta(D_2 - 3\Delta_2|\tau_2) \prod_i \left(\frac{\vartheta_1(z_i - p_1)}{\vartheta_1(z_i - p_2)} \right)^{-m_i} \end{aligned} \tag{5.48}$$

as $t \to 0$. Combining this with the dependence $\eta_t \sim t^{-1}$ above we reproduce the famous t^{-2} pole of the bosonic string [47].

We now consider the superconformal ghost system for an even spin structure $\delta = \left[\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right]$. From the bosonization formula for the superconformal ghost correlator [6] we see that we must estimate

$$\begin{aligned} \vartheta[\delta](D - 2\Delta) &\sim t^{1/2} \vartheta[\delta_2](D_2 - 2\Delta_2) \prod_i \left(\frac{\vartheta_1(z_i - p_1)}{\vartheta_1(z_i - p_2)} \right)^{-m_i} & \text{if} \quad a_1 = 0 \\ &\sim t^{-3/8} \vartheta[\delta_2](D_2 - 2\Delta_2) \prod_i \left(\frac{\vartheta_1(z_i - p_1)}{\vartheta_1(z_i - p_2)} \right)^{\frac{-m_i}{2}} & \text{if} \quad a_1 = 1/2 \end{aligned} \tag{5.49}$$

Again D is a divisor determined by the insertion points of the fields $\xi, \eta, e^{q\phi}$. The bosonization formula involves one more theta function in the denominator than in the numerator so the contribution of the superconformal ghosts will be the inverse of the powers in (5.49).

Finally, we consider the correlator of the fermionic matter fields. These do not involve Δ so they behave like

$$\begin{aligned} \vartheta[\delta](D) &\sim \vartheta\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}(D|\tau_2) + \mathcal{O}(t^{1/2}) & a_1 = 0 \\ &\sim t^{1/8} \vartheta\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}(D|\tau_2) \left(\prod_i \left(\frac{\vartheta_1(z_i - p_1)}{\vartheta_1(z_i - p_2)} \right)^{\frac{m_i}{2}} + \prod_i \left(\frac{\vartheta_1(z_i - p_1)}{\vartheta_1(z_i - p_2)} \right)^{-\frac{m_i}{2}} \right) & a_1 = 1/2 \end{aligned} \tag{5.50}$$

Combining this with the above estimates we see that the leading order singularity in the Neveu-Schwarz sector $(a_1 = 0)$ is

$$\frac{1}{(log|t|^{-1})^5} \frac{1}{t^{3/2}} \frac{1}{\bar{t}^{3/2}} \tag{5.51}$$

reproducing the well-known result that the Neveu-Schwarz tachyon has a value of m^2 which is half that of the bosonic string tachyon. Moreover, we see that in the Ramond sector, where $a_1 = 1/2$, only massless particles can propagate, as expected.

We can now explain how the dangerous terms behaving like (5.51) cancel. As has already been pointed out, the relative sign between the contribution from different spin structures may be determined by inserting a pair of spin fields $P^+(z)$ and $P^-(w)$ in the correlator, and dragging z around various homology cycles. In this case we want to compare the contributions from $a_1 = 0, b_1 = 0$ and $a_1 = 0, b_1 = \frac{1}{2}$ sectors. This may be done by taking z and w on T_2, and dragging z around the node p_1 or p_2. By examining the relevant correlator we can see that as a function of z it has a square root branch point at p_1 and p_2, due to the $\prod_i \left(\frac{\vartheta_1(z_i - p_1)}{\vartheta_1(z_i - p_2)} \right)^{-m_i}$ factor in (5.49). As a result, the relative contribution from the spin structures $a_1 = 0, b_1 = 0$ and $a_1 = 0, b_1 = \frac{1}{2}$ has opposite signs and they cancel after the sum over spin structures is performed. It is worth remarking that our discussion of Δ_0 applies at arbitrary genus with only small modifications.

This completes our proof that the boundary terms are indeed zero.

C. *Evaluation of boundary terms through factorization: arbitrary backgrounds*[30]

In the above demonstration of the vanishing of the cosmological constant in flat space, it was sufficient to examine the behaviour of the antiholomorphic sector. In arbitrary backgrounds this is not expected to be the case. The reason for this is that by assumption we are considering backgrounds which preserve only $N = 1$ supersymmetry. In our case the space-time supersymmetry comes from a right-handed current $J_\alpha(x)$. Since many correlators are expected to vanish as a consequence of supersymmetry non-renormalization theorems, we don't expect to arrive at definite results before analysing the holomorphic sector. Another thing we shall bear in mind in our arbitrary background calculations is that the matter sector is now an interacting theory where left and right movers are coupled in a non-trivial fashion above and beyond their coupling through zero modes.

We shall start by writing down the densities M_t and $M_{\bar t}$ that need to be considered. As was shown in subsection A the only terms in (4.29) and (4.30) that apriori have the potential of contributing are given by:

$$M_t^{(1)} = \sum_{l=1}^{8}\langle \int d^2y \oint_{r_l(y)} \frac{dx}{2\pi i}\xi(\tilde z_1)\xi(z_1)J_\alpha(x)V^\alpha(y)(\eta_1,b)(\eta_2,b) \\ Y(z_2)(e^{\bar\phi}\bar T_F)(z_3)(e^{\bar\phi}\bar T_F)(z_4)\prod_{j=1}^{3}(\bar\eta_j,\bar b)\rangle \tag{5.52}$$

and

$$\begin{aligned} M_{\bar t} \equiv & M_{\bar t}^{(1)} + M_{\bar t}^{(2)} + M_{\bar t}^{(3)} \\ = & \sum_{l=1}^{8}\Big\langle \int d^2y \oint_{r_l(y)} \frac{dx}{2\pi i}\xi(\tilde z_1)\xi(z_1)\bar\xi(\bar z_0)J_\alpha(x)V^\alpha(y) \\ & \times\Big(Y(z_2)(\eta_1,b)(\eta_2,b)\big[\bar Y(z_3)\partial_t\bar\xi(z_4)(\bar\eta_1,\bar b)(\bar\eta_2,\bar b) + z_3\leftrightarrow z_4\big] \\ & + Y(z_2)(\eta_1,b)(\eta_t,b)\bar Y(z_3)\partial_{\tau_2}\bar\xi(z_4)(\bar\eta_1,\bar b)(\bar\eta_2,\bar b) \\ & + Y(z_2)(\eta_t,b)(\eta_2,b)\bar Y(z_4)\partial_{\tau_1}\bar\xi(z_3)(\bar\eta_1,\bar b)(\bar\eta_2,\bar b)\Big)\Big\rangle \end{aligned} \tag{5.53}$$

Recall that in arriving at this conclusion in section A we have utilized properties of the ghost system which are valid in arbitrary backgrounds.

When we calculate (5.52) and (5.53) we must integrate over y and sum over contour integrals in x around the poles $r_l(y)$. When we consider these expressions on a surface which has degenerated into tori T_1 and T_2, there are therefore four distinct terms we must consider:

$1 \cdot a:\ y \in T_1,\ r_\ell(y) \in T_1$

$1 \cdot b: y \in T_1,\ r_\ell(y) \in T_2$

$2 \cdot a: y \in T_2,\ r_\ell(y) \in T_1$

$2 \cdot b: y \in T_2,\ r_\ell(y) \in T_2$

To see that all cases occur recall that the spurious poles $\{r_\ell(y)\}$ are the zeros of the function $f(x) = \prod_\delta \vartheta[\delta](\frac{1}{2}\vec y - \frac{1}{2}\vec x + \sum_{a=1}^{2}\vec z_a - 2\vec\Delta)$ where the product runs over all spin structures [31]. Using the formula $2\vartheta_1(x)\vartheta_2(x)\vartheta_3(x)\vartheta_4(x) = \vartheta_1(2x)\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$ for $g = 1$ theta functions we see that if $y \in T_1$, then in the limit $t \to 0$

$$\begin{aligned} f(x) &\to \tfrac{1}{2}\vartheta_1(y - x + 2z_1 - 2p_1|\tau_1)\eta^3(\tau_1)\prod_{\alpha=1}^{4}\vartheta_\alpha(z_2 - p|\tau_2) \quad x \in T_1 \\ f(x) &\to \prod_{\alpha=1}^{4}\vartheta_\alpha(\tfrac{1}{2}y + z_1 - \tfrac{1}{2}p_1|\tau_1)\tfrac{1}{2}\vartheta_1(-x + 2z_2 - p_2|\tau_2)\eta^3(\tau_2) \quad x \in T_2 \end{aligned} \tag{5.54}$$

[30] We would like to emphasize at this point that although we use factorization to study the behaviour of various correlators near the boundary of moduli space, the ghost correlator on $g = 2$ surface may be computed explicitly in terms of theta functions and the limit $t \to 0$ of this correlator may be taken. It is only for the interacting part of the theory involving matter fields that one needs to use the factorization hypothesis. This was the procedure adopted in ref. [73] in the calculation of the heterotic string two loop cosmological constant.

[31] We are not asserting that all the zeros lead to poles of correlation functions, only that the poles of correlation functions lie in this set.

Hence, for $y \in T_1$, of the eight zeros $\{r_l(y)\}$, four degenerate to $r_l(y) = y + 2z_1 - 2p_1$ on T_1 and four degenerate to $r_l(y) = 2z_2 - p_2$ on T_2. Similar considerations hold for $y \in T_2$. Thus we must consider all four cases.

The number of cases listed above can be cut by half by observing that (5.52) and (5.53) possess a $z_1 \leftrightarrow z_2$ symmetry. To prove this we consider the holomorphic part of (5.52) and (5.53) (for a fixed value of y) which is essentially the same for both and is given by:

$$I \sim \oint_{r_l(y)} \frac{dx}{2\pi i} \langle \xi(\tilde{z}_1)\xi(z_1)J_\alpha(x)V^\alpha(y)Y(z_2)(\eta_i,b)(\eta_j,b)\cdots\rangle \tag{5.55}$$

for some i and j $(i \neq j)$, where the ellipsis denotes suppressed operators which are irrelevant for the present argument. We shall continue to use this notation throughout this section. At this stage we can replace $V^\alpha(y)$ by

$$\tilde{V}^\alpha(y) = V^\alpha(y) + \partial(c\xi e^{-\phi/2}\hat{S}^- S_{\dot\beta}\bar{\partial}X^\mu(\gamma^\mu)^{\alpha\dot\beta}) \tag{5.56}$$

without changing the answer: The added term has the wrong ghost charge. $\tilde{V}^\alpha(y)$ satisfies $[Q_B, \tilde{V}^\alpha(y)] = 0$ point by point in y, where Q_B is the right-handed $BRST$ charge. We can then write $Y(z_2) = \oint \frac{dz}{2\pi i} j_{BRST}(z)\xi(z_2)$ and deform the $BRST$ contour away from z_2 and attempt to shrink it to zero. In doing so we pick up the residues at the poles of the $BRST$ current at the various other vertex insertions. The pole at $\tilde{z}_1$ does not contribute since the resulting correlator as a function of x has no poles at $r_l(y)$. The residue at (η_i, b) on the other hand vanishes by ghost charge conservation. Finally the pole at $\xi(z_1)$ yields $Y(z_1)$. This means that (5.55) becomes

$$I \sim \oint_{r_l(y)} \frac{dx}{2\pi i} \langle \xi(\tilde{z}_1)\xi(z_2)J_\alpha(x)V^\alpha(y)Y(z_1)(\eta_i,b)(\eta_j,b)\cdots\rangle \tag{5.57}$$

where at this stage we have dropped the term in $\tilde{V}$ with the wrong ghost charge. This establishes the desired symmetry.

We should also notice that the correlator in (5.52) and (5.53) is independent of $\tilde{z}_1$: For the x-integration in these expressions to be nonvanishing the poles in the correlator

have to occur at $r_l(y)$ and not at $r_l'(y)$. The latter happens if $\xi(\tilde{z}_1)$ soaks up the ξ-zero mode and hence no dependence on $\tilde{z}_1$ survives.

The $z_1 \leftrightarrow z_2$ symmetry and the independence of the correlator on $\tilde{z}_1$ implies a $T_1 \leftrightarrow T_2$ symmetry in factorization. This is clearly the case since the only information about T_1 and T_2 is contained in where z_1 $\tilde{z}_1$ and z_2 lie in the degeneration limit. This symmetry relates case $2 \cdot a$ to $1 \cdot b$ and $2 \cdot b$ to $1 \cdot a$ listed above. From now on we only concentrate on $1 \cdot a$ and $1 \cdot b$.

By examining the holomorphic structure in (5.52) and (5.53) we see that there are two kinds of terms which satisfy ghost charge conservation at $g = 2$:

$$I: \quad \langle J_\alpha(x)V^{\alpha,\frac{3}{2}}(y)\xi(z_1)\xi(\tilde{z}_1)(e^\phi T_F)(z_2)(\eta_i,b)(\eta_j,b)\cdots\rangle$$

$$II: \quad \langle J_\alpha(x)V^{\alpha,\frac{1}{2}}(y)\xi(z_1)\xi(\tilde{z}_1)(\partial\eta e^{2\phi}b + \partial(\eta e^{2\phi}b)(z_2))(\eta_i,b)(\eta_j,b)\cdots\rangle$$

where $V^{\alpha,\frac{3}{2}}$ and $V^{\alpha,\frac{1}{2}}$ are the parts of the dilatino vertex operator with $\frac{3}{2}$ and $\frac{1}{2}$ ϕ-ghost charge respectively. More precisely these are given by:

$$V^{\alpha,\frac{3}{2}} = \bar{\partial}X^\mu(\gamma_\mu)^{\alpha\dot\beta}(\tfrac{1}{2}e^{\frac{3}{2}\phi}\eta b\hat{S}^- S_{\dot\beta}) \tag{5.58}$$

$$V^{\alpha,\frac{1}{2}} = \bar{\partial}X^\mu(\gamma_\mu)^{\alpha\dot\beta}e^{\frac{1}{2}\phi}\partial X^\nu(\gamma_\nu)_{\dot\beta\gamma}\hat{S}^- S^\gamma \tag{5.59}$$

In writing down (5.59) we have dropped the term $e^{\frac{1}{2}\phi}S_{\dot\beta}\lim_{w\to z}((w-z)^{\frac{1}{2}}T_F^{int}(w)\hat{S}^-(z)$ in (4.14). This term does not contribute since it has the wrong charge under the $U(1)$ of the $(2,0)$ superconformal algebra [78][79][80][81]. (Recall the $U(1)$ charge of $\hat{S}^\pm$ is $\pm\frac{3}{2}$ while $T_F = T_F^+ + T_F^-$ has charges $+1$ and -1 respectively.)

We are now ready to examine (5.52) and (5.53) factorized according to configuration $1.a$ and $1.b$ and using cases I and II. To facilitate this rather lengthy analysis we have summarized the result of factorization of all the possible configurations in table (1).

