GGI LECTURE 4:

THE SPECTRUM-GENERATING STOKES MATRIX: FLIPS, TWISTS & POPS

HITCHIN SYSTEMS & FLAT CONN'S

THE H.E.'S $\Longrightarrow$

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi}$$

IS FLAT.

NEAR REG. SING. POINT z_i

$$\mathcal{A} \sim \left(\frac{R}{\zeta}\frac{\rho}{2} + \frac{\alpha}{2i}\right)\frac{dz}{z-z_i} + \left(R\zeta\frac{\bar{\rho}}{2} - \frac{\alpha}{2i}\right)\frac{d\bar{z}}{\bar{z}-\bar{z}_i}$$

SO B.C.'S FIX MONODROMY OF $\mathcal{A}$.

AROUND z_i:

$$M_i \sim \begin{pmatrix} \mu_i & \\ & \mu_i^{-1} \end{pmatrix}$$

$$\mu_i = \exp\left[2\pi i\left(\tfrac{1}{2}\zeta^{-1}Rm_i - m_i^3 - \tfrac{1}{2}\zeta R\bar{m}_i\right)\right]$$

THEOREM OF C. SIMPSON $\Longrightarrow$

IDENTIFY $\mathcal{M}^{\zeta}$, $\zeta \in \mathbb{C}^*$, WITH MODULI OF FLAT $SL(2,\mathbb{C})$ CONNECTIONS WITH PRESCRIBED MONODROMY AT z_i

MOREOVER, THE HOLO. SYMPLECTIC FORM ON $\mathcal{M}^{\zeta}$ HAS THE SIMPLE FORM:

$$\boxed{\varpi_{\zeta} = \int_C \mathrm{Tr}(\delta \mathcal{A}\, \delta \mathcal{A})}$$

2. FOCK–GONCHAROV COORD'S AND CLUSTER TMNS

$\mathcal{A}$ = A FLAT $SL(2,\mathbb{C})$ CONNECTION WITH MONODROMY M_i AROUND z_i

$$M_i \sim \begin{pmatrix} \mu_i & \\ & 1/\mu_i \end{pmatrix}$$

A. DECORATED TRIANGULATIONS

<u>DEF:</u> A "DECORATED TRIANG." T IS AN IDEAL TRIANGULATION OF C WITH VERTICES AT z_i TOGETHER WITH A CHOICE OF MONODROMY EIGENVALUE μ_i OR $1/\mu_i$ AT EACH z_i

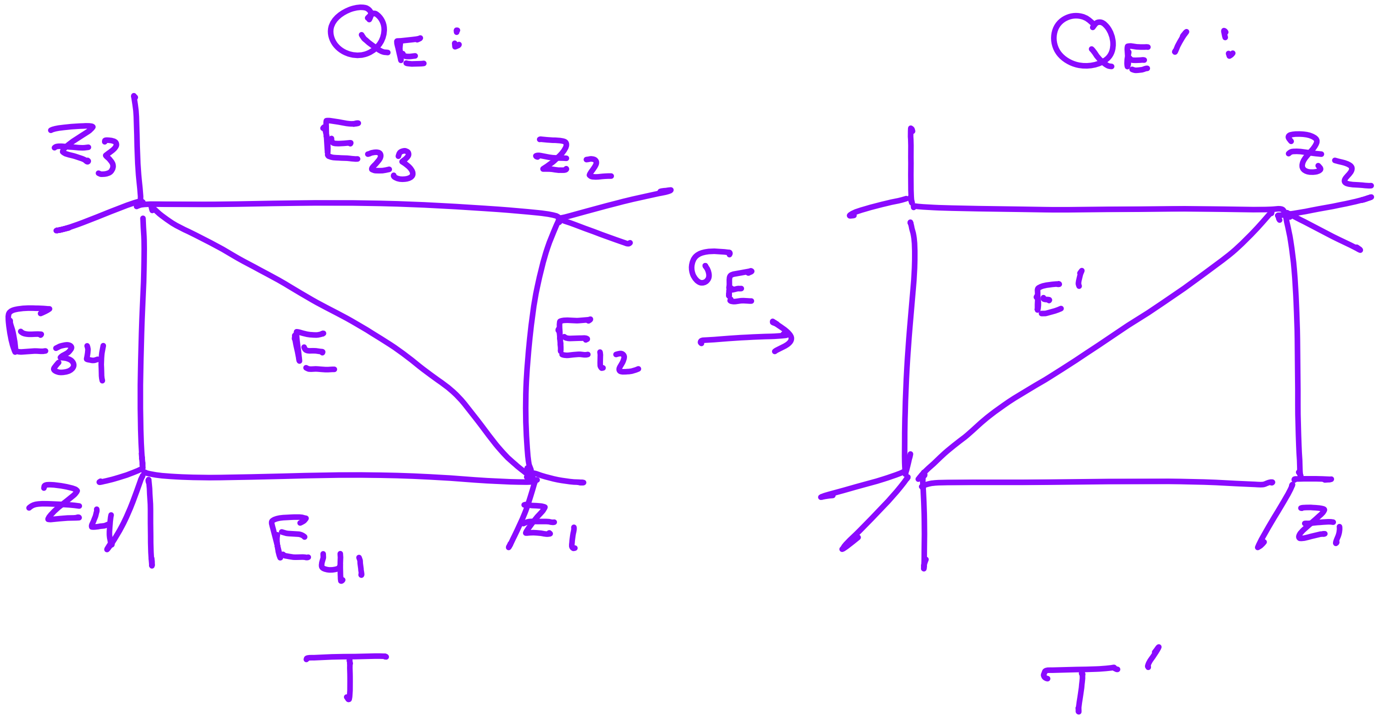
B. FLIPS AND POPS

DEFINE A GROUPOID:

OBJECTS = DECORATED. TRIANGULATIONS

MORPHISMS ARE GENERATED BY FLIPS + POPS

FLIP: σ_E FOR $E \in \mathcal{E}(T)$



POP: π_i FOR $z_i \in V(T)$:

EXCHANGE: $\mu_i \longleftrightarrow \mu_i^{-1}$

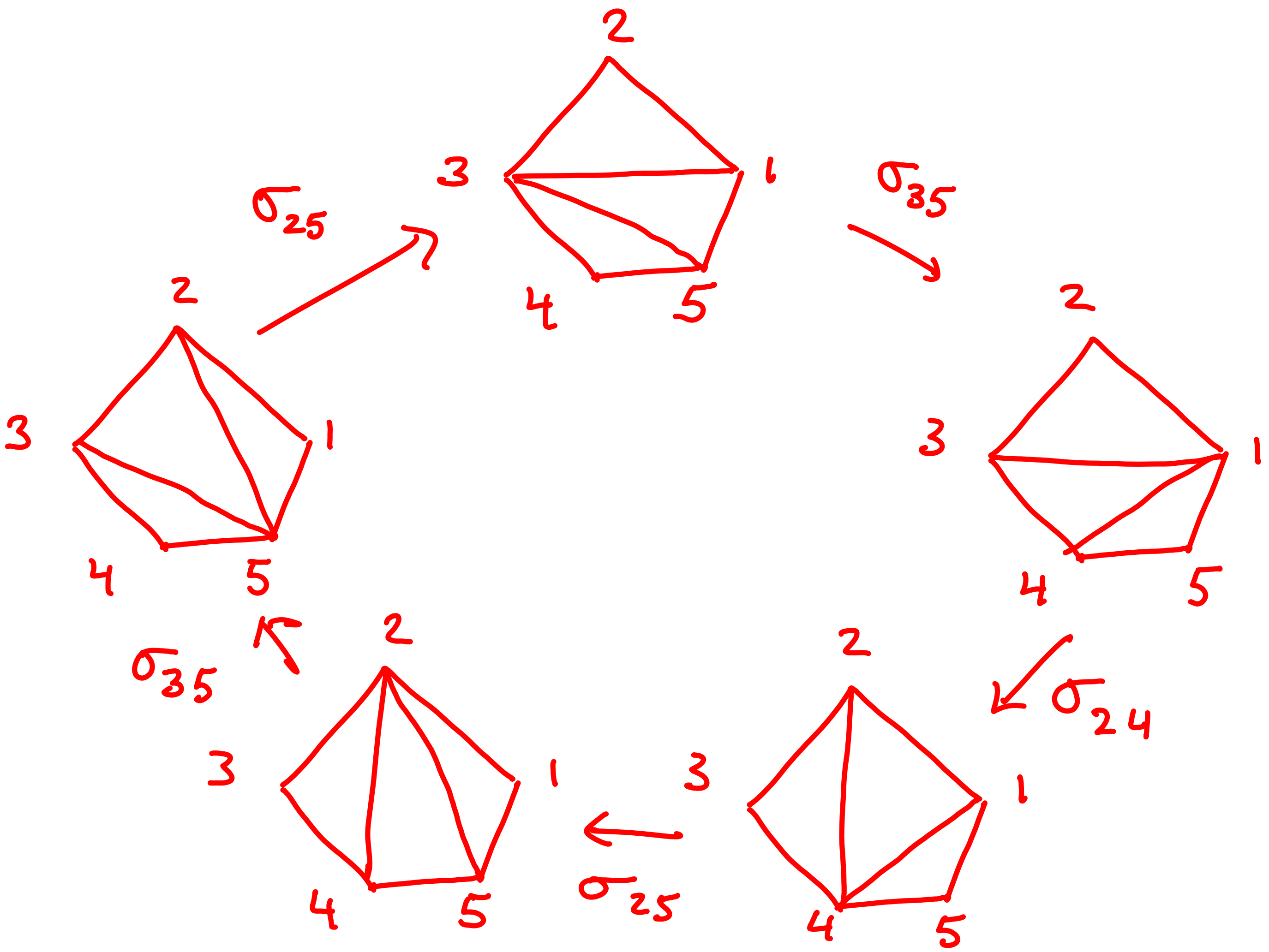
RELATIONS ON FLIPS & POPS

1. $\sigma_E^2 = 1$ AND $\pi_i^2 = 1$

2. POPS COMMUTE

3. $\sigma_E, \sigma_{E'}$ COMMUTE IF $Q_E, Q_{E'}$ DO NOT SHARE A TRIANGLE

4. IF $Q_E, Q_{E'}$ SHARE A TRIANGLE



LATER WE WILL ENHANCE OUR GROUPOID TO INCLUDE "LIMIT TRIANG'S" AND "TWISTS"