	$I\cdot 1\cdot a$ $V^{\frac{3}{2}}; x \in T_1$	$I\cdot 1\cdot b$ $V^{\frac{3}{2}}; x \in T_2$	$II\cdot 1\cdot a$ $V^{\frac{1}{2}}; x \in T_1$	$II\cdot 1\cdot b$ $V^{\frac{1}{2}}; x \in T_2$
$M_t^{(1)}$	$\Phi(p_1) = (c\partial c e^{-\phi}\xi\psi^\mu)$ N.P. + S.S.	$\Phi(p_1) = (c\partial c e^{-\frac{3}{2}\phi}\hat{S}^+ S^{\dot\beta}\xi)$ N.P.	$\Phi(p_1) = c : \eta\xi :$ N.P. + S.S.	D-Terms
$M_{\bar t}^{(1)}$	$\Phi(p_1) = (c\partial c e^{-\phi}\xi\psi^\mu)$ D.C.	$\Phi(p_1) = (c\partial c e^{-\frac{3}{2}\phi} S^{\dot\beta}\hat{S}^-\xi)$ D.C.	$\Phi(p_1) = c : \eta\xi :$ D.C.	$\Phi(p_1) = (c e^{-\frac{1}{2}\phi}\hat{S}^+ S_\gamma)$ D.C.
$M_{\bar t}^{(2)}$	$\Psi^\dagger(p_2) =: bc : e^{-\phi}$ D.C.	$\Psi^\dagger(p_2) =: bc : e^{-\frac{1}{2}\phi}$ D.C.	$\Psi^\dagger(p_2) = (c e^{-2\phi}\partial\xi)$ N.P.	$\Psi^\dagger(p_2) = \xi e^{-\frac{3}{2}\phi} c\hat{S}^- S^\gamma$ N.P. +S.S.
$M_{\bar t}^{(3)}$	$\Psi^\dagger(p_2) = (c e^{-\phi}\psi^\mu)$ N.P. + S.S.	$\Phi(p_1) = \xi e^{-\frac{3}{2}\phi}\hat{S}^+ S^\alpha$ N.P.	$\Phi(p_1) =: bc :$ D.C.	$\Phi(p_1) =: bc : e^{-\frac{1}{2}\phi}$ D.C.

Table 1: *Cases in the factorization analysis*

In this table we identify for every configuration the ghost factors that the operator $\Phi(p_1)$ (or $\Psi^\dagger(p_2)$) going through the neck must have in order for the resulting matrix elements on T_1 and T_2 be nonvanishing. This entails a definite lower bound on the dimension of Φ. Recall however that for $M_{\bar t}$ to yield any contribution on the boundary it has to behave like $\sim t^{-1}(\bar t)^0$ near Δ_1. For M_t on the other hand we need $\sim (\bar t)^{-1} t^0$. Consequently for a given configuration to contribute we have to find an operator of the relevant ghost charge and the relevant dimension. In some cases an operator satisfying these two constraints does not exist. Consequently these configurations will yield vanishing contribution purely by dimension count. We denote these cases in the table by *D.C.*. In other cases there exist operators which could contribute. In many cases however the resulting matrix element vanishes by sum over spin structures in the holomorphic sector on T_1 or T_2 or when we implement the node prescription discussed above ($z_2 \to p_2, z_1 \to p_1$). These cases are denoted in the table by *S.S.* and/or *N.P.* respectively. In some configurations $U(1)$ charge conservation implies the vanishing of the resulting matrix element. One way or another (see below) all configurations can be proven to be vanishing with the exception of one case where we find that the resulting matrix element is nothing other than the square of the

Fayet-Iliopoulos D-term. In what follows we shall present the details of the calculation for every entry in table (1). We present all the cases for completeness. The reader who trusts our results can skip over most of them. However there are cases where we feel that the analysis is instructive. Those we have marked by **.

We shall start the analysis with $M_{\bar t}^{(1)}$. In this case we need to look for Φ of holomorphic dimension $h_\Phi = -1$ with the correct ghost charge.

$M_{\bar t}^{(1)} \cdot I \cdot 1 \cdot a$

By ghost charge conservation and from the structure of the free spin field correlator, we find that $\Phi(p_1)$ must be

$$\Phi(p_1) = (c\partial c e^{-\phi}\xi\psi^\mu) \times O^\mu(p_1)$$

where $O^\mu(p_1)$ is any operator free of holomorphic ghosts with $(h_O \geq 0)$[32]. So $h_\Phi \geq 0$ necessarily and no operator of the required dimension exists.

$M_{\bar t}^{(1)} \cdot I \cdot 1 \cdot b$

In this case[33]

$$\Phi(p_1) = (c\partial c e^{-\frac{3}{2}\phi} S^{\dot\beta}\hat{S}^+\xi)(p_1) \times O(p_1)$$

$h_\Phi \geq 0 \Rightarrow$ no contribution.

$** M_{\bar t}^{(1)} \cdot II \cdot 1 \cdot a$

[32] In this subsection O will stand for an operator neutral under all holomorphic ghost charges having the appropriate structure and dimension in the antiholomorphic sector. We shall not exhibit its antiholomorphic part unless our arguments require it. Furthermore since it is free of holomorphic ghosts it necessarily has $h_O \geq 0$.

[33] Note that we are being slightly sloppy here since instead of having an explicit factor of $\hat{S}^+$ in Φ we could have taken an operator O which has the required $U(1)$ charge $(3/2)$. The important point here is that any such operator must have holomorphic conformal dimension $\geq 3/8$ (see discussion right after eq. (5.65)).

The ghost structure now requires

$$\Phi(p_1) = c(p_1) : \eta(p_1)\xi(p_1) : \times \mathcal{O}$$

This needs some explanation. The reason a factor of $\eta\xi$ is necessary is that in order for the residue of the $J_\alpha(x)$ at $r_l(y)$ to be nonvanishing there should be another factor of ξ in the correlator in addition to $\xi(z_1)$ to soak up the ξ-zero mode. By ξ, η charge conservation however we then need another factor of η. Now $h_\Phi \geq 0 \Rightarrow$ no contribution.

$M_{\bar{t}}^{(1)} \cdot II \cdot 1 \cdot b$

Here

$$\Phi(p_1) = (ce^{-\frac{1}{2}\phi}\hat{S}^+ S_\gamma)(p_1) \times \mathcal{O}(p_1)$$

again $h_\Phi \geq 0$.

This completes the second row in table 1. Next we turn our attention to the more delicate configurations in the first row. Now we are factorizing M_t and a dimension zero operator of the correct ghost charge is needed.

$** M_t^{(1)} \cdot I \cdot 1 \cdot a$

$$\Phi(p_1) = (c\partial ce^{-\phi}\xi\psi^\mu)(p_1) \times \mathcal{O}$$

this could lead to a contribution if $h_\mathcal{O} = 0$. On T_2 the corresponding operator is $\Psi(p_2) = (ce^{-\phi}\psi^\mu)(p_2)\mathcal{O}'(p_2)$ where $\mathcal{O}'$ is some conjugate antiholomorphic operator with $h_{\mathcal{O}'} = 0$. Consider now the resulting correlator on T_2.

$$\sim \langle \xi(\tilde{z}_1)(e^\phi T_F)(z_2)(\eta_2, b)(ce^{-\phi}\psi^\mu)(p_2)\mathcal{O}'(p_2)\cdots\rangle \qquad (5.60)$$

As explained earlier we have to take $z_2 \to p_2$. We therefore examine the correlator in that limit. In this limit the singularities are dictated solely by the operator product expansion. (In principle we should be more careful in the sector with periodic-periodic boundary

conditions on the β, γ fields since $\langle e^{-\phi(p_2)}e^{\phi(z_2)}\rangle$ has a spurious $(\vartheta_1(z_2 - p_2))^{-1}$ type singularity. However the contribution from this sector may be seen to vanish identically due to the zero modes of the free fermions ψ^μ.)

$$\lim_{z_2 \to p_2} (e^\phi T_F)(z_2)(e^{-\phi}\psi^\mu)(p_2)\mathcal{O}'(p_2) \sim \partial X^\mu(p_2)\mathcal{O}'(p_2)$$

In arriving at the RHS we have used the fact that $T_F^{int}(z_2)$ cannot develop a singularity near $\mathcal{O}'(p_2)$. At this stage (5.60) becomes

$$\sim \langle \xi(\tilde{z}_1)\partial X^\mu(p_2)(\eta_2, b)\mathcal{O}'(p_2)\cdots\rangle \qquad (5.61)$$

We finally prove that after summing over the spin structure in the holomorphic sector this matrix element vanishes. The proof utilizes techniques developed in ref. [30] to calculate matrix elements on tori in arbitrary backgrounds. Define the following operators

$$\begin{aligned} P^+ &= e^{-\frac{1}{2}\phi}\hat{S}^+ S_4^+ S_5^+ \\ P^- &= e^{+\frac{1}{2}\phi}\hat{S}^- S_4^- S_5^- \end{aligned} \qquad (5.62)$$

Note that P^+ is just a particular component of the supersymmetry current J_α. Furthermore

$$P^+(z)P^-(w) \sim \frac{1}{(z-w)} \qquad (5.63)$$

While

$$P^+(z)\mathcal{O}'(p_2)\cdots \sim \quad non-singular \qquad (5.64)$$

To see that (5.64) is true we only need to examine the OPE of $\hat{S}^+$ in P^+ with any operator f of holomorphic dimension zero. In general

$$\hat{S}^+(z)f(z') \sim \frac{\mathcal{O}_h}{(z-z')^{3/8-h}} \qquad (5.65)$$

Since f is $U(1)$ neutral, the $U(1)$ charge of $\mathcal{O}_h$ is $\frac{3}{2}$. However any operator of $U(1)$ charge j_0 has a lower bound on its dimension given by $\frac{1}{6}j_0^2$ [78][30]. So dim $(\mathcal{O}_h) \geq \frac{1}{6}(\frac{3}{2})^2 = \frac{3}{8}$. Therefore no singularity exists. We can now write (5.61) as

$$\begin{aligned}&\langle \xi(\tilde{z}_1)\partial X^\mu(p_2)(\eta_2, b)\mathcal{O}'(p_2)\cdots\rangle \\ &\sim \oint_w \frac{dz}{2\pi i}\langle P^+(z)P^-(w)\xi(\tilde{z}_1)\partial X^\mu(p_2)(\eta_2,b)\mathcal{O}'(p_2)\cdots\rangle\end{aligned} \tag{5.66}$$

After summing over spin structures the correlator on the RHS is periodic in z so we can deform the contour away from w. Since no other singularity nor spurious poles exist we can shrink it to zero. It follows that (5.61) is actually zero.

$**M_t^{(1)} \cdot I \cdot 1 \cdot b$

In this case we find[34]

$$\Phi(p_1) = (c\partial c e^{-\frac{3}{2}\phi}\hat{S}^+ S^{\dot\beta}\xi)(p_1) \times \mathcal{O}(p_1)$$

with the required dimension $h_\Phi = 0$. However in the limit $z_1 \to p_1$ this vanishes. (There is no problem in this case in taking this limit before the x integration since $J_\alpha(x)$ is on T_2. The same argument cannot be used to show for example that $M_t \cdot I \cdot 1 \cdot a$ is vanishing as $z_1 \to p_1$ since in that limit the spurious poles of $J_\alpha(x)$ will coincide with the physical pole. In this case we need to integrate over x first and then take the limit indicated.)

$***M_t^{(1)} \cdot II \cdot 1 \cdot a$

Ghost charge conservation on T_1 gives

$$\Phi(p_1) = c(p_1) : \eta(p_1)\xi(p_1) : \mathcal{O}'$$

this means that on T_2 the relevant operator is

$$\Psi^\dagger(p_2) = (c\partial c e^{-2\phi}\partial\xi)(p_2)\mathcal{O}(p_2)$$

[34] Again the result is unchanged if instead of taking the operator $\hat{S}^+\mathcal{O}$ we consider any other operator of $U(1)$ charge 3/2

The relevant matrix element on T_2 is then given by

$$L \sim \langle \xi(\tilde{z}_1)(\partial\eta e^{2\phi}b + \partial(\eta e^{2\phi}b)(z_2))(\eta_2, b)(e^{-2\phi}c\partial c\partial\xi)(p_2)\mathcal{O}(p_2)\cdots\rangle_{T_2} \tag{5.67}$$

We shall first calculate the ghost correlator in a spin structure ν. For that it is more convenient to exhibit the correlator in question in the following form:

$$\begin{aligned}&\langle \xi(\tilde{z}_1)(\partial\eta e^{2\phi}b + \partial(\eta e^{2\phi}b)(z_2)(\eta_2, b)(e^{-2\phi}c\partial c\partial\xi)(p_2)\rangle \\ &= \langle(\eta_2,b)b(z_2)c(p_2)\partial c(p_2)\rangle \Xi\langle\xi(\tilde{z}_1)\eta(z_2')e^{2\phi(z_2)}e^{-2\phi(p_2)}\partial\xi(p_2)\rangle_\nu \\ &+ \langle(\eta_2,b)\partial b(z_2)c(p_2)\partial c(p_2)\rangle\langle\xi(\tilde{z}_1)\eta(z_2)e^{2\phi(z_2)}e^{-2\phi(p_2)}\partial\xi(p_2)\rangle_\nu\end{aligned} \tag{5.68}$$

where the operator Ξ is defined to be

$$\Xi = \lim_{z_2' \to z_2}(2\partial_{z_2'} + \partial_{z_2}) \tag{5.69}$$

Using (5.42) in subsection B above we can easily calculate the superconformal ghost correlator. The answer is[35]

$$\begin{aligned}&\langle\xi(\tilde{z}_1)\eta(z_2')e^{2\phi(z_2)}e^{-2\phi(p_2)}\partial\xi(p_2)\rangle_\nu \\ &\sim \frac{\vartheta_1^4(z_2 - p_2)}{\vartheta_1^2(z_2' - p_2)}\frac{\vartheta_\nu(2(z_2 - z_2'))}{\vartheta_\nu^2(2z_2 - z_2' - p_2)}\end{aligned} \tag{5.70}$$

The key observation at this point is that

$$\Xi\langle\xi(\tilde{z}_1)\eta(z_2')e^{2\phi(z_2)}e^{-2\phi(p_2)}\partial\xi(p_2)\rangle_\nu = 0 \tag{5.71}$$

for all even spin structures ν. As was remarked earlier we are summing over even spin structures only. In the odd spin structure the answer is vanishing by the ψ^μ zero modes. We can then drop the first term in the RHS of (5.68). Going back to (5.67) and using (5.70) we see that the amplitude on T_2 now has the form:

$$\begin{aligned}L \sim &\langle(\eta_2,b)\partial b(z_2)c(p_2)\partial c(p_2)\rangle\vartheta_1^2(z_2 - p_2) \\ &\sum_\nu \epsilon_\nu\Big(\frac{\vartheta_\nu(0)}{\vartheta_\nu^2(z_2 - p_2)}\Big)\langle\mathcal{O}(p_2)\cdots\rangle_\nu\end{aligned} \tag{5.72}$$

[35] ϑ_ν is what we previously denoted as $\vartheta[\delta]$

where $\langle \mathcal{O}(p_2)\cdots\rangle_\nu$ involves purely antiholomorphic fields (so $h_{\mathcal{O}}=0$), which can include both matter and ghosts. In ref. [30]the expectation value $\langle \mathcal{O}\rangle_\nu$ of any antiholomorphic field in the holomorphic spin structure ν was shown to be given by:

$$\langle \mathcal{O}(p_2)\rangle_\nu = K\vartheta_\nu(-A_1)\vartheta_\nu(-B_1)\vartheta_\nu(-C_1)\vartheta^2_\nu(0) \tag{5.73}$$

for any arbitrary background admitting right handed supersymmetry. In (5.73) K, A_1, B_1, C_1 are unknown constants which contain all the background dependence. In our analysis we shall not need to know what they are, we will only need to know that they satisfy the following constraint [30]

$$A_1 + B_1 + C_1 = 0 \tag{5.74}$$

Substituting (5.73) into (5.72) we can then carry out the sum over spin structures in the limit of interest using the riemann theta identity:

$$\begin{aligned}\lim_{z_2\to p_2}\sum_\nu \epsilon_\nu \frac{1}{\vartheta^2_\nu(z_2-p_2)}\vartheta_\nu(-A_1)\vartheta_\nu(-B_1)\vartheta_\nu(-C_1)\vartheta^3_\nu(0)\\ = \vartheta_1(A_1+B_1+C_1)\vartheta_1(-A_1)\vartheta_1(B_1)\vartheta_1(C_1)\\ + O((z_2-p_2)^2)\end{aligned} \tag{5.75}$$

The first term on the RHS vanishes because of (5.74). So the leading order in the spin structure sum is actually $O((z_2-p_2)^2)$. Finally we should notice that the b,c correlator can have at most a triple pole as $z_2 \to p_2$. However the superconformal ghost correlator together with the leading term in the sum over spin structures develops a fourth order zero in (5.72). The net result is that the matrix element vanishes after sum over spin structures and taking $z_2 \to p_2$.