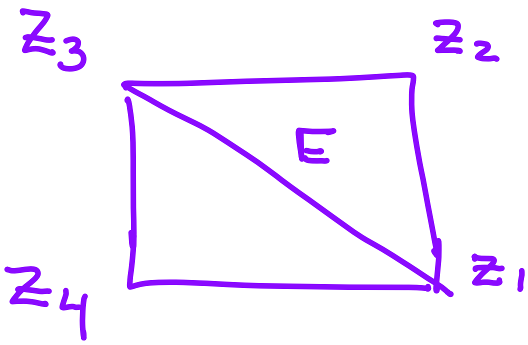
C. FG COORDINATES

GIVEN A DECORATED TRIANGULATION OF C, F & G DEFINE A COLLECTION OF FUNCTIONS ON $\mathcal{M}$:

$$\mathcal{X} : T \longrightarrow \{\mathcal{X}_E^T\}_{E \in \mathcal{E}(T)}$$

DEFINITION:

Q_E:



- CHOOSE FLAT SECTIONS s_i OF SPECIFIED MONODROMY NEAR z_i

- $$\boxed{\mathcal{X}_E^T := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}}$$

$$\boxed{\chi_E^T := -\frac{(s_1\wedge s_2)(s_3\wedge s_4)}{(s_2\wedge s_3)(s_4\wedge s_1)}}$$

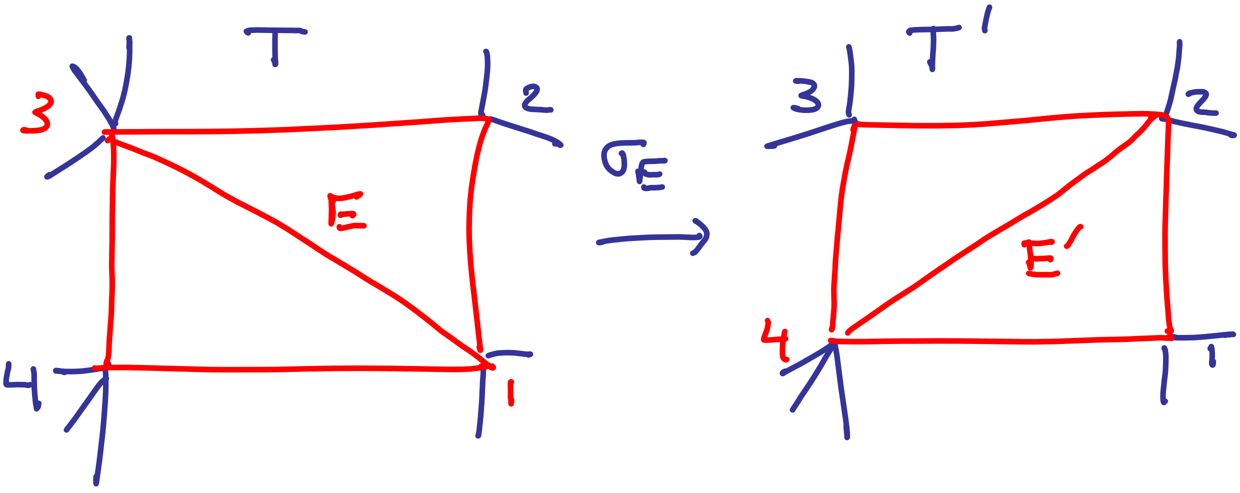
- $s_i \wedge s_j \in \Lambda^2 E$ = LINE BUNDLE
- PARALLEL TRANSPORT TO ANY POINT $Q \in Q_E$
- NORMALIZATION OF s_i CANCELS

THEOREM: (F & G) $\{\chi_E^T\}_E$ PROVIDE HOLO. COORDINATES ON OPEN SET U_T OF $\mathcal{M}$.

D. COORDINATE TMN'S

NOW DESCRIBE THE COORD. TMNS AS WE CHANGE THE DECORATED TRIANGULATION $T \to T'$

TRANSFORMATION UNDER FLIPS



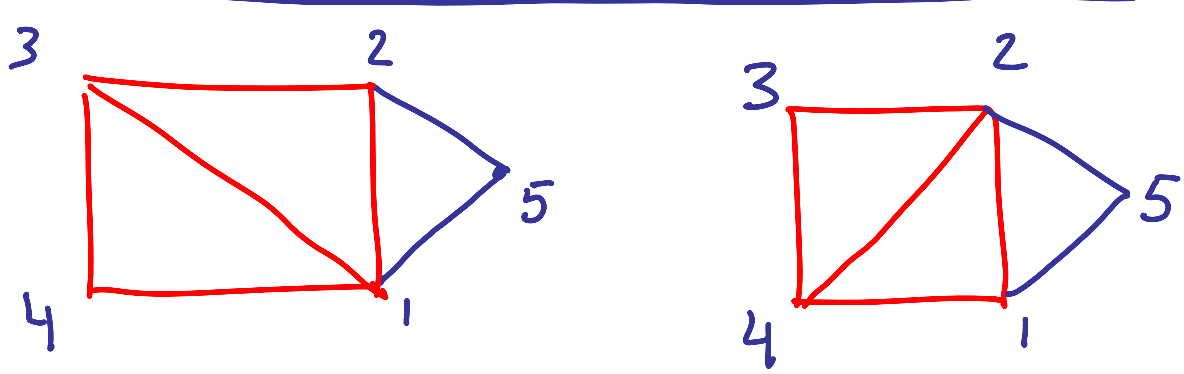
ONLY THE EDGES IN RED CHANGE

$$\chi^{T'}_{E'} = -\frac{S_4 \wedge S_1 \; S_2 \wedge S_3}{S_1 \wedge S_2 \; S_3 \wedge S_4} = \frac{1}{\chi^T_E}$$

$$\chi^{T'}_{E_{12}} = \chi^T_{E_{12}}\left(1 + \chi^T_E\right)$$

do this computation

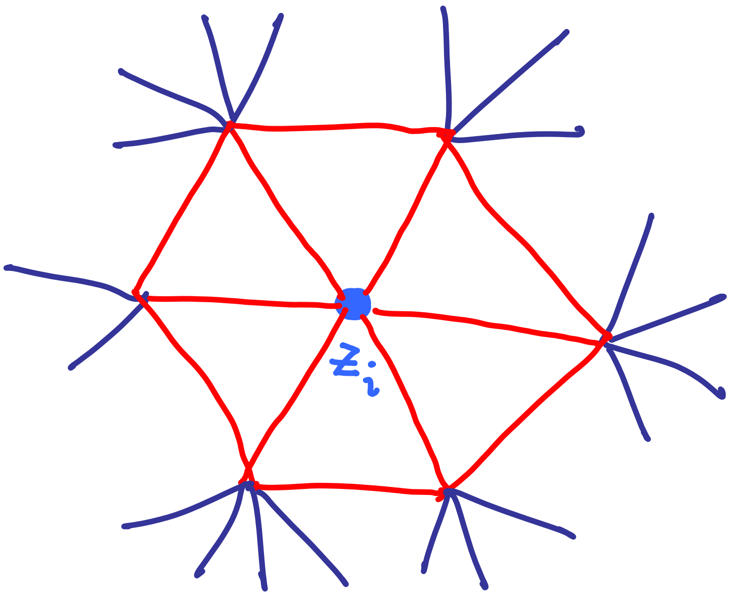
$\ldots$



"CLUSTER TRANSFORMATIONS"

TRANSFORMATION UNDER POPS

A POP AT VERTEX z_i CHANGES THE EDGE COORDINATES IN RED



IT IS POSSIBLE TO WRITE EXPLICIT FORMULAE FOR THE POP TRANSFORMATION, BUT THEY ARE COMPLICATED....

IMPORTANTLY!

IT TURNS OUT THAT THE PRODUCT OF ALL POPS $\prod_v \pi_v$ IS RELATIVELY SIMPLE...

E. SYMPLECTIC STRUCTURE

- USING THE SYMPLECTIC STRUCT $\omega_\mathfrak{T}$ ONE CAN SHOW

$$\{\chi^T_E, \chi^T_{E'}\} = \langle E, E' \rangle \chi^T_E \chi^T_{E'}$$

- TRANSFORMATIONS UNDER FLIPS & POPS ARE POISSON

Sketch proof

3. WKB TRIANGULATIONS

A. MOTIVATION

RECALL THAT A KEY PROPERTY OF $\mathcal{X}_\gamma(\zeta)$ ARE THE $\zeta \to 0$ ASYMPTÓTS:

$$\lim_{\zeta \to 0} \mathcal{X}_\gamma(\zeta)\, e^{-\frac{\pi R}{\zeta} Z_\gamma(u)} \sim \text{FINITE}$$

$\Rightarrow$ WE NEED TO USE VERY SPECIAL TRIANGULATIONS FOR WHICH WE CAN PROVE SUCH ASYMPTOTICS.