$* * * M^{(1)}_t \cdot II \cdot 1 \cdot b$

In this case we find nonvanishing matrix elements of Fayet-Iliopoulos D-terms. The analysis for this case is presented below, after we prove that all the other configurations lead to no nonvanishing contribution.

We now proceed with the third row in table (1). In analysing $M^{(2)}_{\bar t}$ we have a factor of (η_t, b) which yields $\sim \frac{1}{t}$. Any dimension zero operator with the right ghost charges could potentially contribute to factorization.

$M^{(2)}_{\bar t} \cdot I \cdot 1 \cdot a$

For this factorization the operator must be:

$$\Psi^\dagger(p_2) =: bc : e^{-\phi} \times \mathcal{O}$$

since there are no other factors of b or c needed to soak up the b,c ghost zero modes on the torus. However $h_\Psi \geq \frac{3}{2}$ so the dimension is too large.

$M^{(2)}_{\bar t} \cdot I \cdot 1 \cdot b$

For the same reason as in the preceeding case:

$$\Psi^\dagger(p_2) =: bc : e^{-\frac{1}{2}\phi} \times \mathcal{O}$$

$h_\Psi = 1 + \frac{3}{8} > 0 \Rightarrow$ no contribution.

$M^{(2)}_{\bar t} \cdot II \cdot 1 \cdot a$

For this configuration

$$\Psi^\dagger(p_2) = (ce^{-2\phi}\partial\xi)(p_2) \times \mathcal{O}(p_2)$$

which has the right dimension and ghost charge. However in the limit $z_2 \to p_2$ the matrix element on T_2 vanishes since we must use $c(p_2)$ and the factor of b from $Y(z_2)$ to absorb the b,c zero modes on T_2.

$* * M^{(2)}_{\bar t} \cdot II \cdot 1 \cdot b$

The analysis in this case is more intricate than the other cases in this category. The relevant matrix element that needs to be calculated on T_2 is

$$I \sim \langle \xi(\tilde z_1)e^{-\frac{1}{2}\phi(x)}\hat S^+(x)S_\alpha(x)(\partial\eta e^{2\phi}b + \partial(\eta e^{2\phi}b))(z_2)\Psi^\dagger(p_2)\rangle \tag{5.76}$$

with

$$\Psi^\dagger(p_2) = \xi(p_2)e^{-\frac{3}{2}\phi(p_2)}c(p_2)\hat{S}^-(p_2)S^\gamma(p_2) \times \mathcal{O}(p_2)$$

where in this case $\mathcal{O}$ turns out to be $\bar{c} : \bar{\xi}\bar{\eta}$: as is easily verified. (To arrive at the factor of ξ in $\Psi^\dagger$ above form one needs to consider T_1 as well). We can rewrite (5.76) in the following convenient form:

$$I \sim \Xi\langle\xi(\tilde{z}_1)\eta(z_2')e^{-\frac{1}{2}\phi(x)}e^{2\phi(z_2)}e^{-\frac{3}{2}\phi(p_2)}\xi(p_2)\rangle$$
$$\langle\hat{S}^+(x)S_\alpha(x)\hat{S}^-(p_2)S^\alpha(p_2)\rangle\langle b(z_2)c(p_2)\rangle \tag{5.77}$$

We now show that the correlator (5.77) vanishes in the limit $z_2 \to p_2$. However in this case we cannot take the limit before we carry out the x-integration as was explained earlier. Thus we examine the superconformal ghost correlator in a given spin structure ν. The answer can be easily seen to be given by:

$$\langle\xi(\tilde{z}_1)\eta(z_2')e^{-\frac{1}{2}\phi(x)}e^{2\phi(z_2)}e^{-\frac{3}{2}\phi(p_2)}\xi(p_2)\rangle$$
$$\sim\left(\frac{\vartheta_\nu(-2z_2' + \tilde{z}_1 - \frac{1}{2}x + 2z_2 - \frac{1}{2}p_2)}{\vartheta_\nu(\tilde{z}_1 - z_2' + 2z_2 - \frac{1}{2}x - \frac{3}{2}p_2)\vartheta_\nu(-z_2' + 2z_2 - \frac{1}{2}x - \frac{1}{2}p_2)}\right)$$
$$\left(\frac{\vartheta_1(\tilde{z}_1 - p_2)\vartheta_1(x - z_2)\vartheta_1^3(z_2 - p_2)}{\vartheta_1(\tilde{z}_1 - z_2')\vartheta_1(p_2 - z_2')(\vartheta_1(x - p_2))^{\frac{3}{4}}}\right) \tag{5.78}$$

In carrying out the x-integration we pick up the residue of the supercurrent $J_\alpha(x)$ at the point r_l. From (5.78) we see that the spurious pole is located at

$$\tfrac{1}{2}r_l \equiv \tfrac{1}{2}x = 2z_2 - z_2' - \tfrac{1}{2}p_2 + \Delta_\nu \tag{5.79}$$

where Δ_ν is any one particular half period on the torus. (notice that depending on the relative positions of z_2 and p_2 the function $\vartheta_\nu(-z_2' + 2z_2 - \frac{1}{2}x - \frac{1}{2}p_2)$ develops a zero only in one particular spin structure ν). In the $z_2 \to p_2$ limit it is not difficult to see from the

above equations with x given by (5.79) that the superconformal ghost correlator behaves as

$$\sim \frac{(z_2 - p_2)^2}{(z_2 - p_2)^{\frac{3}{4}}}$$

The behaviour of the spin field correlator can be inferred from the the general expression [30][76]

$$\langle\hat{S}^+(x)S_\alpha(x)\hat{S}^-(p_2)S^\alpha(p_2)\rangle \sim$$
$$\frac{1}{(\vartheta_1(x - p_2))^{\frac{5}{4}}}\vartheta_\nu^2(\frac{x}{2} - \frac{p_2}{2})\vartheta_\nu(\frac{x - p_2}{2} - \tilde{A}_1)\vartheta_\nu(\frac{x - p_2}{2} - \tilde{B}_1)\vartheta_\nu(\frac{x - p_2}{2} - \tilde{C}_1) \tag{5.80}$$

Substituting for $\frac{1}{2}x$ using (5.79) we see that in the relevant spin structure the spin field correlator behaves as

$$\sim \frac{\vartheta_1^2(z_2 - p_2)}{(\vartheta_1(z_2 - p_2))^{\frac{5}{4}}}$$

Finally putting all factors together and counting powers of $(z_2 - p_2)$ we discover that the matrix element vanishes in the limit $z_2 \to p_2$ as advertised. Notice we could not have concluded this until after we carried out the x-integration.

This completes our analysis for the third row in table (1). We next turn our attention to the last row. Again from (5.53) we see the presence of a factor of (η_t, b) which yields $\sim \frac{1}{t}$. Now any operator of dimension zero and correct ghost charge could contribute.

$M_t^{(3)} \cdot I \cdot 1 \cdot a$

$$\Psi^\dagger(p_2) = (ce^{-\phi}\psi^\mu)(p_2) \times \mathcal{O}(p_2)$$

which yields a vanishing matrix element by the same analysis as for $M_t^{(1)} \cdot I \cdot 1 \cdot a$.

$M_t^{(3)} \cdot I \cdot 1 \cdot b$

In this configuration there exists an operator

$$\Phi(p_1) = (\xi e^{-\frac{3}{2}\phi}\hat{S}^+S^\alpha)(p_1) \times \mathcal{O}(p_1)$$

However the matrix element vanishes in the limit $z_1 \to p_1$.

$M_{\bar{t}}^{(3)} \cdot II \cdot 1 \cdot a$

The only operators that can propagate through the neck are of the form

$$\Phi(p_1) =: bc : (p_1) \times \mathcal{O}(p_1)$$

which however have high conformal dimensions.

$M_{\bar{t}}^{(3)} \cdot II \cdot 1 \cdot b$

Similarly, the only operators that conserve ghost charge and absorb the b, c ghost zero modes, are of the form:

$$\Phi(p_1) = (: bc : e^{-\frac{1}{2}\phi})(p_1) \times \mathcal{O}(p_1)$$

with dimension $h_\Phi \geq \frac{3}{2}$ and hence cannot contribute to the boundary term.

This completes our analysis of all the entries of table (1) except for $M_t \cdot II \cdot 1 \cdot b$. All terms considered so far have been carefully proven to yield no boundary contribution. This last case however turns out to give rise to a non-vanishing boundary term. We analyse this next.

Fayet-Iliopoulos D-Terms

Applying factorization to (5.52) for the configuration $II \cdot 1 \cdot b$ we arrive at the following expression:

$$\begin{aligned} M_t \sim & t^{h_\Phi} \bar{t}^{\bar{h}_\Phi - 1} \\ & \langle V^{\alpha, \frac{1}{2}}(y)(\eta_1, b)(\bar{\eta}_1, \bar{b})\xi(z_1)\bar{\xi}(\bar{z}_0)(e^{\bar{\phi}}\bar{T}_F)(z_3)\Phi(p_1)\rangle_{T_1} \langle 0|\Phi^\dagger \bar{b}_0 \Psi|0\rangle_C \\ & \langle \Psi^\dagger(p_2) J_\alpha(x)(\partial\eta e^{2\phi} b + \partial(\eta e^{2\phi} b))(z_2)(e^{\bar{\phi}}\bar{T}_F)(z_4)\xi(\bar{z}_1)(\eta_2, b)(\bar{\eta}_2, \bar{b})\rangle_{T_2} \end{aligned} \qquad (5.81)$$

By ghost charge conservation in the antiholomorphic sector on T_2, the relevant operator $\Psi^\dagger$ has the form

$$\Psi^\dagger(p_2) \sim \bar{c}(p_2) e^{-\bar{\phi}(p_2)} f^{(a)}(p_2) \qquad (5.82)$$

with $f^{(a)}(p_2)$ an operator of dimension $(0, \frac{1}{2})$ and does not contain any ghost fields in the antiholomorphic ghost sector. Applying ghost charge conservation on the cylinder we arrive at the conclusion that Φ on T_1 has to be:

$$\Phi \sim \bar{c}(p_1) e^{-\bar{\phi}(p_1)} \tilde{f}^{(a)}(p_1) \qquad (5.83)$$

where $\tilde{f}^{(a)}$ is the operator of dimension $(0, \frac{1}{2})$ conjugate to $f^{(a)}$ with no factors of anti-holomorphic ghost fields.

As discussed previously we must implement the node prescription, and take the limit $z_4 \to p_2$ and $z_3 \to p_1$.[36] Define,

$$\begin{aligned} F^{(a)}(p_2) &= \lim_{z_4 \to p_2} (\bar{z}_4 - \bar{p}_2)\bar{T}_F(\bar{z}_4) f^{(a)}(p_2) \\ \tilde{F}^{(a)}(p_1) &= \lim_{z_3 \to p_1} (\bar{z}_3 - \bar{p}_1)\bar{T}_F(\bar{z}_3) \tilde{f}^{(a)}(p_1) \end{aligned} \qquad (5.84)$$

so that $F^{(a)}, \tilde{F}^{(a)}$ are dimension $(0,1)$ operators free from any anti-holomorphic ghosts, and are conjugates of each other. (5.81) may then be written as

$$\begin{aligned} M_t \sim & \frac{1}{\bar{t}} \langle V^{\alpha, \frac{1}{2}}(y)(\eta_1, b)(\bar{\eta}_1, \bar{b})\xi(z_1)\bar{\xi}(\bar{z}_0)\tilde{F}^{(a)}(p_1)\bar{c}(p_1)\rangle_{T_1} \\ & \langle \bar{c}(p_2) F^{(a)}(p_2) J_\alpha(x)(\partial\eta e^{2\phi} b + \partial(\eta e^{2\phi} b))(z_2)\xi(\bar{z}_1)(\eta_2, b)(\bar{\eta}_2, \bar{b})\rangle_{T_2} \end{aligned} \qquad (5.85)$$

The above expression is identical to the one obtained in the case of the heterotic string theory [73], and may be analysed in the same way. In particular the operators $F^{(a)}(\tilde{F}^{(a)})$ that give a non-vanishing contribution to (5.85) are of the form:

$$\begin{aligned} F^{(a)} &= c\partial c \xi \hat{S}^- S_4^- S_5^- e^{-\frac{3}{2}\phi} U^{(a)} \\ \tilde{F}^{(a)} &= c\hat{S}^+ S_4^+ S_5^+ e^{-\frac{1}{2}\phi} U^{(a)} \end{aligned} \qquad (5.86)$$

[36] So far in our analysis in this subsection we have always taken the $z_1 \to p_1$, $z_2 \to p_2$ limit before taking $z_3 \to p_3$, $z_4 \to p_4$ limit. We may continue to do that here if we replace $F^{(a)}$ in subsequent discussion by $(e^{\bar{\phi}(z)}\bar{T}_F(z) e^{-\bar{\phi}(p)} f^{(a)}(p))$. Using this we may arrive at eq. (5.87) (5.88) below with $U^{(a)}$ in (5.88) replaced by $e^{\bar{\phi}(z)}\bar{T}_F(z) e^{-\bar{\phi}(p)} u^{(a)}(p)$ where $u^{(a)}$ is related to $f^{(a)}(p)$ through $f^{(a)} = c\partial c \xi \hat{S}^- S_4^- S_5^- e^{-\frac{3}{2}\phi} u^{(a)}$. At this stage we may take the limit $z_3 \to p_1$ and $z_4 \to p_2$ to recover (5.87) and (5.88).

where $U^{(a)}$ is a dimension $(0,1)$ operator constructed out of the operators of the conformal field theory describing the compact dimensions. (In principle $U^{(a)}$ may also be constructed from the fields $\bar{\psi}^{\mu}$ as $\bar{\psi}^{\mu}\bar{\psi}^{\nu}$, but the corresponding matrix element may be seen to vanish on T_1 and T_2 seperately in every spin structure). The corresponding matrix elements may be calculated in the same way as for the heterotic string and the contribution to Λ may be shown to be:

$$\Lambda \sim \sum_a c^{(a)} c^{(a)} \tag{5.87}$$

where,

$$c^{(a)} \propto \langle\langle U^{(a)} \rangle\rangle_{PP} = \int D\bar{\psi}^{\mu} D\bar{\beta} D\bar{\gamma} D\varphi e^{-S} U^{(a)} \tag{5.88}$$

φ stands for the fields describing the conformal field theory associated with the compact dimensions. In evaluating (5.88) we sum over all spin structures in the anti-holomorphic sector, but only over those spin structures in the holomorphic sector which give a periodic boundary condition on the holomorphic fermionic stress tensor $T_F(z)$. This matrix element in turn may be evaluated in the same way as in the heterotic string theory following [30] with the final result,

$$c^{(a)} = \frac{g}{192\pi^2} \sum_i n_i q_i^{(a)} h_i \tag{5.89}$$

where n_i is the number of massless fermions carrying $U^{(a)}(1)$ charge $q_i^{(a)}$ and helicity h_i, g is the four dimensional gauge coupling constant.

Thus we conclude that the final answer for the two loop partition function of the type II string compactified on arbitrary backgrounds is given by (5.87) with $c^{(a)}$ given by (5.89): The vacuum amplitude is just the square of the Fayet-Iliopoulos D-term induced at one loop, just as it turned out in the heterotic string theory [73]. It is worth mentioning that so far there is no known type II vacuum which possesses anomolous $U(1)$ factors (i.e. with $c^{(a)}$ in (5.89) nonzero). This means that for all the known four dimensional type II vacua which preserve tree level supersymmetry, our result in this section shows

that the cosmological constant vanishes at two loops. However at this point there is no general reason to believe that Fayet-Iliopoulos D-terms cannot be generated in any type II model.[37] If this happens then the vacuum is destabilized at two loops as we can see from (5.87).

6. Conclusion

In this paper we have discussed some global issues involved in choosing a gauge slice in superstring theories. Working with a specific class of gauge slices in which the metric is independent of the odd coordinates of the supermoduli space, and the gravitino has delta function support, we have shown that the requirement of modular invariance and transversality of the slice to the gauge directions (a good modular invariant slice) prevents us from choosing a holomorphic slice. We have further shown that given a good, modular invariant slice, the superstring partition function calculated with this slice is a total divergence in the moduli space. This result is true for superstring theories formulated on R^{10}, as well as compactified superstring theories with at least an $N = 1$ supersymmetry in four dimensions. Thus the cosmological constant may be expressed purely in terms of boundary integrals. The final answer is independent of how we choose the slice away from the boundary, but does, in general, depend on the choice of the slice at the boundary. At genus two, the correct choice of the slice at the boundary may be determined by using $BRST$ invariance. With this choice of slice we can calculate the genus two partition function. The relevant boundary that contributes turns out to be Δ_1, where the genus two surface breaks up into two genus one surfaces. The boundary contribution is shown to vanish for uncompactified superstring theory, and is proportional to the square of the Fayet-Iliopoulos D-term induced at one loop (if any) for the compactified theory.

We would also like to mention some open problems and speculations. First, although we have shown that the partition function is independent of the choice of slice (except

[37] We wish to thank L. Dixon for a discussion on this point

at the boundary) as long as it is a good modular invariant slice, we have not explicitly constructed, or even shown the existence of such a slice. (We have shown that all conditions other than modular invariance can be simultaneously satisfied.) Since we have relaxed the criterion of holomorphy of the slice, we expect that it should be possible to construct such a slice, unless there is a purely topological obstruction. We hope that this gap will soon be filled in. A more serious problem concerns the choice of the slice at the boundary. At genus two, a prescription for choosing such a slice at the boundary is obtained by demanding BRST invariance, although an understanding of this prescription based on a more geometrical notion would certainly constitute progress. For $g \geq 3$ we do not yet have a prescription for choosing the slice at the boundary, and hence we cannot calculate the partition function. In short, although we can express the partition function as a sum of boundary terms, these depend on the choice of the slice at the boundary, and hence are ambiguous.

In fact, we do not even have to go beyond genus two surfaces to see the origin of the problem. It already exists in the computation of higher point functions on genus two surfaces. There are two ways of seeing this problem. First, we may regard the n-point function on a genus two surface as a functional integral over a punctured (super)-riemann surface. In this case the natural vertex operators to be used are the ones in the -1 or $-\frac{1}{2}$ picture depending on whether they are bosonic or fermionic. Each extra bosonic vertex operator, or pair of fermionic vertex operators, introduces an extra supermodulus, [12][82] which, in turn, introduces extra factors of picture changing operators $Y(w_i)$ (if we use delta function basis for these extra super-beltrami differentials). The answer for the correlator then not only depends on how we choose the points z_a (associated with the original super-beltrami differentials) at the boundary of the moduli space, but also how we choose the w_i at the boundary. Furthermore, the boundary now not only includes the one where the genus two surface breaks up into two genus one surfaces, but also where the punctures

collide. A particularly dangerous boundary is the one where all the punctures approach each other.

Another prescription for calculating amplitudes involving physical external states is implicit in the work of Friedan, Martinec and Shenker [1] on picture changing. According to this formalism, the same physical state may be represented by any member of an infinite set of vertex operators, related to each other by picture changing operators. In order to compute an amplitude, we pick up one representative vertex operator for each of the external states in such a way that the total ghost charge of all the vertex operators adds up to the right amount so as to conserve the ghost charge. In order for this prescription to make sense, we must ensure that the final result is independent of which representative vertex operator we choose for a given state. This may be shown using BRST contour deformation arguments, but these leave total derivative terms, which include total derivatives in the moduli, as well as total derivatives in the location of the vertex operators. These are precisely the terms that give rise to an ambiguity in the final answer.

Thus we see that we must develop a general framework which allows us to determine the right choice of basis for the super-beltrami differentials at the boundary of the moduli space. Unless this problem is solved, there is no hope for computing amplitudes or higher loop partition functions in fermionic string theory. The situation may not be any better in string field theory. It is conceivable that similar ambiguities exist within this formalism and that a careful analysis of "global issues" would expose such subleties. Also, it will be interestring to see if there is a connection between these subleties and the contact interactions discussed in ref. [83](see also [84][85].)

On the more speculative side, we should mention that there have been suggestions that the ambiguities in string perturbation theory arise from the (conjectured) nonsplit nature of supermoduli space. It is well-known that superteichmuller space is split, and supermoduli space is the quotient of superteichmuller space by the action of nontrivial diffeomorphisms. If we use the fiber bundle definition of supermoduli space, then, since in WZ gauge diffeomorphisms act on (e, χ) by pullback and do not mix $O(\chi^2)$ terms into the metric, one might expect that supermoduli space is also split, at least before compactification. It is quite possible that this argument is too naive and contains a flaw; whether or not this is the case must be left to future work.

Finally it must be clear to the reader that the present formalism, while efficient enough to allow some limited computation, cannot be the best one. The result that the superstring measure is a total derivative suggests that it might be some kind of characteristic class. (A relation to BRS cohomology is suggested in [86].) If this is the case, the most beautiful proof of the vanishing of the cosmological constant would be the demonstration of the vanishing of a certain characteristic class, based entirely on the geometry of supermoduli space.

Note added: After completion of this work we received a paper [87] in which some related issues are discussed from another point of view.

Acknowledgements

We would like to thank P. Deligne, V. Della Pietra, A. Marden, P. Nelson, N. Seiberg and H. Sonoda for conversations. We are especially indebted to C. Earle, for his help in proving the theorems in appendix B, and to E. Witten for useful suggestions.

Appendix A. The theorems of Earle and Hubbard

In this appendix we merely sketch some of the ideas that lie behind some of the theorems used in the text. Let us begin by reviewing a little teichmuller theory [43][88].

The space of frames modulo weyl and lorentz groups is just the space of beltrami differentials. If we uniformize $S = U/\Gamma$ with a fuchsian group Γ we may pull the beltrami differentials up to U and therefore identify the space of beltrami differentials on S with the space

$$M(\Gamma) = \{\mu : U \to C \mid \mu \circ \gamma \bar{\gamma}'/\gamma' = \mu \quad \forall \gamma \in \Gamma, \text{and} \quad \| \mu \|_\infty < 1\}$$

Here $\| \mu \|_\infty = sup\{|\mu(z)| : z \in U\}$. The condition $\| \mu \|_\infty < 1$ arises from the requirement that the metric $|dz + \mu d\bar{z}|^2$ be nonsingular. Thus, the space of conformal structures may be thought of as an infinite dimensional generalization of the open unit disk Δ. The group $Diff_0$ of diffeomorphisms of S homotopic to one acts on the beltrami differentials on S by pullback. We can obtain the equivalent action on $M(\Gamma)$ by pulling back small diffeomorphisms on S to diffeomorphisms of U. This is accomplished by the following

Lemma: *For $S = U/\Gamma$, let $D_0(S)$ be the group of diffeomorphisms homotopic to one, and $D_0(\Gamma)$ be the group of diffeomorphisms $U \to U$ commuting with Γ. Then $D_0(\Gamma) \cong D_0(S)$ are isomorphic as topological groups.*

For a proof of this lemma see [88] and references therein. We will simply describe the idea of the correspondence. If $f : S \to S$ is homotopic to one we may lift it to $\tilde{f} : U \to U$ and by the covering homotopy theorem we can lift the homotopy. Since $\tilde{f} \circ \gamma = \gamma' \circ \tilde{f}$ and Γ is discrete it follows that $\gamma = \gamma'$ and $\tilde{f}$ commutes with Γ. Suppose on the other hand that we have a map $f : U \to U$ commuting with Γ. Consider the geodesics in the poincare metric connecting z to $f(z)$. Since f commutes and since Γ is a group of isometries of this metric we can construct a homotopy by flowing for a fractional distance t along the geodesics. This homotopy commutes with Γ and hence descends to a homotopy of the projection of f to 1.

The action of $D_0(\Gamma)$ allows us to define an equivalence relation $\mu \sim \nu$ if there is an $f \in D_0(\Gamma)$ with $\mu = f^*(\nu)$. The set of equivalence classes $[\mu]$ is teichmuller space T. Denote by $\Phi : M(\Gamma) \to T$, the projection $\mu \to [\mu]$. We can now describe the idea of Earle's theorem, which states that

Theorem(Earle): *When $dimT \geq 2$ then $\Phi : \mathcal{F}/C \to T$ has no holomorphic cross-section.*

If there were a cross-section $\sigma : T \to M$ then $s = \sigma\Phi : M \to M$ is a self-map and, by changing the group Γ appropriately we can arrange that $s(0) = 0$. Thus we may apply Schwarz's lemma. Recall that this states that if $f : \Delta \to \bar{\Delta}$ is a holomorphic function satisfying $f(0) = 0$, then $|f'(0)| \leq 1$. In our case we learn that the operator $P = s'(0) : TM \to TM$ satisfies $\| P\mu \|_\infty \leq \| \mu \|_\infty$. (Note that TM is itself a space of beltrami differentials with the condition $\| \mu \|_\infty < 1$ removed.) Moreover, since σ is a cross-section, $\sigma^2 = \sigma$, so $P^2 = P$. Moreover, $ker P$ is just the fiber above $t = 0$, i.e. $ker\ P$ consists of beltrami differentials which are pure gauge transformations of the metric on S_0. Let us rephrase this last condition.

The holomorhic quadratic differentials on S_0 can also be pulled back to U and consist of holomorphic functions $\psi : U \to C$ such that $\psi(\gamma \cdot z)(\gamma'(z))^2 = \psi(z)$ for all $\gamma \in \Gamma$ together with the condition

$$\| \psi \| = \int \sqrt{\psi \bar{\psi}} < \infty$$

Denote the space of such ψ by $A(\Gamma)$. $A(\Gamma)$ is naturally isomorphic to the cotangent space of T at S_0. We can characterize $ker\ P$ as the space of μ for which

$$(\psi, \mu) = \int \psi\mu = 0 \quad \forall \psi \in A(\Gamma) \tag{A.1}$$

Thus, a holomorphic section defines an operator $P : TM \to TM$ such that

1. $P^2 = P$
2. $ker\ P$ is pure gauge
3. $\| P\mu \|_\infty \leq \| \mu \|_\infty$

The next part of the proof shows that no such operator exists. A crucial role is played by the teichmuller differentials. These are the phases of quadratic differentials:

$$\mu(z) = k|\varphi(z)|/\varphi(z) \quad k > 0 \quad \varphi \in A(\Gamma) \tag{A.2}$$

Teichmuller differentials satisfy two useful properties (for a proof see [45] and references therein). The first property is that they are of minimal norm in their gauge orbit, i.e. $\| \mu + \lambda \|_\infty < \| \mu \|_\infty$, if λ is pure gauge. The second property will be described below. Using

the first property and properties (1-3) of P above one finds $P\mu = \mu$ for all teichmuller differentials.

If $dimT \geq 2$ then $dimA(\Gamma) \geq 2$ so, choosing linearly independent quadratic differentials ψ_1, ψ_2 we can find points z_1, z_2 such that

$$\psi_1(z_1)\psi_2(z_2) \neq \psi_1(z_2)\psi_2(z_1) \tag{A.3}$$

The second useful property of teichmuller differentials is that we can find teichmuller differentials μ_1, μ_2, μ such that, for all $\psi \in A(\Gamma)$,

$$\begin{aligned} \psi(z_1) &= (\psi, \mu_1) \\ \psi(z_2) &= (\psi, \mu_2) \\ \psi(z_1) + \psi(z_2) &= (\psi, \mu) \end{aligned} \tag{A.4}$$

Note that μ, being a teichmuller differential is of the form $(\varphi(z_1) + \varphi(z_2))|\varphi(z)|/\varphi(z)$ for some $\varphi(z) \in A(\Gamma)$. It is clear from (A.4) that $\mu - \mu_1 - \mu_2$ is pure gauge, so $P\mu = P\mu_1 + P\mu_2$, but these are teichmuller differentials, so $\mu = \mu_1 + \mu_2$. This means $|\mu_1 + \mu_2|$ is a constant, since μ, being a teichmuller differential is a pure phase. But μ_i, $i = 1, 2$ are also teichmuller differentials so $\mu_i(z) = \varphi_i(z_i)|\varphi_i(z)|/\varphi_i(z)$, and the relative phase of μ_1 and μ_2 is just the phase of $\varphi_1(z)/\varphi_2(z)$, which must be constant. Since the $\varphi_i(z)$ are holomorphic this means $\varphi_1(z) = c\varphi_2(z)$, but this means that $\psi(z_1) = c\psi(z_2)$ for all $\psi \in A(\Gamma)$, contradicting (A.3). This concludes the argument.