IDEA: USE THE WKB APPROXIMATION TO DESCRIBE THE FLAT SECTIONS:

$$(d + \mathcal{A})\, s = 0$$

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\overline{\varphi}$$

FOR $\zeta \to 0$, $\zeta \sim \hbar$

$$\left(\zeta\, d + R\varphi + O(\zeta)\right) s = 0$$

RECALL: $\varphi \sim \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$

WKB:

$$s \underset{\zeta \to 0}{\sim} \exp\left(-\frac{R}{\zeta}\int^{z} \lambda\ \sigma^3\right) \underset{\text{Const.}}{s_0}$$

FROM THIS GET ASYMPT'S OF FG COORD.

B. WKB CURVES

HOWEVER THE WKB APPXT. IS NOTORIOUSLY SUBTLE.

EXPONENTIALLY SMALL CORRECTIONS CAN GROW IN Z AND INVALIDATE COMPUTATIONS

FOR VALIDITY OF WKB APPXT. WE MUST RESTRICT TO VERY SPECIAL TRIANGULATIONS $T(\vartheta, \lambda)$ WHOSE EDGES ARE <u>WKB CURVES</u>

<u>DEF</u>: <u>WKB CURVE WITH ANGLE ϑ</u>:

CURVE ON C WITH

$$\boxed{\langle \lambda, \partial_t \rangle = \pm e^{i\vartheta}}$$

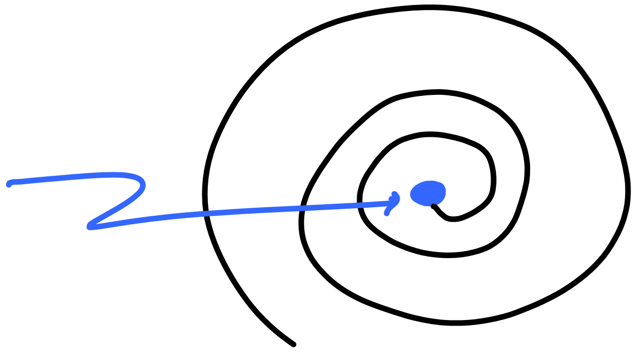
$\Rightarrow$ WKB FOLIATION OF C.

NOTE: WKB CURVES GET TRAPPED BY SINGULARITIES:

$$\lambda = \frac{m}{2}\frac{dz}{z} \Longrightarrow z(t) = z_0 \exp\left(-\frac{e^{i\vartheta}}{m} t\right)$$

put in σ

$z_i = 0$



THREE KINDS OF WKB CURVES:

GENERIC: BOTH ENDS ON z_i, z_j

SEPARATING: CONNECTS BRANCH POINT w_a TO SINGULAR POINT z_i

FINITE: CLOSED, OR BOTH ENDS ON TURNING POINTS w_a, w_b

NOTE THAT OUR RULE FOR BPS STATES WAS THAT $\exists\, \vartheta_*$ FOR WHICH THERE IS A FINITE WKB CURVE

<u>IMPORTANT FACT</u>: FOR GENERIC VALUES OF ϑ THERE ARE NO FINITE WKB CURVES. BUT AT SPECIAL CRITICAL VALUES OF ϑ THERE ARE FINITE WKB CURVES.

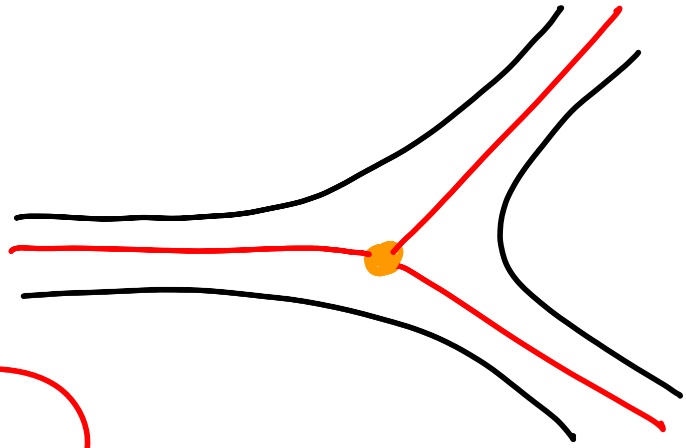
RECALL THAT FOR FINITE WKB CURVES $\langle \lambda, \partial_t \rangle = e^{i\vartheta_*}$ WITH $\vartheta_* = \arg Z$.

SO: THE CRITICAL VALUES ARE THE PHASES ϑ_* OF BPS STATES.

C. DEFINITION

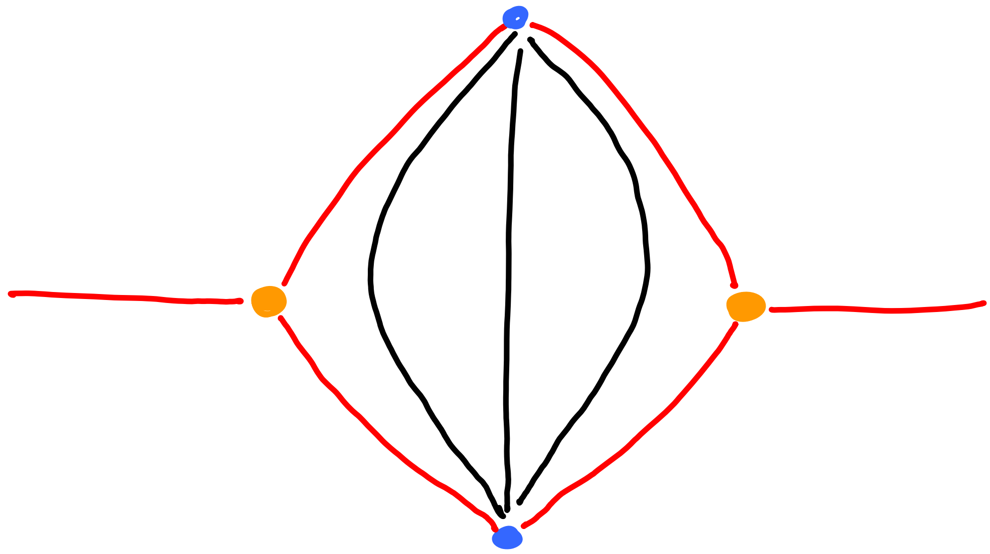
TO DEFINE OUR TRIANGULATION WE FIRST USE THE SEPARATING CURVES TO SPLIT C INTO WKB CELLS

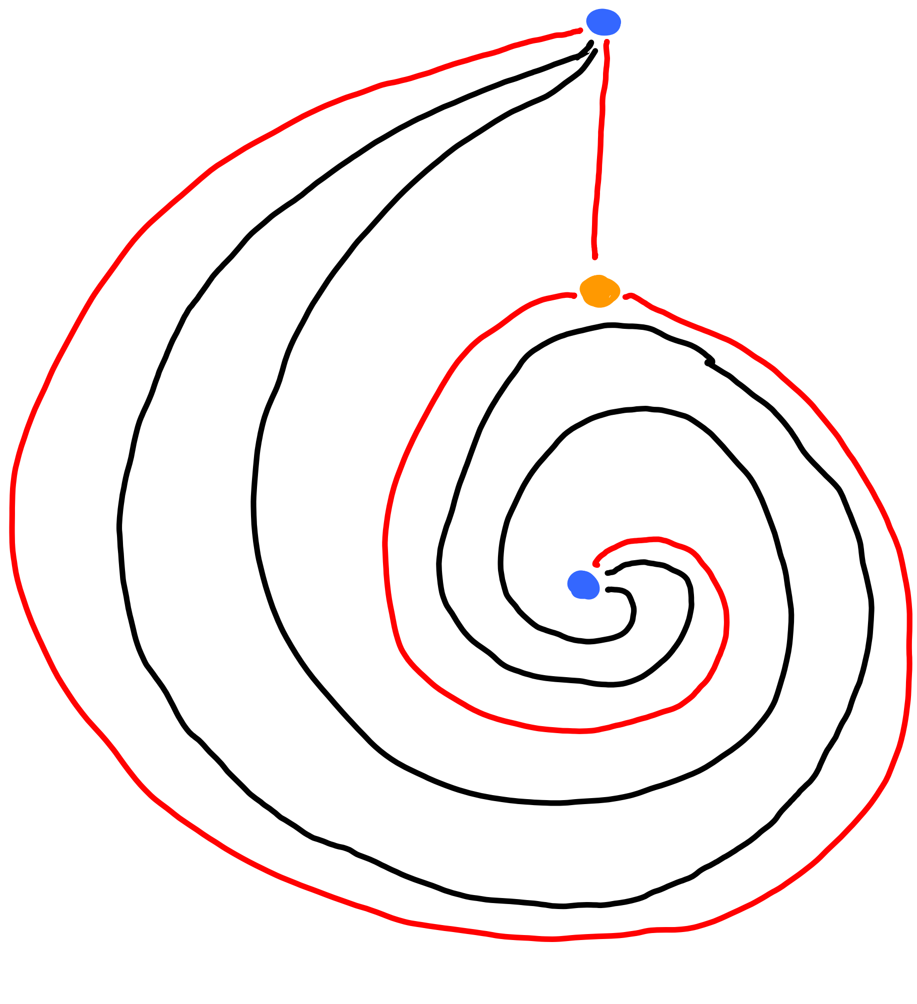
LOCALLY:



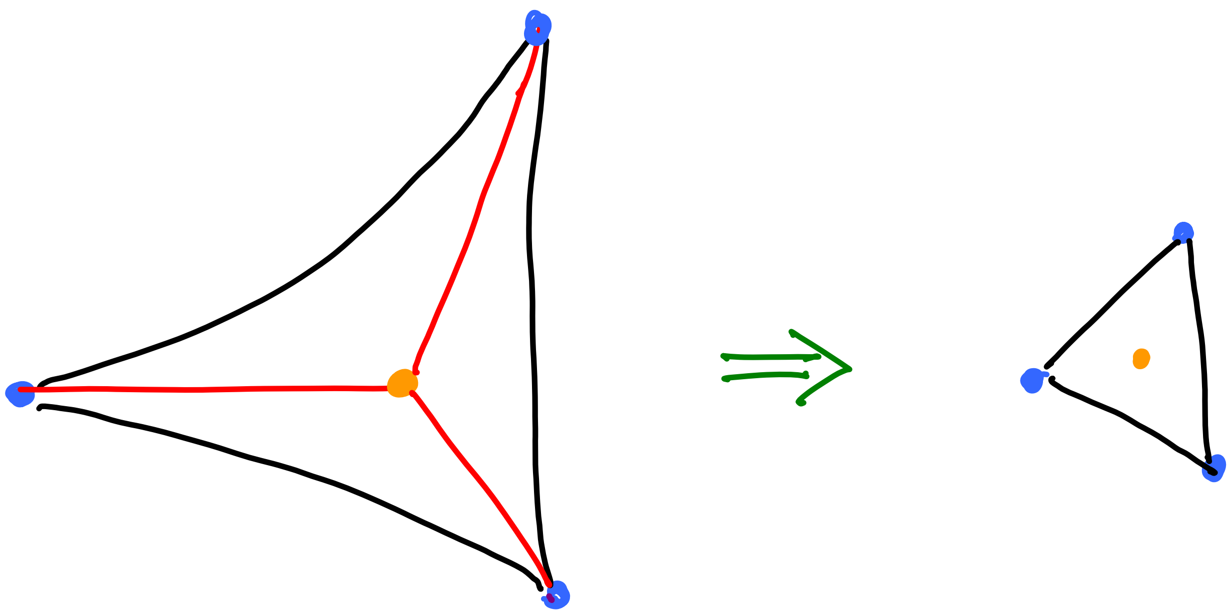
more detail

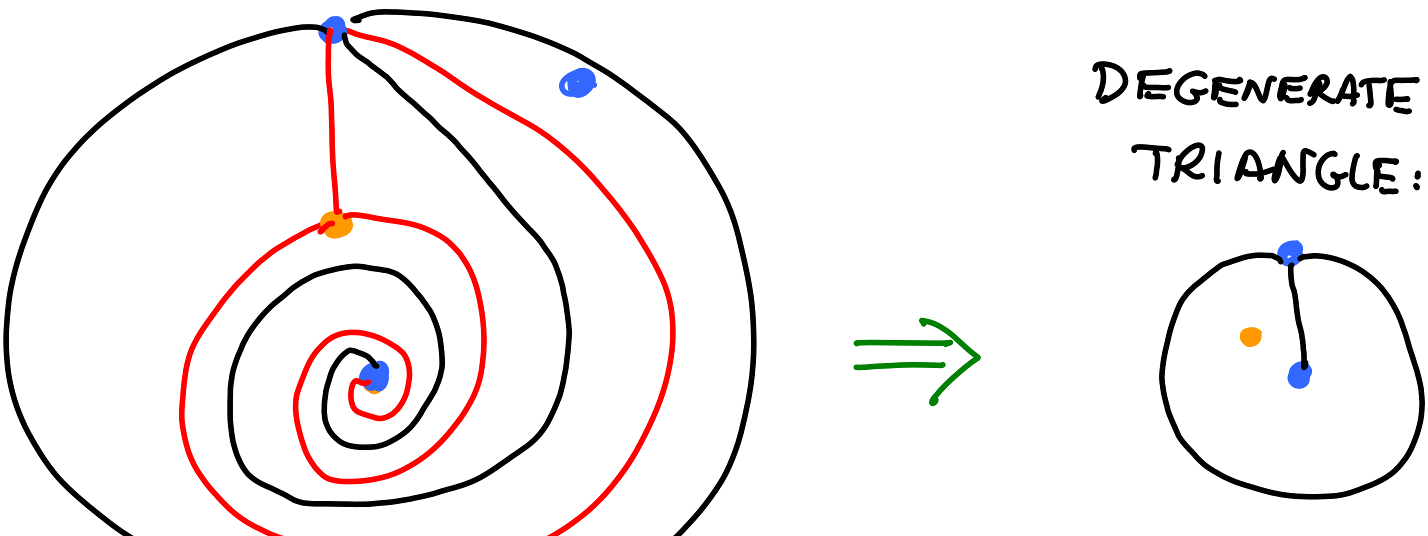
FOR GENERIC λ, ϑ IT TURNS OUT THERE ARE ONLY TWO KINDS OF CELLS:





FOR THE WKB TRIANGULATION WE CHOOSE A GENERIC WKB CURVE IN EACH CELL:





CHOICE OF μ_i

RECALL THAT WE MUST DEFINE A "DECORATED TRIANGULATION."

$(\lambda, \vartheta) \Rightarrow$ DISTINGUISHED EIGENVALUE OF M_i

"SMALL FLAT SECTION": THE FLAT SECTION WHICH DECAYS ALONG THE WKB CURVE GOING INTO THE SINGULARITY

THESE ARE THE SECTIONS FOR WHICH WE HAVE GOOD CONTROL IN WKB APPXT.

more detail

DENOTE THE RESULTING DECORATED TRIANGULATION $T(\vartheta, \lambda)$

D. MORPHISMS OF WKB TRIANG'S

VARY ϑ $\Longrightarrow$ (HOMOTOPY CLASS OF) $T(\vartheta, \lambda)$ IS UNCHANGED

EXCEPT AT CRITICAL VALUES ϑ_c WHERE FINITE WKB CURVES DEVELOP.

$\boxed{\text{WHEN VARYING } \vartheta, \ T(\vartheta,\lambda) \text{ JUMPS PRECISELY AT THE VALUES OF PHASES OF BPS STATES!}}$

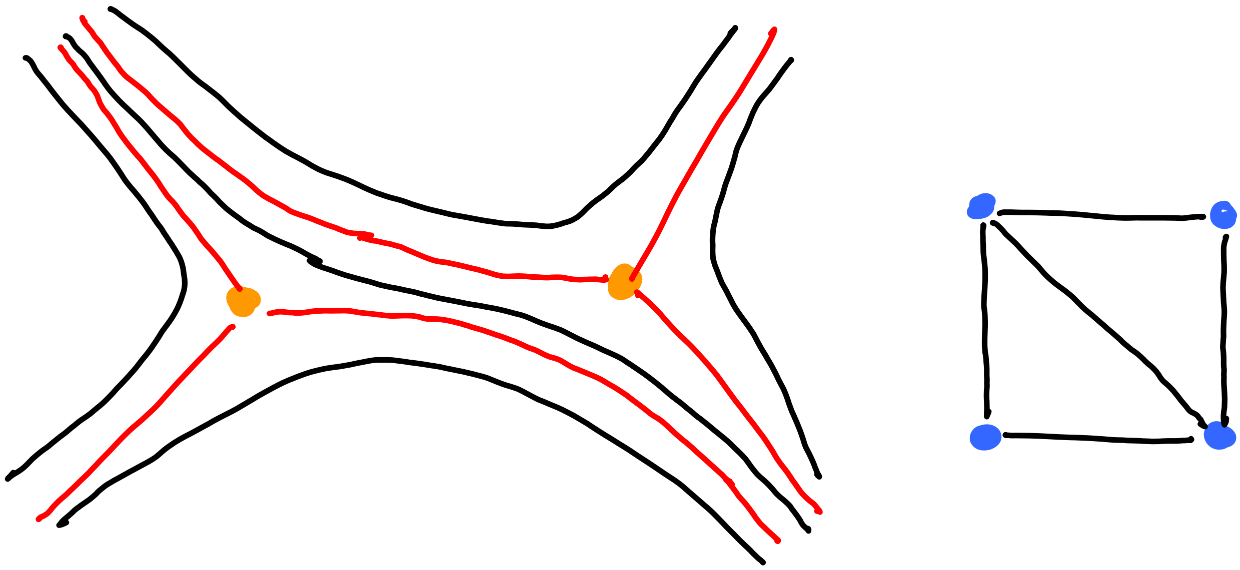
FOR GENERIC λ A JUMP IN $T(\vartheta, \lambda)$ ONLY HAPPENS WHEN A SEPARATING CURVE DEGENERATES TO A FINITE WKB CURVE JOINING TURNING POINTS w_a, w_b.