Hubbard's theorem is a stronger version of Earle's theorem and may be related to it as follows. Consider the subgroup $Diff_0(x) \subset Diff_0$ of diffeomorphisms fixing x: $f(x) = x$. Then $Diff_0/Diff_0(x)$ is in bijective correspondence with the surface S through $[f] \leftrightarrow f(x)$. We thus have the diagram:

$$\begin{array}{ccc} & & M(\Gamma) \\ & \swarrow & \\ M/Diff_0(x) & & \downarrow \pi \\ & \searrow & \\ & & T \end{array} \tag{A.5}$$

$M/Diff_0(x)$ can be endowed with a complex structure so that it becomes the universal teichmuller curve. Earle's theorem states that the vertical arrow has no holomorphic cross-section. Hubbard's theorem states that (with the exception of the six Weierstrass sections at $g = 2$) in fact, no holomorphic cross-section of $M/Diff_0(x)$ exists.

The above characterization of the universal curve makes clear the relation to Earle's theorem, but obscures the holomorphic structure. Thus we give another description of the universal curve [43] which we will use in the following subsection.

The equivalence relation $\mu \sim \nu$ used above has another useful characterization, which is based on the fact that all $f \in D_0(\Gamma)$ extend to the real line and are the identity $f(x) = x$ there [88]. Given $\mu \in M(\Gamma)$ we can construct $w^\mu : \hat{C} \to \hat{C}$ by solving the beltrami equation $\bar{\partial} w^\mu = \mu \partial w^\mu$ in U and $\bar{\partial} w^\mu = 0$ in $\bar{U}$, the lower half plane. The equivalence relation can now be phrased $w^\mu \sim w^\nu$ if $w^\mu = w^\nu$ on the real axis. The particular mapping w^μ depends on which μ we choose, but the region $w^\mu(U)$ and the group $w^\mu \Gamma (w^\mu)^{-1}$ (called a quasiconformal deformation of Γ) depend only on the equivalence class $\Phi(\mu)$. The Bers fiber space may therefore be defined by

$$F(\Gamma_0) = \{(\Phi(\mu), z) \in T(\Gamma_0) \times C | \mu \in M(\Gamma_0), z \in w^\mu(U)\} \tag{A.6}$$

The definition makes sense since $w^\mu(U)$ only depends on the equivalence class of μ. Finally, Γ_0 acts on F, namely, $\gamma \cdot (\Phi(\mu), z) = (\Phi(\mu), \gamma^\mu(z))$ where $\gamma^\mu = w^\mu \gamma (w^\mu)^{-1}$. For fixed μ the quotient by the group action is just a copy of the curve with complex structure $\Phi(\mu)$. The universal curve is defined to be $V(\Gamma) = F(\Gamma)/\Gamma$, and $F(\Gamma)$ is its universal cover.

The proof of Hubbard's theorem again relies on the absence of certain projection operators dictated by the geometry of the Banach spaces of quadratic and beltrami differentials, and makes essential use of Royden's theorem equating the teichmuller metric with the Kobayashi metric.

Appendix B. Some theorems on quasiconformal maps

In this appendix we prove the theorem used in section II.D. The proof of theorem A.1 was suggested by C. Earle, who also helped with the proof of theorem A.2.

First we establish some notation. Let $H_g \subset T_g$ be the hyperelliptic locus. For $t \in H$ we have the corresponding hyperelliptic curve $y^2 = \prod(z - e_i(t))$ with projection $\pi_t : S_t \to C$ and involution j. If $w : S_1 \to S_2$ is a quasiconformal map commuting with the involution it induces a map $\tilde{w}$ on the plane. For a hyperelliptic curve there is a fuchsian group G containing Γ as a subgroup of index two. G has elliptic transformations of order two, and the fixed points are the Weierstrass points. One can prove that any quasiconformal deformation of Γ comes from one of G. U/G is a sphere with six branch points of order two. In this way one establishes the isomorphism $T_{2,0} \cong T_{0,6}$. We begin with

Theorem A.1: *Let $p(t) \in C$ be any family of points $p : H_g \to C$. Define*

$$H'_g = \{t \in H_g | p(t) \neq e_i(t), i = 1, \dots 2g + 2\}$$

For any connected and simply connected subset $W \subset H'_g$ there is a continuous family of quasiconformal maps $w_t : S_0 \to S_t$, $t \in W$, commuting with the hyperelliptic involution, such that $\tilde{w}_t(p(0)) = p(t)$.

Proof:

a.) By a standard lifting criterion, since W is simply connected, we can lift

$$\begin{array}{ccc} & & F(G_0) \\ & & \downarrow \\ W & \to & H_g \times C \end{array} \tag{B.1}$$

where the bottom arrow is $t \mapsto (t, p(t))$.

b.) Since $M(G_0) \to T(G_0)$ has a real analytic cross section there is a real-analytic family $w_t^{(1)} : U \to w^{\mu_t}(U)$.

c.) Thus $(w_t^{(1)})^{-1}(\tilde{p}(t)) = p^\sharp(t)$ is a family of points in U. We need a diffeomorphism $w_t^\sharp \in D_0(G_0)$ satisfying $w_t^\sharp(\tilde{p}(0)) = p^\sharp(t)$. It will then descend to the appropriate small diffeomorphism with which to modify $w_t^{(1)}$.

d.) To find this consider the fibration $\varphi : D_0(G_0) \to U$ defined by the evaluation map:$\varphi(f) = f(\tilde{p}(0))$. One can show that $im\varphi$ is open, and that its complement is open, so since U is connected it is onto. Also it is continuous, and locally trivial, so we do have a fibration. The group of the fibre bundle is $D_0(G_0, \tilde{p}(0))$, i.e. those diffeomorphisms fixing the specified point. Since the bundle is trivial there exists a continuous cross-section σ: $z \mapsto \sigma_z$. Since $\varphi\sigma_z = z$ we see that the desired map is obtained by defining $w_t^\sharp \equiv \sigma_{p^\sharp(t)}$. ∎

Remark: This only shows that $w_t^\sharp$ depends continuously on t, but we should be able to strengthen this to real-analytic dependence.

Next we prove

Theorem A.2: *Let $p_a(t) \in C$, $a = 1, \ldots n$ be any family of points for $t \in H_g$ with $p_a(t) \neq p_b(t)$ for $a \neq b$. Define*

$$H_g' = \{t \in H_g | p_a(t) \neq e_i(t), i = 1, \ldots 2g+2, a = 1, \ldots n\}$$

For any connected and simply connected subset $W \subset H_g'$ there is a continuous family of quasiconformal maps $w_t : S_0 \to S_t$ $t \in W$, commuting with the hyperelliptic involution, such that $\tilde{w}_t(p_a(0)) = p_a(t)$.

Proof: We proceed by induction on n. Theorem A.1 is the case $n = 1$. Let us assume the theorem is true up to $n-1$. Again we may lift $t \mapsto (t, p_a(t))$, and choosing $w_t^{(1)}$ as before we may define $(w_t^{(1)})^{-1}(\tilde{p}_a(t)) = p_a^\sharp(t)$ for $a = 1, \ldots n$. We can find $w_t^\flat$ with $w_t^\flat(\tilde{p}_a(0)) = p_a^\sharp(t)$ $a = 1, \ldots n-1$ by the induction hypothesis. Now let $(w_t^\flat)^{-1}(p_n^\sharp(t)) = p_n^\flat(t)$. We seek $w_t^\sharp$ with in the subgroup $D_0(G_0; \tilde{p}_1(0), \ldots, \tilde{p}_{n-1}(0))$ of $D_0(G_0)$ fixing the indicated points such that $w_t^\sharp(\tilde{p}_n(0)) = p_n^\flat(t)$. We now let φ be the evaluation map at $\tilde{p}_n(0)$ and obtain a fibration

$$D_0(G_0; \tilde{p}_1(0), \ldots, \tilde{p}_{n-1}(0)) \xrightarrow{\varphi} U - G_0 \cdot \{\tilde{p}_1(0), \ldots \tilde{p}_{n-1}(0)\} \tag{B.2}$$

Again one can argue that this is a fibration. Note that the base of the fibration is not topologically trivial. On the other hand it is known, for example for $n = 2$ [41] that the total space is topologically trivial. Thus the fibration is nontrivial and no cross-section exists. However, we can take the universal cover $U \to U - G_0 \cdot \{\tilde{p}_1(0), \ldots \tilde{p}_{n-1}(0)\}$ and pull the bundle back to this space. The base is now trivial and a cross-section σ exists. This means that $\sigma_z(\tilde{p}_n(0)) = \pi(z)$. For $t \in W$ we can lift the map $t \mapsto p_n^\flat(t)$ along the bottom row of

$$\begin{array}{ccc} & & U \\ & & \downarrow \\ W & \to & U - G_0 \cdot \{\tilde{p}_1(0), \ldots \tilde{p}_{n-1}(0)\} \end{array} \tag{B.3}$$

to $t \mapsto \tilde{p}_n^\flat(t)$, again because W is simply connected. Thus we choose $w_t^\sharp = \sigma_{\tilde{p}_n^\flat(t)}$ which satisfies the required properties. ∎

Finally we prove "theorem 2" of section 2.F. Actually, all we have to do is formulate the statement more precisely; its truth will then be evident. We uniformize the basepoint riemann surface $S_0 = U/\Gamma$ as usual. We are now considering any riemann surface at any genus. Consider an open covering U_α of T and a set of compatible slices $(e^\alpha(t), \bar{\chi}^\alpha(t, \varsigma))$. The frame $e^\alpha(t)$ defines a family of beltrami differentials $\mu_t^\alpha \in M(\Gamma)$ and these, in turn, define maps $w_t^\alpha \equiv w^{\mu_t^\alpha}$ as in appendix A. For each t we may think of the gravitino supports as a set of points $p_a^\alpha(t) \in w_t^\alpha(U)/\Gamma_t^\alpha$ where Γ_t^α is the quasiconformal deformation of Γ described previously and these lift to a discrete set of points $p_{a,i}^\alpha(t) \in w_t^\alpha(U)$. For $t \in U_\alpha \cap U_\beta$, $w_t^\alpha(U) = w_t^\beta(U)$ and $\Gamma_t^\alpha = \Gamma_t^\beta$ since these only depend on the equivalence class of $e(t)$. Thus the maps $(w_t^\alpha)^{-1} w_t^\beta$ make sense for $t \in U_\alpha \cap U_\beta$. As mentioned in section 2.F the condition that τ_α and τ_β be related by a diffeomorphism means that $(w_t^\beta)^{-1}(p_{a,i}^\alpha(t)) = (w_t^\beta)^{-1}(p_{a,i}^\beta(t))$ so $p_{a,i}^\alpha(t) = p_{a,i}^\beta(t)$. Moreover, since $p_{a,i}^\alpha(t) = w_t^\alpha(p_{a,i}^\alpha(0))$, by the condition (2.42) , and since w_t^α varies holomorphically $p_{a,i}^\alpha(t)$ vary holomorphically so $t \mapsto (t, p_a^\alpha(t))$ are holomorphic section of $V(\Gamma)$ over U_α. We have just seen that they patch together to make a global holomorphic section.

Appendix C. Weierstrass sections and pointwise vanishing

In this appendix we give an explanation of why the choice of slice where we take the points z_a to a Weierstrass point gives a measure in uncompactified superstring theory which vanishes pointwise, even though this choice does not give a modular invariant slice. Let T be the teichmuller space, M the modular group, and G a subgroup of the modular group that leaves a particular Weierstrass point (say e_1 in the notation of subsection F) invariant (in the sense of eq.(2.41)). Let us now choose a holomorphic slice S given by the Weierstrass section e_1. Also, instead of defining the integration over t^r to be over the fundamental region T/M, let us take it to be over the domain T/G. (This is equivalent to summing over prescriptions discussed in [11]). Then, following the steps of sec.IV we may calculate the measure for this particular choice of slice, and show that it is a total derivative in the teichmuller space. Let us denote this by $\frac{\partial M^r}{\partial t^r}$. Following the analysis of appendix G we can show that M^r is a globally defined vector density on the space T/G, since the choice of slice S, while not modular invariant, is invariant under the subgroup G of M. Hence dispensing with Δ_0 in the usual way, $\int_{T/G} \frac{\partial M^r}{\partial t^r}$ will receive contribution only from the boundary where the genus two surface degenerates into two genus one surfaces. (There will be many copies of this boundary in T/G but that is irrelevant).

At the boundary, the point z_3 and z_4 (as well as z_1 and z_2) either all lie on the torus T_1 or all lie on the torus T_2, since they all approach the same Weierstrass point. (The six Weirstrass points on the $g=2$ surface in the $t \to 0$ limit are the points $\frac{1}{2}$, $\frac{\tau_1}{2}$ and $\frac{1+\tau_1}{2}$ on the torus T_1, and $\frac{1}{2}$, $\frac{\tau_2}{2}$ and $\frac{1+\tau_2}{2}$ on the torus T_2, taking the origin at the nodes p_1 and p_2 on the tori T_1 and T_2 respectively.) Let us, for definiteness, take the points z_a to lie on the torus T_1. Let us also, for definiteness, insert the operator $\bar{\xi}(\bar{z}_0)$ on T_2. (The final answer is independent of $\bar{z}_0$). We now need to evaluate the contribution to M_t defined in eq.(4.29), $M_{\bar{t}}$ vanishes for a holomorphic section. The contribution of M_t may be analyzed by using the factorization hypothesis (5.18). In order to get a non-vanishing boundary contribution, M_t must be at least of order $\frac{1}{t}$, and this happens if the operator $\Psi^\dagger(p_2)$ in (5.18) has antiholomorphic conformal dimension 0 or less. The relevant correlator on T_2 involving the antiholomorphic fields is,

$$\langle \Psi^\dagger(p_2)\bar{\xi}(\bar{z}_0)(\bar{\eta}_2, b)\ldots\rangle_{T_2} \tag{C.1}$$

where $\ldots$ denote a possible $\bar{\partial}X^\mu(y)$ term coming from $V^\alpha(y)$ if $y \in T_2$. The relevant operators $\Psi^\dagger$ are $\bar{c}$, $\bar{c}\bar{\partial}X$, $\bar{c}\bar{\psi}^\mu\bar{\psi}^\nu$ and $\bar{c}\bar{\partial}\bar{\phi}$. Matrix elements of each of these operators on T_2 may be shown to vanish after summing over spin structures in the anti-holomorphic sector.

Thus we see that $\int_{T/G} \frac{\partial M^r}{\partial t^r}$ must vanish. On the other hand, since the slice S is holomorphic, the measure is positive semidefinite (see sec.III). Combining these two results we see that the measure calculated with this slice must vanish at every point in the teichmuller space.

Appendix D.

In proving that the partition function is a total derivative in moduli in section (4) we encountered additional terms which are total derivatives in y. These arise from commuting the $BRST$ current through the dilatino vertex operator. In order to prove that the complete superstring partition function is a total derivative in the moduli space, we must show that these additional terms also integrate to total derivatives in the moduli space. In this appendix we shall give the careful analysis needed to show that.

A. The terms in question arise in the process of deforming the $BRST$ contour and in particular from

$$[Q_B, V^\alpha] = \partial_{\bar{y}} V_1^\alpha(y) + \partial_y V_2^\alpha(y) \tag{D.1}$$

as defined in (4.20) and (4.21). The contribution from these terms is of the form (taking $z_0 = \tilde{z}_1$),

$$\int \prod_{i=1}^{6g-6} dt^i \int_{\Sigma - \{z_a\}} d^2y (\partial_{\bar{y}} K_1 + \partial_y K_2) \tag{D.2}$$

where,

$$K_i = \sum_l \oint_{r_l(y)} \frac{dx}{2\pi i} \int D[XBC]\bar{\xi}(\bar{z}_0)\xi(z_1)\xi(\tilde{z}_1)J_\alpha(x)V_i^\alpha(y) \prod_{a=2}^{4g-4}(\check{Y}(z_a) + \partial_i\check{\xi}(z_a)D_i) \prod_{k=1}^{6g-6}(\eta_k, b) \tag{D.3}$$

In writing down (D.3) we have taken ∂_y, $\partial_{\bar{y}}$ operators outside $\oint_{r_l(y)} dx$. This is not a completely obvious step, since the locations of $r_l(y)$ depend on y. To see how this can be done, note that at any point y_0 in the y plane the x integral in (D.3) may be taken to be along any contour C enclosing the points $r_l(y_0)$, but not the points $r'_l(y_0)$ (as defined in sec.IV) and y_0. As long as none of the $r'_l(y_0)$ or y_0 is close to any of the $r_l(y_0)$, we may take the contour C to be a finite distance away from the points $r_l(y_0)$, $r'_l(y_0)$ and y_0. It then follows that there exists a local neighborhood U in the y plane around the point y_0, such that the same contour C will enclose the points $r_l(y)$, and exclude the points $r'_l(y)$ and y for $y \in U$. Since the position of the contour C is independent of y for $y \in U$, we can surely take the ∂_y, $\partial_{\bar{y}}$ operators outside $\oint_C dx$. This, in turn, may be repeated in every local patch in the y plane. In the region of overlap of two such patches, the corresponding contours C and C' may not match, but may be deformed continuously to each other without hitting any singularities, and hence give the same value of $K_i(y)$ $(i = 1, 2)$.

This procedure breaks down if either y, or one of the points $r'_l(y)$ approaches one of the points $r_l(y)$ for any value of y, since then it is not possible to deform the contour C a finite distance away from the points $r_l(y)$ without hitting any of the $r'_l(y)$ or y. However, as we shall show now, this can never happen. First of all, note that $\tilde{z}_1$ is chosen in a way so that none of the poles $r'_l(y)$ coincide with any of the $r_l(y)$. (This may, in turn, require $\tilde{z}_1$ to be a function of y, and one may worry about taking the ∂_y, $\partial_{\bar{y}}$ terms outside the $\xi(\tilde{z}_1)$ term in eqs.(D.2), (D.3). However, note that if we replace $\xi(\tilde{z}_1)$ by $\partial_y \xi(\tilde{z}_1) = \partial\xi(\tilde{z}_1)\partial_y \tilde{z}_1$ (or $\partial\xi(\tilde{z}_1)\partial_{\bar{y}}\tilde{z}_1$) in (D.3), the resulting correlator has no pole at $r_l(y)$, and vanishes after x integration.) In order to see that the point y can never approach $r_l(y)$, let us remember that $r_l(y)$ is the zero of $\vartheta[\delta](\frac{1}{2}(\vec{y} - \vec{x}) + \sum_{a=1}^{2g-2} \vec{z}_a - 2\vec{\Delta})$ in the x plane. Thus if $y = r_l(y)$ at any value of y, $\vartheta[\delta](\sum_{a=1}^{2g-2} \vec{z}_a - 2\vec{\Delta})$ must vanish. As we have remarked after eq.(4.4), the transversality condition (2.36) on the points z_a precisely guarantees that $\vartheta[\delta](\sum_{a=1}^{2g-2} \vec{z}_a - 2\vec{\Delta})$ never vanishes at any point in the moduli

space. This justifies taking the ∂_y, $\partial_{\bar{y}}$ operators outside $\oint_C dx$ in writing down eqs.(D.2), (D.3).

B. Without too much effort we can see that the contribution of the term $\partial_y K_2$ is identically zero: Such a term may receive contribution from the boundary of the y-integration only if the correlator involving V_2 has a simple pole $(\bar{y} - \bar{z}_a)^{-1}$. However the only $\bar{y}$ dependence comes from the $\bar{\partial}X(y)$ term inside V_2 and in the rest of the terms X appears through the combination $\partial X(z_a)$ or $\bar{\partial}X(z_a)$. Consequently in any contraction the only possible singularity as a function of $\bar{y}$ is of the form $\bar{\partial}X(y)\bar{\partial}X(z_a) \sim (\bar{y} - \bar{z}_a)^{-2}+$ non-singular terms; i.e there are no simple poles as a function of $\bar{y}$. As a result the total derivative in y vanishes identically after the y-integration.

C. The analysis of the $\partial_{\bar{y}} K_1$ term is more intricate. In this case K must develop singularities of the form $(y - y_0)^{-1}$ at some point y_0 in the y plane for it to contribute at the boundary of the y integral. Looking at eq.(D.3) we see that for fixed x, y can develop poles either at x, z_a $(a = 1, \ldots 2g - 2)$ or at the zeros of $\vartheta[\delta](\frac{1}{2}(\vec{y} - \vec{x}) + \sum_{a=1}^{2g-2} \vec{z}_a - 2\vec{\Delta})$, $\vartheta[\delta](\frac{1}{2}(\vec{y} - \vec{x}) + \sum_{a=2}^{2g-2} \vec{z}_a + \vec{\tilde{z}}_1 - 2\vec{\Delta})$. But the x contour in (D.3) is chosen in a way so as to avoid the points at $x = y$, as well as all configurations for which either $\vartheta[\delta](\frac{1}{2}(\vec{y} - \vec{x}) + \sum_{a=1}^{2g-2} \vec{z}_a - 2\vec{\Delta})$ or $\vartheta[\delta](\frac{1}{2}(\vec{y} - \vec{x}) + \sum_{a=2}^{2g-2} \vec{z}_a + \vec{\tilde{z}}_1 - 2\vec{\Delta})$ vanishes. The possible poles in y are then at $z_1, \tilde{z}_1$ and at z_a $(a \neq 1)$. Therefore, after performing the y integration we have a double sum

$$\sim \sum_{y_0 = z_1, \tilde{z}_1, z_a} \sum_l \oint_{r_l(y_0)} \frac{dx}{2\pi i} Res(J_\alpha(x) V_1^\alpha(y_0) \bar{\xi}(\bar{z}_0) \xi(\tilde{z}_1) \xi(z_1)$$
$$\prod_{a=2} (\check{Y}_a + \partial_i \check{\xi}_a D_i) \prod_{k=1}^{6g-6} (\eta_k, b)) \tag{D.4}$$

where Res denotes the residue at the argument of the dilatino vertex. Let us now consider in turn the contribution of $\tilde{z}_1$,z_1 and z_a.

D. The part of $V_1^\alpha(y)$ which can develop poles near $\xi(z)$ is $\frac{1}{2}(\gamma_\mu)^{\alpha\dot{\beta}}(\bar{c}\bar{\partial}X^\mu + \bar{\gamma}\bar{\psi}^\mu)e^{\frac{3}{2}\phi}\eta b\hat{S}^- S_{\dot{\beta}}$ and the residue is $\frac{1}{2}(\gamma_\mu)^{\alpha\dot{\beta}}(\bar{c}\bar{\partial}X^\mu + \bar{\gamma}\bar{\psi}^\mu)e^{\frac{3}{2}\phi} b\hat{S}^- S_{\dot{\beta}} \equiv \hat{V}^\alpha$. First of all if we consider the residue at $\bar{z}_1$ it can easily be seen that the resulting correlator as a function of x has poles at $\{r_l'(\bar{z}_1)\}$, but not at $\{r_l(\bar{z}_1)\}$ and hence vanishes after the x-contour integral is performed. The residue at z_1 on the other hand, has poles at $\{r_l(z_1)\}$, but not at $\{r_l'(z_1)\}$. We may now deform the x-contour away from the points $\{r_l(z_1)\}$ and try to shrink it to a point. The only possible obstruction comes from deforming the x contour past z_1. To see if this contributes we examine the OPE $J_\alpha(x)\hat{V}^\alpha(z_1)$, and this is given (up to irrelevant factors) by

$$\begin{aligned} &e^{-\frac{1}{2}\phi(x)}\hat{S}^+(x)S_\alpha(x)e^{\frac{3}{2}\phi(z_1)}\hat{S}^-(z_1)S_{\dot\beta}(z_1) \\ &\qquad\sim (x-z_1)^{\frac{3}{4}}\frac{1}{(x-z_1)^{\frac{3}{4}}} \sim \mathcal{O}(1) \end{aligned} \tag{D.5}$$

Thus there is no singularity and the resulting contribution vanishes.

E. Finally we have to worry about possible poles near $\{z_a, a = 2, \ldots 2g-2\}$ in the y plane. Since $Y(z_a) = \{Q_B, \xi(z_a)\}$ we may write

$$V_1^\alpha(y)Y(z_2) = -[Q_B, V_1^\alpha(y)]\xi(z_a) + \{Q_B, V_1^\alpha(y)\xi(z_a)\} \quad a = 2, \ldots 2g-2. \tag{D.6}$$

where Q_B is the the full $BRST$ charge (of the left and right movers). Now,

$$[Q_B, V_1^\alpha(y)] = \partial_y[(\bar{c}\bar{\partial}X^\mu + \bar\gamma\bar\psi^\mu)(\gamma^\mu)^{\alpha\dot\beta}\lim_{w\to y}\{Y(w)c(y)e^{-\frac{1}{2}\phi(y)}\hat{S}^-(y)S_{\dot\beta}(y)\}] \tag{D.7}$$

we may calculate (D.7) explicitly using the known expression for Y. But the main point here is that the right hand side of (D.7) is a total derivative in y. As a result, near $\xi(z_a)$ it never develops a single pole, since the term inside the bracket in (D.7) never develops logarithmic singularities near $\xi(z_a)$. On the other hand, the second term on the right hand side of (D.6) has a $(y-z_a)^{-1}$ pole with a residue $\{Q_B, \hat{V}^\alpha(z_a)\}$ where, as before, $\hat{V}^\alpha = \frac{1}{2}(\gamma_\mu)^{\alpha\dot\beta}(\bar{c}\bar{\partial}X^\mu + \bar\gamma\bar\psi^\mu)e^{\frac{3}{2}\phi}b\hat{S}^-S_{\dot\beta}$.

We also have,

$$V_1^\alpha(y)\partial\xi(z_a) \sim \frac{1}{y-z_a}\partial\hat{V}^\alpha(z_a) \quad a = 2, \ldots 2g-2, \tag{D.8}$$

and hence after y integration we get a residue of $\partial\hat{V}^\alpha(z_a)$.

F. Combining all the terms together, we may express the result of y integration in (D.2) as,

$$\begin{aligned} &\int\prod_{i=1}^{6g-6}dt^i\sum_{b=2}^{2g-2}\sum_l\oint_{r_l(z_b)}\frac{dx}{2\pi i}\int D[XBC]\bar\xi(\bar{z}_0)\xi(z_1)\xi(\bar{z}_1)J_\alpha(x) \\ &(\{Q_B, \hat{V}^\alpha(z_b)\} + \partial_i\hat{V}^\alpha(z_b)D_i)\prod_{\substack{a=2\\a\neq b}}^{4g-4}(\check{Y}(z_a) + \partial_i\check{\xi}(z_a)D_i)\prod_{k=1}^{6g-6}(\eta_k, b) \end{aligned} \tag{D.9}$$

Now we try to deform away the $BRST$ contour in (D.9). Deformation through the products of picture changing–and (η, b)–insertions produces a total derivative in the moduli in the usual way. Since $J_\alpha(x)$ is $BRST$ invariant up to a total derivative and since $J_\alpha(x)$ is being integrated over a closed contour we need only worry about the $BRST$ residues at $\bar\xi(\bar{z}_0)\xi(z_1)\xi(\bar{z}_1)$. The residue at $\bar\xi(\bar{z}_0)$ is $\bar{Y}(\bar{z}_0)$, and this vanishes since nothing remains to soak up the $\bar\xi$ zero mode. We are left with $\{Q_B, \xi(z_1)\xi(\bar{z}_1)\} = Y(z_1)\xi(\bar{z}_1) - \xi(z_1)Y(\bar{z}_1)$. The correlator involving $Y(\bar{z}_1)$ has no poles at $r_l(z_b)$ as a function of x and hence vanishes after the x integration. The correlation involving $Y(z_1)$, on the other hand, has poles in x only at $r_l(z_b)$, since, as we have seen there is no singularity in the OPE of $J_\alpha(x)$ with $\hat{V}^\alpha(z_b)$. The sum of the residues at the poles $r_l(z_b)$ can be seen to vanish, because we can deform back the supersymmetry (x-contour) integral, and there are no singularities of $J_\alpha(x)$ at $\hat{V}^\alpha(z_b), \check{Y}(z_a)$ or $\partial_i\check{\xi}(z_a)$. Thus after these contour deformations we can write the final contribution to (D.9) as:

$$\int\prod_{i=1}^{6g-6}dt^i\left(\frac{\partial}{\partial t^j}F_j - G\right) \tag{D.10}$$

where,

$$\begin{aligned} F_j = &\sum_{b=2}^{2g-2}\sum_l\oint_{r_l(z_b)}\frac{dx}{2\pi i}\int D[XBC]\bar\xi(\bar{z}_0)\xi(z_1)\xi(\bar{z}_1)J_\alpha(x) \\ &\hat{V}^\alpha(z_b)D_j\prod_{\substack{a=2\\a\neq b}}^{4g-4}(\check{Y}(z_a) + \partial_i\check{\xi}(z_a)D_i)\prod_{k=1}^{6g-6}(\eta_k, b) \end{aligned} \tag{D.11}$$

and,

$$G = \sum_{b=2}^{2g-2} \sum_{l} \sum_{j=1}^{6g-6} \oint_{r_l(z_b)} \frac{dx}{2\pi i} \int D[XBC] \bar{\xi}(z_0) (\partial_j \xi(z_1) \xi(\tilde{z}_1) + \xi(z_1) \partial_j \xi(\tilde{z}_1)) J_\alpha(x)$$
$$\hat{V}^\alpha(z_b) D_j \prod_{\substack{a=2 \\ a\neq b}}^{4g-4} (\check{Y}(z_a) + \partial_i \check{\xi}(z_a) D_i) \prod_{k=1}^{6} (\eta_k, b) \tag{D.12}$$

Again, the term involving $\partial_i \xi(z_b) = \partial_i z_b \partial \xi(z_b)$ in (D.12) has no pole at $x = r_l(z_b)$, and vanishes after x integration. The term involving $\partial_i \xi(\tilde{z}_1)$, on the other hand, has poles only at $r_l(z_b)$ in the x plane, and hence may be shown to vanish by deforming back the x contour as before. Consequently G is identically zero. This completes our proof of section (4) that the partition function is indeed a total derivative in the moduli space. Note also that F_j defined in (D.11) vanishes at genus two by ϕ-ghost charge conservation.