THUS, FOR GENERIC λ THERE ARE ONLY TWO KINDS OF JUMPS:

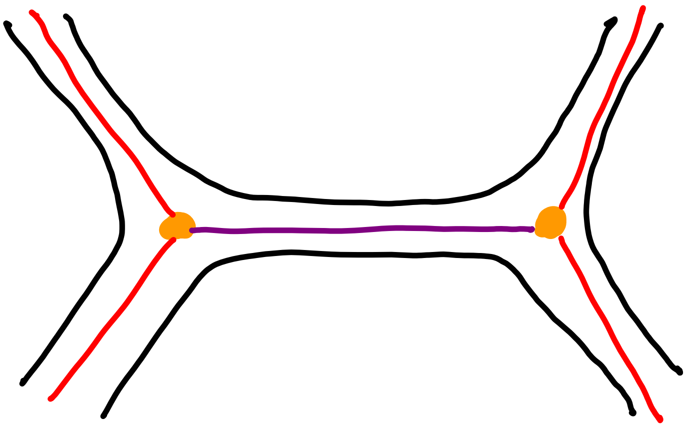
- EITHER $w_a \neq w_b$
- OR $w_a = w_b$

HYPERMULTIPLET JUMP: $W_a \neq W_b$

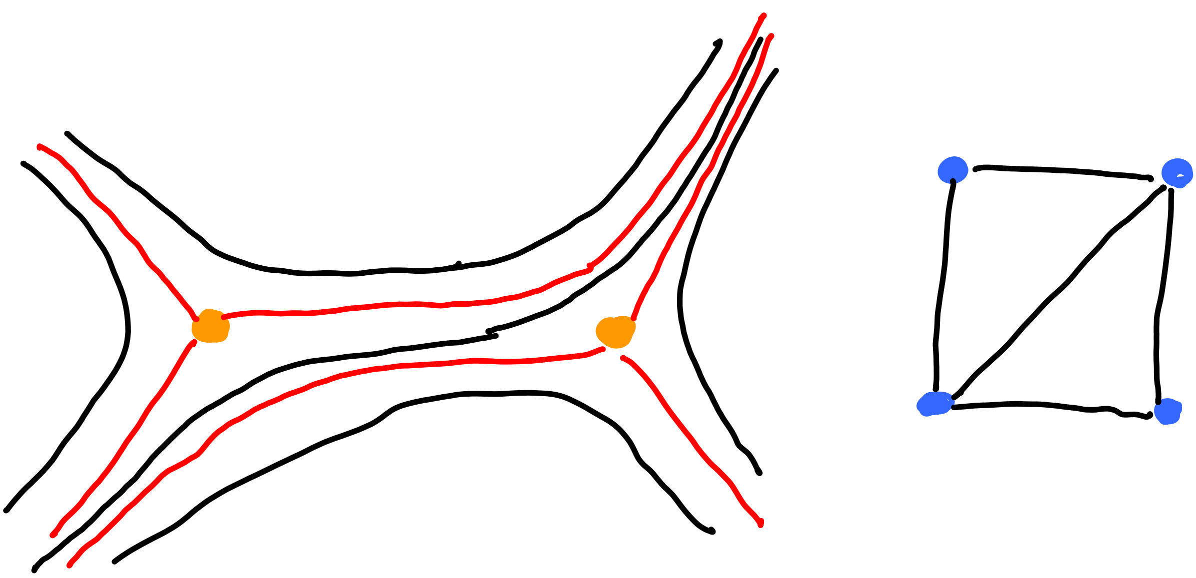
$\vartheta < \vartheta_c$



$\vartheta = \vartheta_c$

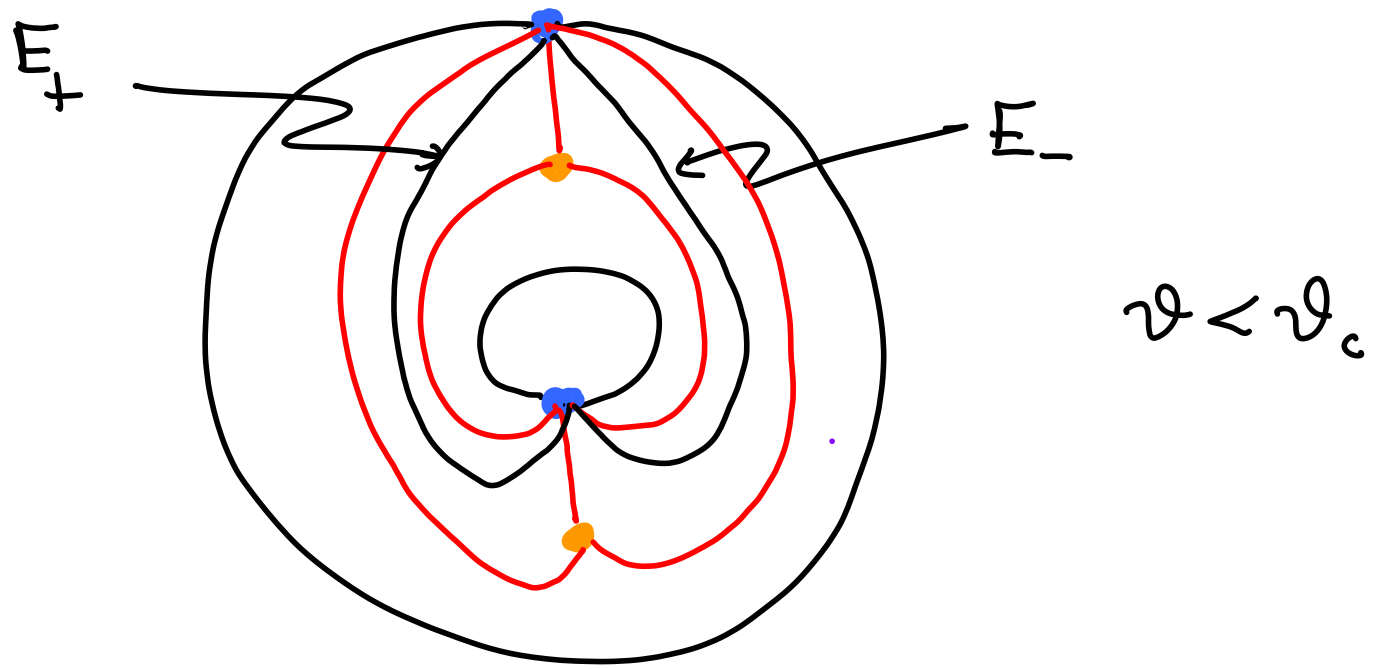


$\vartheta > \vartheta_c$



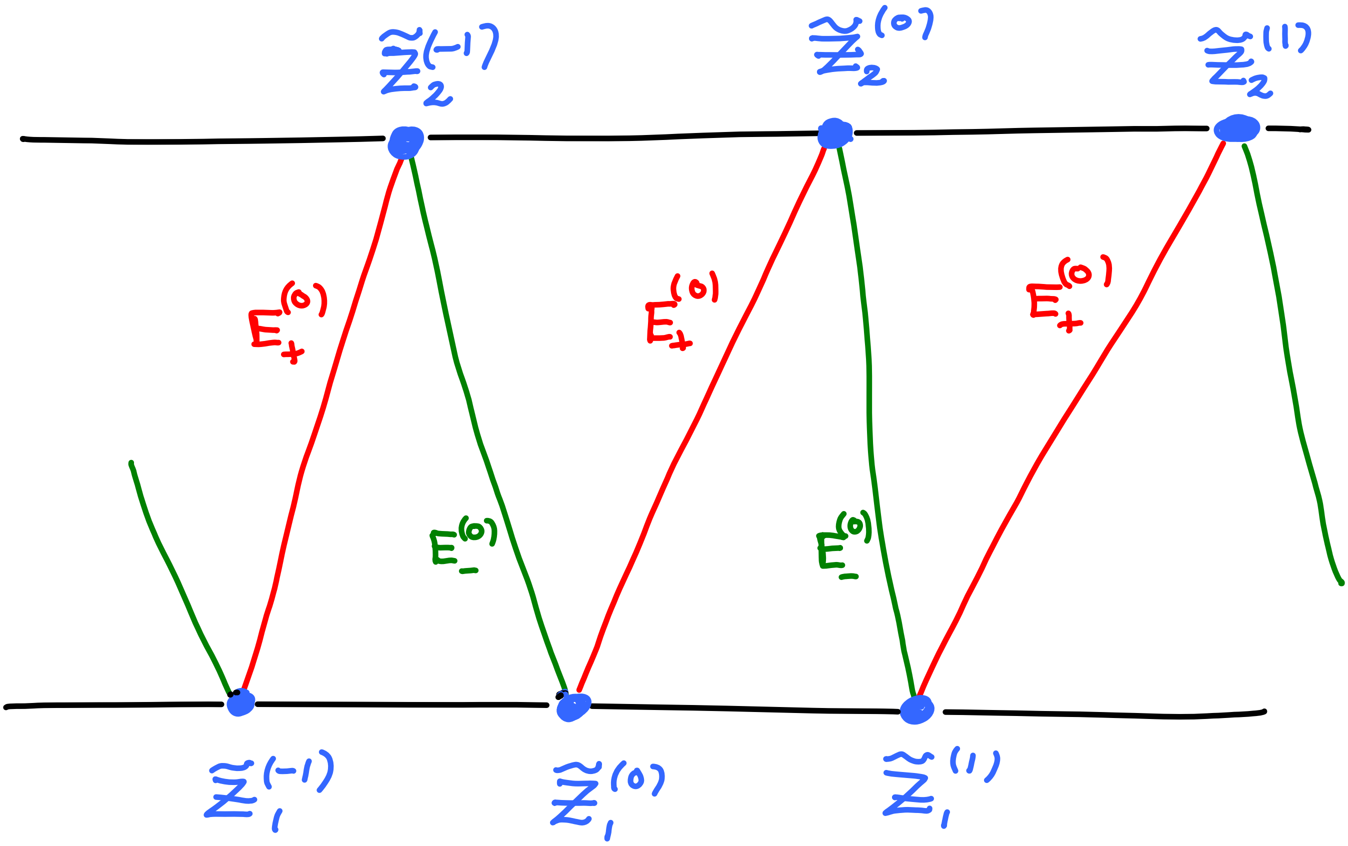
THIS IS JUST A FLIP

VECTORMULTIPLET JUMP: $W_a = W_b$



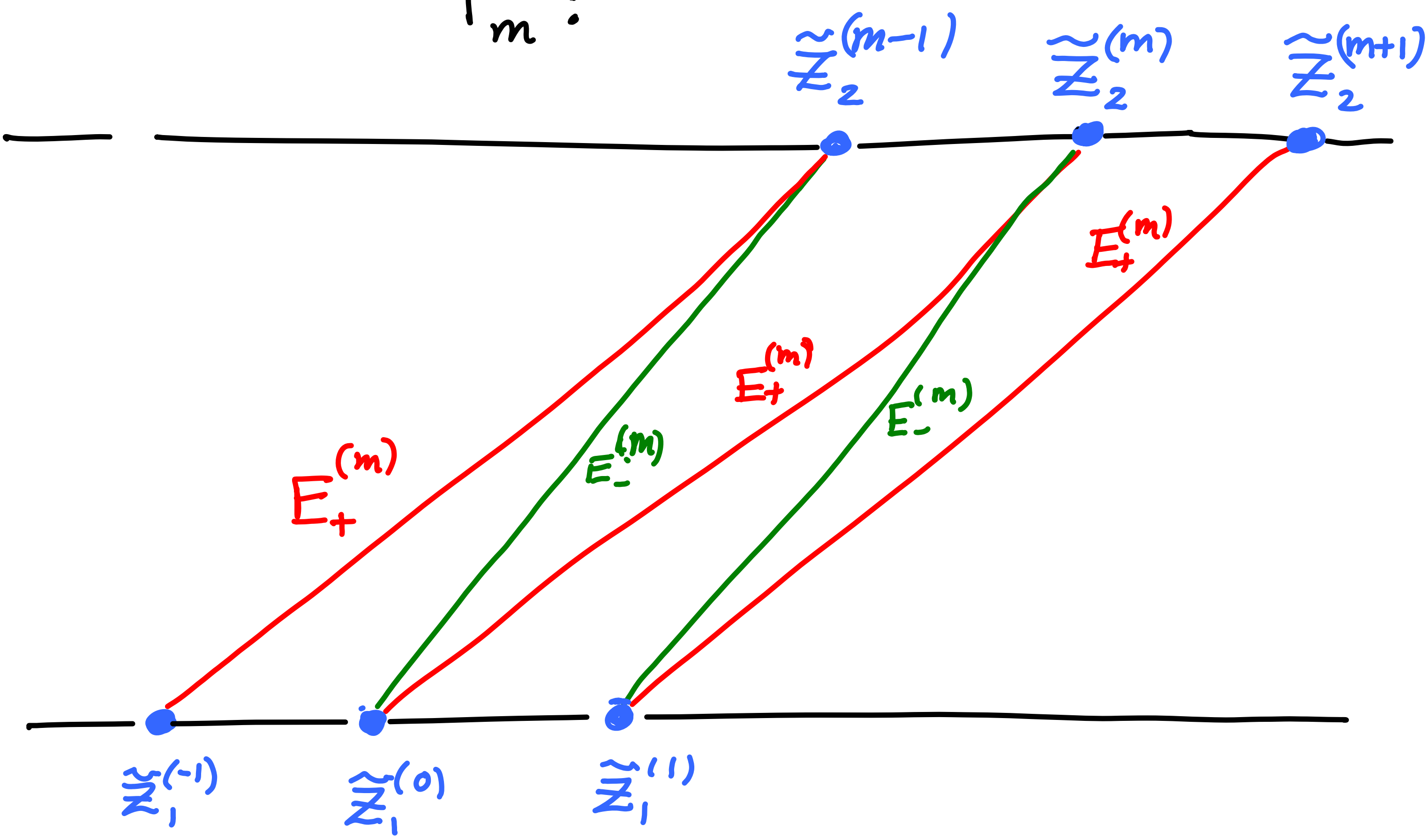
AS $\vartheta \to \vartheta_c^-$ $T(\vartheta, \lambda)$ HAS AN INFINITE SEQUENCE OF FLIPS

$E_+ E_- E_+ E_- \cdots\cdots$



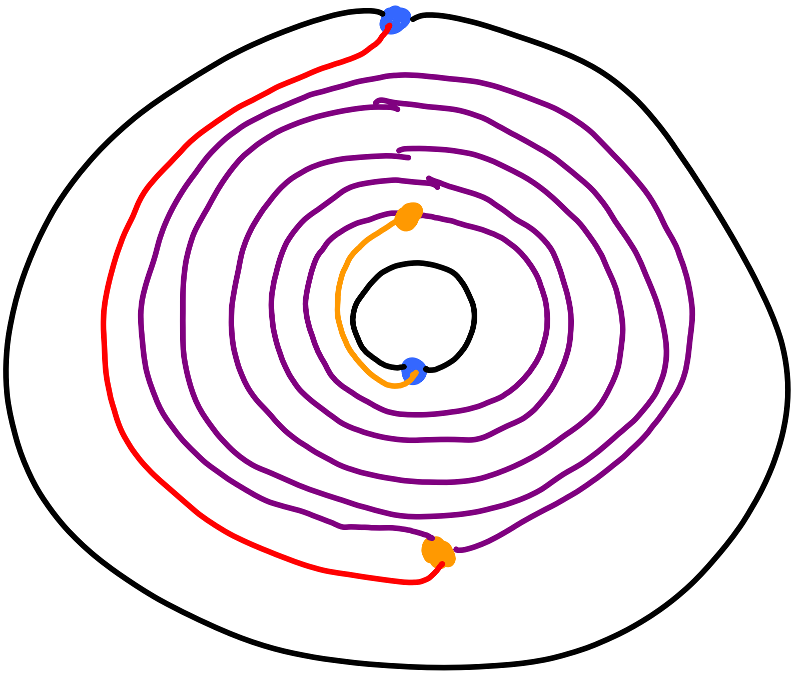
$\tilde{\tilde{Z}}_2^{(-1)}$
$\tilde{\tilde{Z}}_2^{(0)}$
$\tilde{\tilde{Z}}_2^{(1)}$
$E_+^{(0)}$
$E_+^{(0)}$
$E_+^{(0)}$
$E_-^{(0)}$
$E_-^{(0)}$
$\tilde{\tilde{Z}}_1^{(-1)}$
$\tilde{\tilde{Z}}_1^{(0)}$
$\tilde{\tilde{Z}}_1^{(1)}$

T_m :



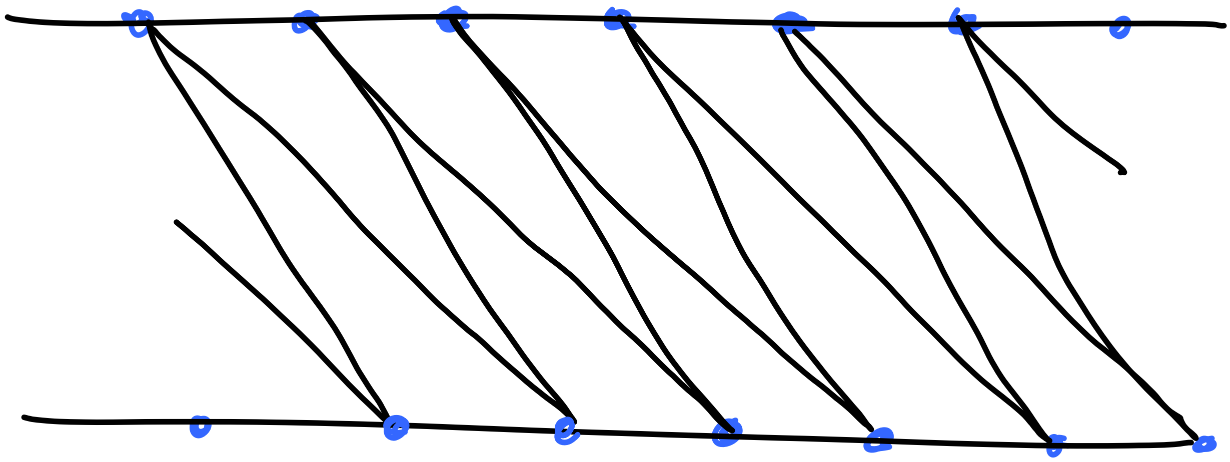
$\tilde{\tilde{Z}}_2^{(m-1)}$
$\tilde{\tilde{Z}}_2^{(m)}$
$\tilde{\tilde{Z}}_2^{(m+1)}$
$E_+^{(m)}$
$E_+^{(m)}$
$E_+^{(m)}$
$E_-^{(m)}$
$E_-^{(m)}$
$\tilde{\tilde{Z}}_1^{(-1)}$
$\tilde{\tilde{Z}}_1^{(0)}$
$\tilde{\tilde{Z}}_1^{(1)}$

AT $\vartheta = \vartheta_c$



$\vartheta = \vartheta_c$

AT $\vartheta > \vartheta_c$



T_{-m}

SUITABLE COMBINATIONS OF

$\chi_{E_+}^{T_m}$, $\chi_{E_-}^{T_m}$ HAVE LIMITS

FOR $m \to \infty$:

$$\chi_A^{T_{+\infty}} = \lim_{m \to \infty} \chi_{E_+}^{T_m} \chi_{E_-}^{T_m}$$

$$\chi_A^{T_{-\infty}} = \lim_{m \to -\infty} \chi_{E_+}^{T_m} \chi_{E_-}^{T_m}$$

$\Rightarrow$ ADD NEW OBJECTS $T_{\pm\infty}$ TO THE GROUPOID, CALL THEM "LIMIT TRIANGULATIONS"

NEW MORPHISMS:

$$T_{+\infty} \longrightarrow T_{-\infty}$$

CALLED "TWISTS"