Appendix E.

In this appendix we shall show that the cosmological constant Λ as defined in eq.(4.4) is invariant under simultaneous transformations of $\partial_i z_a^\mu$, $\eta_{i\bar{z}}{}^{z}$ and $\eta_{iz}{}^{\bar{z}}$ induced by a reparametrization v^μ:

$$\begin{aligned} \delta(\partial_i z_a^\mu) &= \partial_i v^\mu(z_a) \\ \delta(\eta_{i\bar{z}}^{z}) &= \partial_i \partial_{\bar{z}} v^{z} \\ \delta(\eta_{iz}^{\bar{z}}) &= \partial_i \partial_z v^{\bar{z}} \end{aligned} \tag{E.1}$$

Under these transformations Λ changes to,

$$\begin{aligned} \Lambda' = &\prod_a (\check{Y}_a + \sum_i \partial_i z_a^\mu \partial_\mu \check{\xi}^a D_i + \sum_i \partial_i v^\mu(z_a) \partial_\mu \check{\xi}^a D_i) \\ &\prod_{j=1}^{6g-6} \Big((\eta_j, b) + \int d^2 w (\partial_{\bar{w}} \partial_j v^w b_{ww} + \partial_w \partial_j v^{\bar{w}} b_{\bar{w}\bar{w}}) \Big), \\ = &\prod_a (\check{Y}_a + \sum_i \partial_i z_a^\mu \partial_\mu \check{\xi}^a D_i + + \sum_i \partial_i v^\mu(z_a) \partial_\mu \check{\xi}^a D_i) \\ &\prod_{k=1}^{6g-6} (1 + \int d^2 w (\partial_{\bar{w}} \partial_k v^w b_{ww} + \partial_w \partial_k v^{\bar{w}} b_{\bar{w}\bar{w}}) D_k) \prod_{j=1}^{6g-6} (\eta_j, b) \end{aligned} \tag{E.2}$$

In each term in the second product we may now do the w integration by parts. It receives contribution only from the regions where b, $\bar{b}$ have poles, namely, near the locations z_a. Noting that,

$$\begin{aligned} b_{ww}(w) Y(z_a) &\sim \frac{1}{w - z_a} \partial \xi(z_a) \quad (a = 1, \cdots, 2g-2) \\ \bar{b}_{\bar{w}\bar{w}}(w) \bar{Y}(z_a) &\sim \frac{1}{\bar{w} - \bar{z}_a} \bar{\partial} \bar{\xi}(z_a) \quad (a = 2g-1, \cdots, 4g-4) \end{aligned} \tag{E.3}$$

we may evaluate the contribution from the residues at z_a. The result may be expressed in the compact form,

$$\Lambda' = \prod_a \Big(\check{Y}_a + \sum_i \partial_i z_a^\mu \partial_\mu \check{\xi}^a D_i + \sum_i \partial_i v^\mu(z_a) \partial_\mu \check{\xi}_a D_i - \sum_k \partial_k v^\mu(z_a) \partial_\mu \check{\xi}_a D_k \Big) \prod_{j=1}^{6g-6} (\eta_j, b) \tag{E.4}$$

but this is just Λ, thus we see the invariance of Λ under reparametrizations connected to the identity. One consequence of this invariance is that by choosing appropriate v^μ we may set $\partial_i z_a^\mu = 0$, at the cost of changing the η's.

Appendix F.

In this appendix we shall sketch how the insertion of the stress tensor in given correlator accounts for the full variation of the correlator in moduli, i.e. variation due to explicit dependence but also to implicit dependence on the moduli. These matters have already been discussed in [12] and in [89][90]; here we take a point of view which ties in with the treatment in sections 2 and 4. As explained in sec. 4 the latter dependence is due to the fact that the coordinate system in which we are exhibiting our vertex operators, which is the coordinate system that diagonalizes the metric at a given point in moduli space, changes as we move in moduli. Let t be a specific point in the moduli space, while z denotes the coordinate system in which the metric at t is diagonal, and set $V_k(z_k)$ $(k = 1, \dots N)$ to be several vertex operators inserted at points z_k on the riemann surface such that $\frac{\partial z_k}{\partial t} = 0$ in the sense defined in section two. We shall first assume that all the vertex operators are of dimension (0,0), since this is the simplest case. Let us now consider the correlator,

$$\langle \prod_{k=1}^{N} V_k(z_k)(\eta_r, T)\delta t^r\rangle_t \equiv \int d^2 z \langle \prod_{k=1}^{N} V_k(z_k)(\eta_{r\bar{z}}{}^{z} T_{zz} + \eta_{rz}{}^{\bar{z}} \bar{T}_{\bar{z}\bar{z}})\delta t^r\rangle \tag{F.1}$$

Let $g^{\alpha\beta}(z, t+\delta t)$ be the metric at the point $t+\delta t$ in the teichmuller space in the z coordinate system, and w be the coordinate system which makes the metric at $t+\delta t$ diagonal. Then there exists a quasi-conformal map,

$$w = z + v^z, \quad \bar{w} = \bar{z} + v^{\bar{z}} \tag{F.2}$$

where the vector field v^z is discontinuous on the riemann surface. In particular, we may represent the riemann surface at t as a region in the complex z plane bounded by some curve C, with various parts of C identified with each other. The discontinuity in v^z is then reflected in the fact that v^z on two different parts of C which are identified do not match. In terms of $v^z, v^{\bar{z}}$ we may write,

$$\eta_{r\bar{z}}{}^{z}\delta t^r = -\partial_{\bar{z}} v^z, \quad \eta_{rz}{}^{\bar{z}}\delta t^r = -\partial_z v^{\bar{z}} \tag{F.3}$$

Substituting (F.3) in (F.1) we may do the z integral by parts, picking up residues from the poles in $z, \bar{z}$ at z_i. The final answer is,

$$\begin{aligned}\langle \prod_{k=1}^{N} V_k(z_k)(\eta_r, T)\delta t^r\rangle_t =& \langle \sum_j \prod_{k\neq j}^{N} V_k(z_k)(v^z(z_j)\partial V(z_j) + v^{\bar{z}}(z_j)\bar{\partial} V(z_j))\rangle_t \\ &- \frac{1}{2\pi i}\langle (\oint_C dz v^z(z) T(z) + \oint_C d\bar{z} v^{\bar{z}}(z)\bar{T}(\bar{z})) \prod_k V_k(z_k)\rangle_t\end{aligned} \tag{F.4}$$

We may now compare it with the difference,

$$\langle \prod_{k=1}^{N} V_k(w_k)\rangle_{t+\delta t} - \langle \prod_{k=1}^{N} V_k(z_k)\rangle_t \equiv \delta_t \langle \prod_{k=1}^{N} V_k(z_k)\rangle_t \tag{F.5}$$

Since $\frac{\partial z_k}{\partial t} = 0$, $w_k = z_k + v^z(z_k)$. Both these correlators may be identified with the correlators in field theories defined on the complex plane with diagonal metric, but with different boundary conditions on the fundamental fields in the theory. In calculating $\langle\ \rangle_t$, the relevant boundary conditions on the fields are obtained by identifying various parts of the curve C with each other. On the other hand, in calculating $\langle\ \rangle_{t+\delta t}$, the relevant boundary conditions are obtained by identifying various parts of the curve C' with each other, where C' is the image of the curve C in the w plane under the map (F.2). Thus the difference between the two terms in (F.5) may be written as a sum of two terms. One reflects the fact that w_k, regarded as a complex number, is not the same as z_k, but differs from it by $v^z(z_k)$. This term may be easily identified with the first term in (F.4). The second difference is due to the fact that the curve C' constituting the boundary in the w plane differs from the curve C by $(v^z, v^{\bar{z}})$ evaluated on C. It can be shown that the effect of the second term in (F.4) is precisely to move the boundary from C to C'. [38]Thus (F.4) is the same as (F.5), showing that the insertion of a factor of (η_r, T) in a correlator generates a derivative of the correlator with respect to t^r.

[38] More precisely, the effect of $\oint(dz v^z T(z) + d\bar{z} v^{\bar{z}} T(\bar{z}))$ is to convert the boundary condition on various fields obtained by identifying various parts of C with each other to those obtained by identifying various parts of C' with each other.

If some of the vertex operators (say $V_1(z_1)$) are of dimension (1,1) instead of (0,0), the operator product of $T(z), \bar{T}(\bar{z})$ with $V_1(z_1)$ will have extra terms, and as a result, (F.4) will have extra terms on the right hand side. Using these extra terms we can show that,

$$d^2z_1\langle \prod_{k=1}^{N} V_k(z_k)(\eta_r, T)\delta t^r\rangle = d^2w_1\langle \prod_{k=1}^{N} V_k(w_k)\rangle_{t+\delta t} - d^2z_1\langle \prod_{k=1}^{N} V_k(z_k)\rangle_t \tag{F.6}$$

Hence as long as all dimension (1,1) operators are integrated over the riemann surface (and similarly all dimension (1,0) operators are integrated over a contour) the insertion of (η_r, T) in a correlator generates a derivative of the correlator with respect to t^r.

We now come to the case of correlators with insertions of ghosts. As is well known, these should be regarded as differential forms on moduli space. For example, if $\gamma_1, \ldots, \gamma_l$ are vector fields on $\mathcal{M}$, represented by beltrami differentials $\gamma_i = [\eta_i]$, then we can define the l-form Ω by

$$\Omega(\gamma_1, \ldots, \gamma_l) = \langle (b, \eta_1) \cdots (b, \eta_l) \prod V_k(z_k)\rangle \tag{F.7}$$

By the equations of motion the RHS indeed defines a differential form on $\mathcal{M}$. Choosing a good slice so that $\frac{\partial}{\partial t^i} = [\eta_i]$ we have

$$\Omega = \langle (b, \eta_{i_1}) dt^{i_1} \cdots (b, \eta_{i_l}) dt^{i_l} \prod V_k(z_k)\rangle \tag{F.8}$$

The insertion of the stress-tensor now defines the exterior derivative

$$d\Omega = \langle (T, \eta_s) dt^s (b, \eta_{i_1}) dt^{i_1} \cdots (b, \eta_{i_l}) dt^{i_l} \prod V_k(z_k)\rangle \tag{F.9}$$

To prove this we expand

$$\langle (b, \eta_{i_1}(t+\delta t)) \cdots (b, \eta_{i_l}(t+\delta t)) \prod V_k(z_k)\rangle_{t+\delta t} \tag{F.10}$$

in δt. First expand the η_i. If we compute tensors in a fixed frame-index and coordinate system then we may regard the frame e^a_m as a matrix $A(t)$ and then $\eta_i = A^{-1}\frac{\partial}{\partial t^i}A$. Since $A^{-1} = A^{tr}$, we have $\eta_i^{tr} = -\eta_i$. Thus although

$$\partial_i \eta_j = A^{-1}\frac{\partial^2}{\partial t^i \partial t^j}A - \eta_i \eta_j$$

is not symmetric in i, j, it *is* true that

$$tr b \partial_i \eta_j = tr b \partial_j \eta_i$$

since b is symmetric when regarded as a 2×2 matrix in a fixed coordinate system. The analogous identity $(\partial_i \eta_j, T) = (\partial_j \eta_i, T)$ occurs in the proof that $[\partial_i, \partial_j] = 0$ when acting on correlation functions. Therefore, if we expand $\eta_i(t+\delta t) = \eta_i(t) + \delta t^s \partial_s \eta_i$ the contraction with $dt^s \wedge dt^i$ gives no correction term. If we express tensors in the coordinate system diagonalizing the metric at t then $\eta_i(t)$ only has a $(dz)^{-1} \otimes d\bar{z}$ component so we need only consider the $(dz)^2$ term in the pullback of b. This is:

$$b(z+v)(d(z+v))^2 = b(z)(dz)^2 + (v\partial b + 2\partial v b)(dz)^2 + \cdots$$

where v is as in (F.2). On the other hand, writing once more $(T, \eta_s)\delta t^s = (T, \bar{\partial} v) + (\bar{T}, \partial \bar{v})$ we see that the OPE of T with b accounts for the above change in b, exactly as in the previous cases. Putting these facts together we arrive at (F.9).

Appendix G.

In this appendix we shall sketch a proof that the M^r defined in eq.(4.24) are indeed the components of a globally defined $6g-7$-form on moduli space. (Equivalently, this can be thought of as a vector density.) What we must prove is that given any two points t, $\tilde{t}$ in the moduli space related by a modular transformation, we need to show that,

$$M^r(\tilde{t}) = \{det \frac{\partial \tilde{t}^m}{\partial t^n}\}^{-1} \frac{\partial t^r}{\partial t^s} M^s(t) \tag{G.1}$$

Although this analysis may be carried out for the general expression given in (4.24), we may, without any loss of generality, set $\frac{\partial z_a}{\partial t^s}$ to be 0 in M^r, provided we do not impose any restrictions on η_r's. This is a consequence of reparametrization invariance discussed in appendix E. Using transformation (E.1), which can be shown to be an exact symmetry of M^r following steps similar to those in appendix E, we may set $\frac{\partial z_a}{\partial t^i} = 0$ at each point t in the teichmuller space. In this scheme, if $\vec{u}$ denotes a fixed coordinate system on the riemann surface, then $\vec{u}_a(t)$ is independent of t. Furthermore, if t, $\tilde{t}$ are points in the teichmuller space related to each other by a modular transformation, and $g^{\alpha\beta}(\vec{u}, t)$, $g^{\alpha\beta}(\vec{u}, \tilde{t})$the representative metrics at these points, then the global diffeomorphism $\mathcal{D}(t)$ which takes $g^{\alpha\beta}(\vec{u}, t)$ to $g^{\alpha\beta}(\vec{u}, \tilde{t})$ leaves $\vec{u}_a$ fixed. This is true for every point t in the teichmuller space.

The modular transformation properties of M^r may be determined by studying the transformation properties of various terms in the correlator in (4.24) under the global diffeomorphism $\mathcal{D}$. Since $\check{Y}$ are dimension (0,0) operators they remain invariant under diffeomorphism $\mathcal{D}$, as long as their arguments go to their images under $\mathcal{D}$. Similarly, since $J_\alpha(x)$ and $V^\alpha(y)$ are dimension (1,0) and (1,1) operators respectively, $\oint dx J_\alpha(x)$ and $\int d^2y V^\alpha(y)$ are invariant under diffeomorphism. Thus we only need to study the transformation laws of (η_r, b) under $t \to \tilde{t}$. If we can show that,

$$(\eta_r, b) \mid_{\tilde{t}} d\tilde{t}^r = (\eta_r, b) \mid_t dt^r \tag{G.2}$$

then (η_r, b) transforms as a contravariant vector, and so,

$$\prod_{s \neq r} (\eta_s, b) \equiv \varepsilon^{r l_2 \ldots l_{6g-6}} (\eta_{l_2}, b) \ldots (\eta_{l_{6g-6}}, b) \tag{G.3}$$

will transform as a $6g - 7$ form. This would, in turn, prove that M^r is a globally defined $6g - 7$ form in the moduli space.