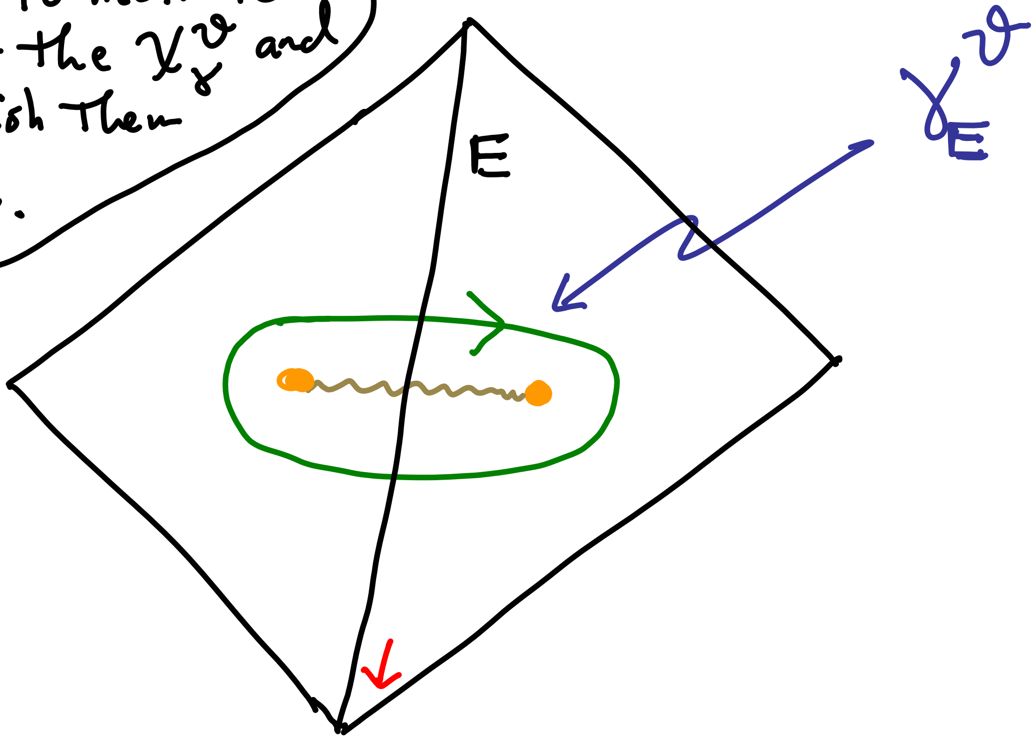
4. DEFINING THE TWISTOR COORD'S

FINALLY, TO DEFINE $\mathcal{X}_\gamma^\vartheta(\cdot,\zeta)$

WE ASSOCIATE TO $E \in \mathcal{E}(T(\vartheta,\lambda))$

CERTAIN CYCLES $\gamma_E^\vartheta \in H_1(\Sigma,\mathbb{Z})$

need to motivate better the $\mathcal{X}_\gamma^\vartheta$ and distinguish them from $\mathcal{X}_\gamma$.



RULE: ORIENT THE LIFTS $\hat{E}$ SO THAT $e^{-i\vartheta}\langle \lambda, \partial_t \rangle > 0$: +VE

DEMAND $\langle \gamma_E^\vartheta, \hat{E} \rangle = +1$

THE $\{\gamma_E^{\vartheta}\}_{E \in \mathcal{E}(T)}$ FORM A (POSITIVE) BASIS FOR Γ.

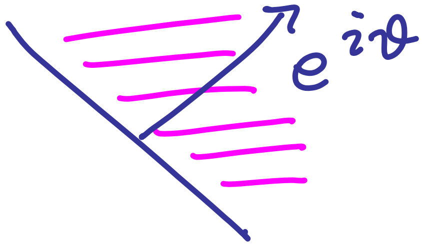
NOW DEFINE:

$$\chi^{\vartheta}_{\gamma_E^{\vartheta}} := \chi_E^{T(\vartheta,\lambda)}$$

$$\chi^{\vartheta}_{\gamma+\gamma'} := \chi^{\vartheta}_{\gamma}\, \chi^{\vartheta}_{\gamma'}$$

__THEOREM 1:__ IF $R \to \infty$ AND

ζ IS IN $\mathbb{H}_\vartheta$:

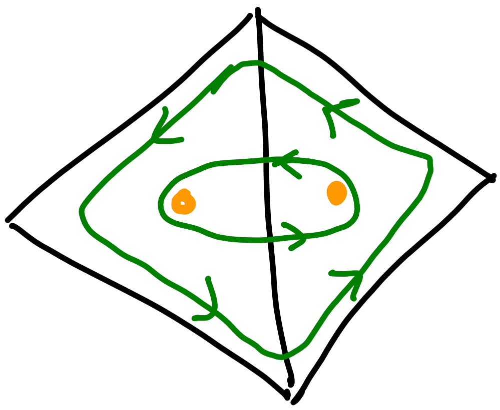


THEN

$$\mathcal{X}_\gamma^\vartheta(\cdot,\zeta) \underset{R\to\infty}{\sim} \exp\left(\frac{\pi R}{\zeta} Z_\gamma + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right)$$

RECOVERS NEITZKE-PIOLINE SEMIFLAT TWISTOR COORDINATES.

__PROOFS:__



$$s_i \sim \exp\left(\pm \frac{R}{\zeta}\int_{z_i}^{z} \lambda\right) \cdot \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

ξ USE RELATION TO 2D Sinh-Gordon ▨

THEOREM 2:

WITH RESPECT TO SYMPLECTIC STRUCTURE:

$$\omega_{\zeta} = \int_C \mathrm{Tr}\, \delta A\, \delta A$$

$$\{ \mathcal{X}^{\vartheta}_{\gamma}, \mathcal{X}^{\vartheta}_{\gamma'} \} = \langle \gamma, \gamma' \rangle \mathcal{X}^{\vartheta}_{\gamma+\gamma'}$$

THEOREM 3:

AT SUFFICIENTLY LARGE R

$$\mathcal{X}_{\gamma}(\cdot\,, \zeta) = \mathcal{X}^{\vartheta = \arg \zeta}_{\gamma}(\cdot\,, \zeta)$$

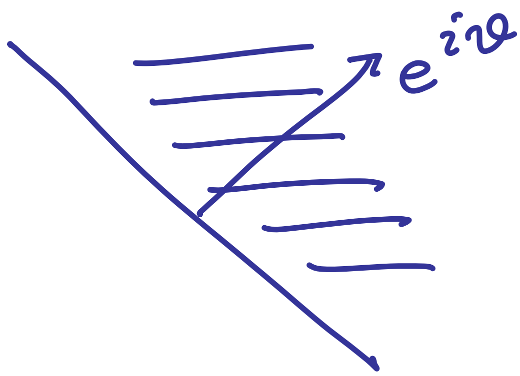
SATISFY THE 5 DEFINING PROPERTIES.

PROOF:

(1) $\mathcal{X}_\gamma(\cdot, \zeta)$ HOLOMORPHIC ON $\mathcal{M}^\zeta$: FOCK & GONCHAROV

(2),(3) FOLLOW EASILY FROM THE DEFINITION

(4B) FOR $\zeta \to 0$ IN THE HALF-PLANE

$H_\vartheta =$ 

$$\lim_{\zeta \to 0} \mathcal{X}_\gamma^\vartheta(\zeta) \exp\left(-\frac{\pi R}{\zeta} Z_\gamma\right) \quad \text{EXISTS}$$

FOLLOWS FROM WKB ASYMPTOTICS AS WITH $R \to \infty$

(5) IF $\vartheta = \vartheta_c$ IS THE PHASE OF A BPS STATE OF CHARGE γ_0 THEN, DEFINING

$$\mathcal{X}^{\pm}_{\gamma} = \lim_{\vartheta \to \vartheta_c^{\pm}} \mathcal{X}^{\vartheta}_{\gamma}$$

$$\boxed{\mathcal{X}^{+}_{\gamma} = \mathcal{X}^{-}_{\gamma}\left(1 - \sigma(\gamma_0)\mathcal{X}^{-}_{\gamma_0}\right)^{\Omega(\gamma_0)\langle\gamma,\gamma_0\rangle}}$$