In order to study the transformation law of (η_r, b), let us express all the relevant correlators in the fixed coordinate system. Then we may write,

$$(\eta_r, b) \mid_t \delta t^r = \int d^2\vec{u}\sqrt{g}\frac{\partial g^{\alpha\beta}}{\partial t^r}\delta t^r b_{\alpha\beta}(\vec{u}, t) = \int d^2\vec{u}\sqrt{g} g^{\alpha\beta}(\vec{u}, t + \delta t) b_{\alpha\beta}(\vec{u}, t) \tag{G.4}$$

where $b_{\alpha\beta}(t)$ denotes that in writing down the ghost action we use the metric $g^{\alpha\beta}(t)$, and use the constraint $g^{\alpha\beta}(t) b_{\alpha\beta}(t) = 0$. Similarly,

$$(\eta_r, b) \mid_{\tilde{t}} \delta\tilde{t}^r = \int d^2\vec{u}\sqrt{g} g^{\alpha\beta}(\tilde{t} + \delta\tilde{t}) b_{\alpha\beta}(\tilde{t}) \tag{G.5}$$

where $\tilde{t} + \delta\tilde{t}$ is the image of $t + \delta t$ under modular transformation. Now we may write,

$$g^{\alpha\beta}(\tilde{t}) = (\mathcal{D}_1(t))^{\alpha\beta}_{\gamma\delta} g^{\gamma\delta}(t) \tag{G.6}$$

where $\mathcal{D}_1$ is the operator representing the action of $\mathcal{D}$ on g. We may write a similar equation at $t + \delta t$. Since $b_{\alpha\beta}$ transforms in the same way as $g_{\alpha\beta}$, we have,

$$b_{\alpha\beta}(\tilde{t}) = (\mathcal{D}_1^{-1}(t))^{\gamma\delta}_{\alpha\beta} b_{\gamma\delta}(t) \tag{G.7}$$

Using (G.6) and (G.7) we get,

$$\int d^2\vec{u}\sqrt{g} g^{\alpha\beta}(\tilde{t} + \delta\tilde{t}) b_{\alpha\beta}(\vec{u}, \tilde{t}) = \int d^2\vec{u}\sqrt{g} g^{\alpha\beta}(t + \delta t)(\mathcal{D}_1^{-1}(t)\mathcal{D}_1(t + \delta t))^{\gamma\delta}_{\alpha\beta} b_{\gamma\delta}(\vec{u}, t) \tag{G.8}$$

Since $\mathcal{D}_1^{-1}(t)\mathcal{D}_1(t + \delta t)$ represents an infinitesimal diffeomorphism, we may express (G.8) as,

$$\int d^2\vec{u}\sqrt{g} g^{\alpha\beta}(t + \delta t) b_{\alpha\beta}(t) + \int d^2\vec{u}\sqrt{g}(\nabla^\alpha v^\beta + \nabla^\beta v^\alpha) b_{\alpha\beta}(\vec{u}, t). \tag{G.9}$$

The first term gives $(\eta_r, b) \mid_t \delta t^r$. The second term may be made to vanish by doing integration by parts and using equations of motion of $b_{\alpha\beta}(\nabla^\alpha b_{\alpha\beta} = \nabla^\beta b_{\alpha\beta} = 0)$, except for possible poles in the argument of $b_{\alpha\beta}$. These may occur at the points $\vec{u}_a$ since $\check{Y}(\vec{u}_a)$ contains factors of $c^\alpha(\vec{u}_a)$. Note, however, from the requirement of modular invariance, and from the independence of $\vec{u}_a$'s on t, $\mathcal{D}(t)$ as well as $\mathcal{D}(t + \delta t)$ leaves $\vec{u}_a$ fixed. As a result $v^\alpha(\vec{u}_a)$ must vanish identically, showing the vanishing of the second term in (G.9). This proves the required relation,

$$(\eta_r, b) \mid_t \delta t^r = (\eta_r, b) \mid_{\tilde{t}} \delta\tilde{t}^r \tag{G.10}$$

and consequently proves that M^r is a globally defined $6g - 7$ form in the moduli space. A similar analysis can be used to show that F^r defined in appendix D is also a globally defined $6g - 7$ form on moduli.

Note that if we had not worked in this special choice of gauge where $\partial_i z_a = 0$, then we would get non-vanishing residues from the points z_a while trying to integrate the second term in (G.9) by parts. In this case, however, we would also be left with some non-vanishing contribution when we try to transform $\partial\xi(z_a)$ or $\bar{\partial}\bar{\xi}(z_a)$ appearing in the expression for M^r under a global diffeomorphism, since these are dimension (1,0) or (0,1) operators, and hence no longer invariant under diffeomorphism. These two effects would then cancel each other. This may be verified by explicit computation, however, there is no need to do so, due to the reparametrization invariance of M^r discussed in appendix E. Finally we would like to mention that at least for $g = 2$, M_r remains a globally defined $6g - 7$ form if modular transformations do not take the points q_a to the images in the sense of eq. (2.41) but instead permutes them. This is because as showed in sec. 5.C the final expression for M_r is explicitly invariant under $z_1 \leftrightarrow z_2$.

References

[1] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B271** (1986) 93.

[2] G. Moore, P. Nelson and J. Polchinski, Phys. Lett. **B169** (1986) 47.

[3] P. Nelson and G. Moore, Nucl. Phys. **B274** (1986) 509.

[4] E. D'Hoker and D. Phong, Nucl. Phys. **B278** (1986) 225.

[5] E. D' Hoker and D. Phong, Nucl. Phys. **B292** (1987) 317

[6] E. Verlinde and H. Verlinde, Phys. Lett. **B192** (1987) 95.

[7] M. Baranov and A. Schwarz, "On the multiloop contribution to the string theory" preprint

[8] P. Deligne, unpublished letter to D. Kazhdan

[9] A. A. Voronov, "A formula for the Mumford measure in superstring theory," preprint.

[10] J. J. Atick, J. M. Rabin and A. Sen, preprint IASSNS-HEP-87/45 (SLAC-PUB-4420) Sept. 1987, to appear in Nucl. Phys. B.

[11] G. Moore and A. Morozov, preprint IASSNS-HEP-87/47 Sept. 1987.

[12] E. Martinec, Nucl. Phys. **B281** (1987) 157.

[13] B. de Witt, Supermanifolds, Cambridge Univ. Press (1984); A. Rogers, J. Math. Phys. **26** (1985) 385, **27** (1986) 710; J. M. Rabin, Physica **15D** (1985) 65 (Proceedings of the Workshop o n Supersymmetry in Physics, held at Los Alamos, N. Mex., Dec. 1983); M. Rothstein, Trans. AMS **299** (1987) 387.

[14] D. Lebedev and A. Morozov, "Statistical sums of strings on hyperelliptic surfaces," preprint ITEP-16

[15] V. Knizhnik, Phys. Lett. **B196**(1987)473

[16] A. Morozov, preprint ITEP-87, no. 143

[17] A. Morozov and A. Perelomov, preprint ITEP-104

[18] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258** (1985) 46.

[19] L. J. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B261** (1985) 620, **B274** (1986) 285.

[20] H. Kawai, D. C. Lewellen and S.-H. H. Tye, Phys. Lett. **191B** (1987) 63; Nucl. Phys. **B288** (1987) 1.

[21] R. Bluhm, L. Dolan and P. Goddard, Nucl. Phys. **B289** (1987) 364.

[22] K. Narain, M. H. Sarmadi and C. Vafa, Nucl. Phys. **B288** (1987) 551.

[23] L. J. Dixon, V. Kaplunovsky and C. Vafa, Nucl. Phys. **B294** (1987) 45.

[24] I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys. **B289** (1987) 87.

[25] M. Mueller and E. Witten, Phys. Lett. **182B** (1986) 28.

[26] L. Ibanez, H. Nilles and F. Quevedo, Phys. Lett. **187B** (1987) 25.

[27] W. Lerche, D. Lust and A. N. Schellekens, Nucl. Phys. **B287** (1987) 477

[28] W. Lerche, B. E. Nilsson and A. N. Schellekens, Nucl. Phys. **B294** (1987) 136

[29] M. Dine, N. Seiberg and E. Witten, Nucl. Phys. **B289** (1987) 589; M. Dine, I. Ichinose and N. Seiberg, Nucl. Phys. **B293** (1987) 253.

[30] J. J. Atick, L. J. Dixon and A. Sen, Nucl. Phys. **B292** (1987) 109

[31] P. Howe, J. Phys. **A 12** (1979) 393

[32] M. Baranov, I. Frolov, and A. Schwarz, Teor. Mat. Fiz. **70**(1987)92[=Theor. Math. Phys. **70**(1987)64]

[33] S. Giddings and P. Nelson, preprint BUHEP-87-28 (HUTP-87/A062), Aug. 1987.

[34] P. Nelson, preprint BUHEP-87-30, Sept. 1987.

[35] S. Giddings and P. Nelson, "The Geometry of Super Riemann Surfaces," BUHEP-87-31 and HUTP-87/A070

[36] Yu. Manin, Funk. Anal. **20**(1986)88 [=Funct. Anal. Appl. **20**(1987)244; Talk at ICM, Berkeley 1986

[37] C. Le Brun and M. Rothstein, IAS preprint (1987)

[38] P. Deligne, unpublished letter to Manin.

[39] M. Baranov and A. Shvarts, Pisma ZETF **42**(1985)340= [JETP Lett. **42**(1986) 419]; M. Baranov, Yu. Manin, I. Frolov, and A. Shvarts, Yad. Phys. **43**(1986)1053[=Sov. J. Nucl. Phys. **43** (1986)670]

[40] J. Rabin and L. Crane, Comm. Math. Phys., to appear.

[41] C. Earle and J. Eells, J. Diff. Geom. **3**(1969)19.

[42] Yu. Manin, *Gauge fields and complex geometry*, English translation to appear.

[43] L. Bers, Bull. Lond. Math. Soc. **4**(1972)257; Bull. Amer. Math. Soc. **5**(1981)131

[44] H. Grauert, Math. Annalen, **135** (1958) 263

[45] C. Earle, Duke Math. J. **36** (1969) 409

[46] For a description of sheaf theory see, e.g. R. C. Gunning, Lectures on Riemann Surfaces, Princeton, 1966; R. Bott and L. Tu, Differential Forms in Algebraic Topology, Springer-Verlag Graduate texts in Mathematics 82,1983.

[47] A. Belavin and V. Knizhnik, Phys. Lett. **168B** (1986) 201; Sov. Phys. JETP **64** (1986) 214 [≡ ZhETF **91** 364]

[48] M. Bershadsky and A. Radul, Int. J. Mod. Phys. **A2** (1987) 165.

[49] D. Montano, preprint SLAC-PUB-4319 (1987)

[50] H. Farkas and I. Kra, *Riemann Surfaces* Springer GTM 71

[51] O. Lechtenfeld and A. Parkes, preprint CCNY-HEP-87/10

[52] J. Hubbard, C.R. Acad. Sci. Paris Ser. A-B, 274 (1972) A978-A979; Mem. Am. Math. Soc. No. 166

[53] C. Earle and I. Kra, Acta Math. 137 (1976) 49

[54] E. Martinec, Phys. Rev. **D28**(1983)2604

[55] J. Polchinski, Comm. Math. Phys. **104** (1986) 37.

[56] G. Moore and P. Nelson, Nucl. Phys. **B266** (1986) 58; E. D' Hoker and D. Phong, Nucl. Phys. **B269** (1986) 205

[57] J.-B. Bost and Th. Jolicoer, Phys. Lett. **B174** (1986)273

[58] L. Alvarez-Gaumé, G. Moore, and C. Vafa, Comm. Math. Phys. **106**(1986)1

[59] P. Nelson, "Lecture on Moduli Space and Strings," Harvard preprint HUTP-86/A047

[60] J.-B. Bost, Seminaire Bourbaki, 39ème anée, no. 676

[61] V. G. Knizhnik, Institute for theoretical physics (Kiev) preprints, ITP-87-60P, ITP-87-61P, ITP-87-62P, submitted to Sov. Sci. Rev.

[62] G. Moore, J. Harris, P. Nelson, and I. Singer, Phys. Lett. **178B**(1986)167

[63] J. J. Atick and A. Sen, Phys. Lett. **186B** (1987) 339.

[64] H. Sonoda, preprint LBL-23619 June (1987).

[65] L. Dixon, D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B282** (1987) 13.

[66] M. Bershadsky and A. Radul, Int. J. Mod. Phys. **A2**(1987)165

[67] A. Zamolodchikov, ZhETF **90**(1986)1808

[68] L. Alvarez Gaume, J. B. Bost, G. Moore, P. Nelson and C. Vafa, Phys. Lett. **B178** (1986) 41; Commun. Math. Phys. **112** (1987) 503

[69] T. Eguchi and H. Ooguri, Phys. Lett. **187 B** (1987) 127

[70] E. Verlinde and H. Verlinde, Nucl. Phys. **B288** (1987) 357

[71] M. Dugan and H. Sonoda, Nucl. Phys. **B289** (1987) 227.

[72] V.G. Knizhnik, Comm. Math. Phys. **112**(1987)567

[73] J. J. Atick and A. Sen, SLAC-PUB-4292 (1987), Nucl. Phys. B in press.

[74] J. J. Atick and A. Sen, preprint IASSNS-HEP-87/53 (SLAC-PUB-4440), to appear in proceedings of the NATO workshop on Superstrings held at Boulder Colorado, July 1987.

[75] J. D. Fay, *Theta Functions on Riemann Surfaces*, Springer Notes in Mathematics 352, Springer (1973)

[76] J. J. Atick and A. Sen, Nucl. Phys. **B286** (1987) 189

[77] D. Mumford, *Tata Lectures on Theta* vol. I, Birkhauser, Boston 1983

[78] W. Boucher, D. Friedan and A. Kent, Phys. Lett. **172B** (1986) 316

[79] A. Sen, Nucl. Phys. **B278**(1986)289; Nucl. Phys. **B284**(1987)423;

[80] L. Dixon, D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. **B282**(1987)13

[81] T. Banks, L. Dixon, D. Friedan, and E. Martinec, SLAC-PUB-4377

[82] V.G. Knizhnik, Phys. Lett. **178B**(1986)21

[83] M. Green and N. Seiberg, preprint IASSNS-HEP-87/38 (1987)

[84] J. Greensite and F. R. Klinkhamer, preprint NBI-HE-87-58 (1987)

[85] C. Wendt, preprint SLAC-PUB-4442 (1987)

[86] D. Friedan and S. Shenker, Phys. Lett. **175B** (1986)287; Nucl. Phys. **B281**(1987)509

[87] H. Verlinde, "A note on the integral over the fermionic supermoduli," Utrecht preprint, THU-87/26

[88] C. Earle, *in Discrete groups and automorphic forms*, ed. W.J. Harvey (Academic Press), NY, 1977

[89] H. Sonoda, Nucl. Phys. **B281**(1987)546

[90] T. Eguchi and H. Ooguri, Nucl. Phys. **B282** (1987)308