NOTE: $\sigma(\gamma_0) = +1, \ \Omega(\gamma_0) = -2$ VM

$\sigma(\gamma_0) = -1, \ \Omega(\gamma_0) = +1$ HM

1. FOR HM: CLUSTER TRMN.

2. FOR VM: EXPLICIT COMPUTATION OF TWIST TMN:

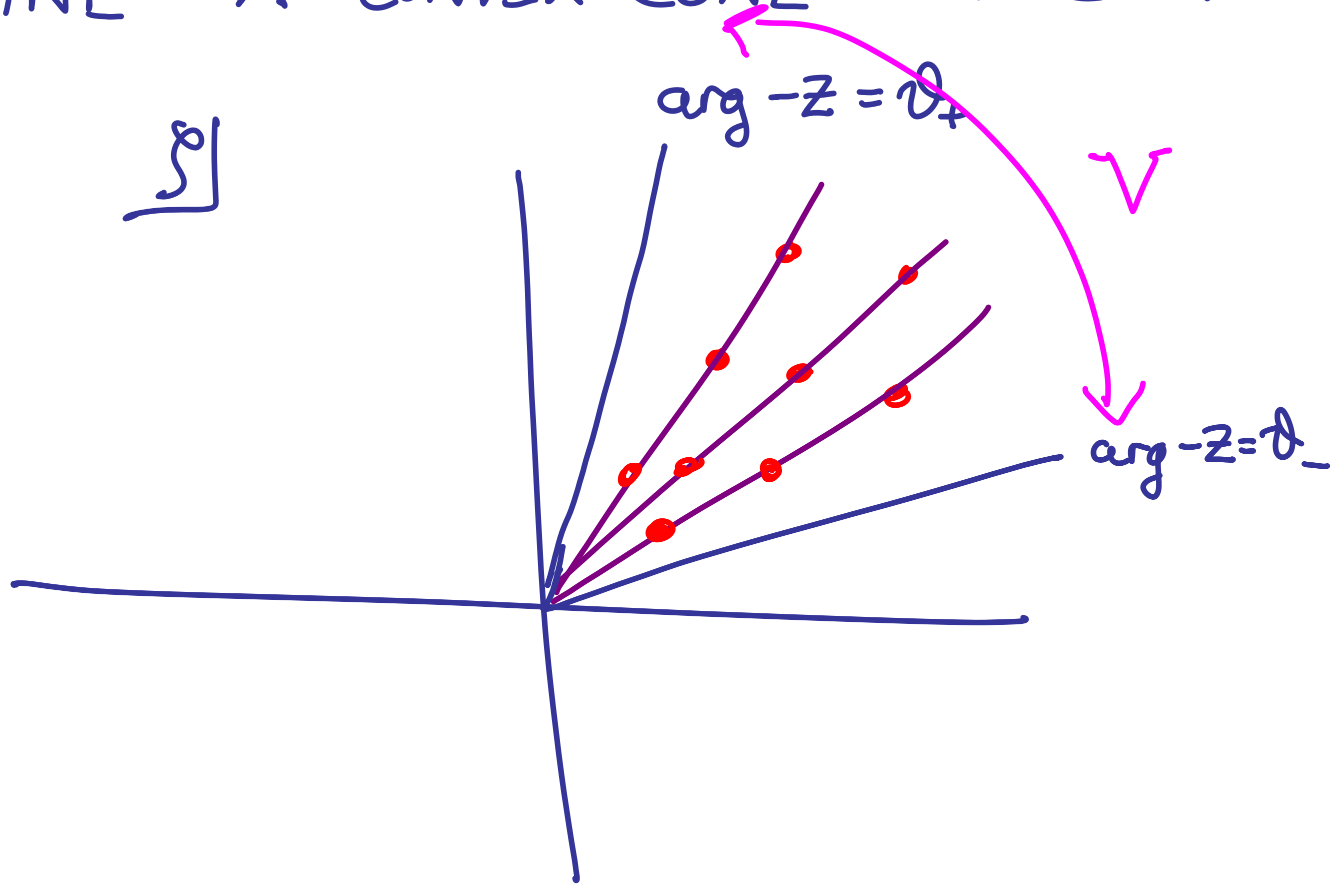
$$\mathcal{X}^{T_{+\infty}} \longrightarrow \mathcal{X}^{T_{-\infty}}$$

■

5. $R \to \infty$ LIMIT & SINH-GORDON

6. WALL CROSSING

CHOOSE $\vartheta_- < \vartheta_+$ TO DEFINE A CONVEX CONE IN COMPLEX



SUPPOSE WE FOLLOW A PATH u_- TO u_+ SO THAT NO BPS RAY CROSSES $\arg(-Z) = \vartheta_\pm$.

THEN $T(\vartheta_\pm, \lambda_-)$ SMOOTHLY EVOLVES TO $T(\vartheta_\pm, \lambda_+)$

ON THE OTHER HAND, EVOLVING ϑ_- TO ϑ_+ AT FIXED λ PRODUCES A SEQUENCE OF FLIPS, TWISTS, AND POPS.

FACT: ALL POPS OCCUR IN DEGENERATE TRIANGLES, AND THE INDUCED TRANSFORMATION IS 1 FOR SUCH POPS.

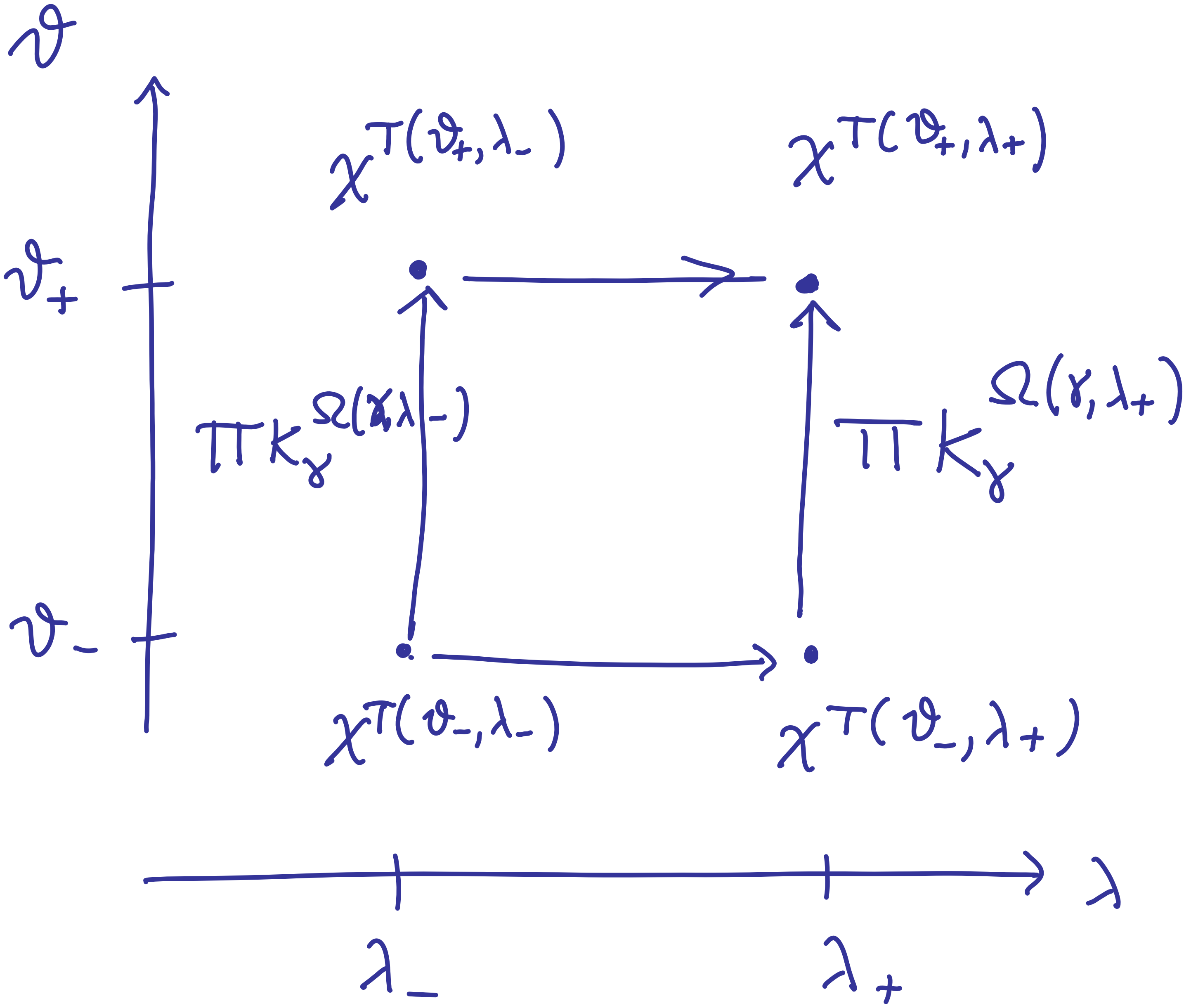
THEREFORE χ^{ϑ_+} IS RELATED TO χ^{ϑ_-} VIA THE IMAGE OF

$$\prod_{\vartheta_- < \vartheta_c < \vartheta_+} \sigma_{\gamma_{E_c}^{\vartheta_c}}$$

i.e. $$\prod_{\vartheta_- < \arg(-Z_\gamma) < \vartheta_+} K_\gamma^{\Omega(\gamma,\lambda)} = A_V$$

BUT THERE IS NO DISCONTINUITY

IN $\chi^T(\vartheta, \lambda_-) \longrightarrow \chi^T(\vartheta, \lambda_+)$



$$\boxed{\prod K_\gamma^{\Omega(\gamma, \lambda_-)} = \prod K_\gamma^{\Omega(\gamma, \lambda_+)}}$$

7. MOVIES & EXAMPLES

8. DETERMINING THE BPS SPECTRUM

NOW LET US VARY ϑ TO $\vartheta+\pi$.

WE CAPTURE ALL THE BPS STATES

$$\prod_{\vartheta_c} \sigma_{\gamma^{\vartheta_c}_{E_c}} \longrightarrow \prod_{\vartheta < -\arg Z < \vartheta+\pi} K_\gamma^{\Omega(\gamma,\lambda)}$$

ON THE OTHER HAND,

$T(\vartheta,\lambda)$ AND $T(\vartheta+\pi,\lambda)$

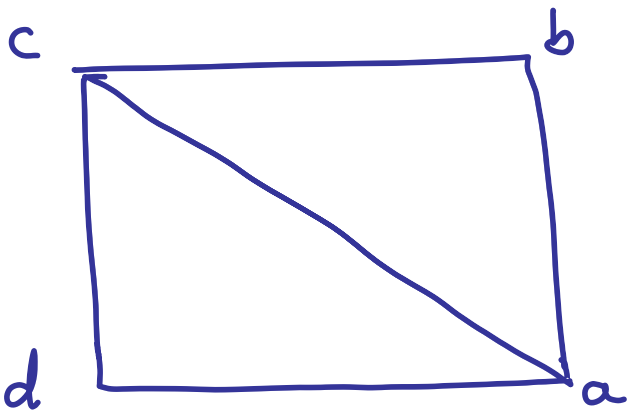
<u>ONLY</u> DIFFER BY

SIMULTANEOUSLY POPPING

ALL THE VERTICES!!

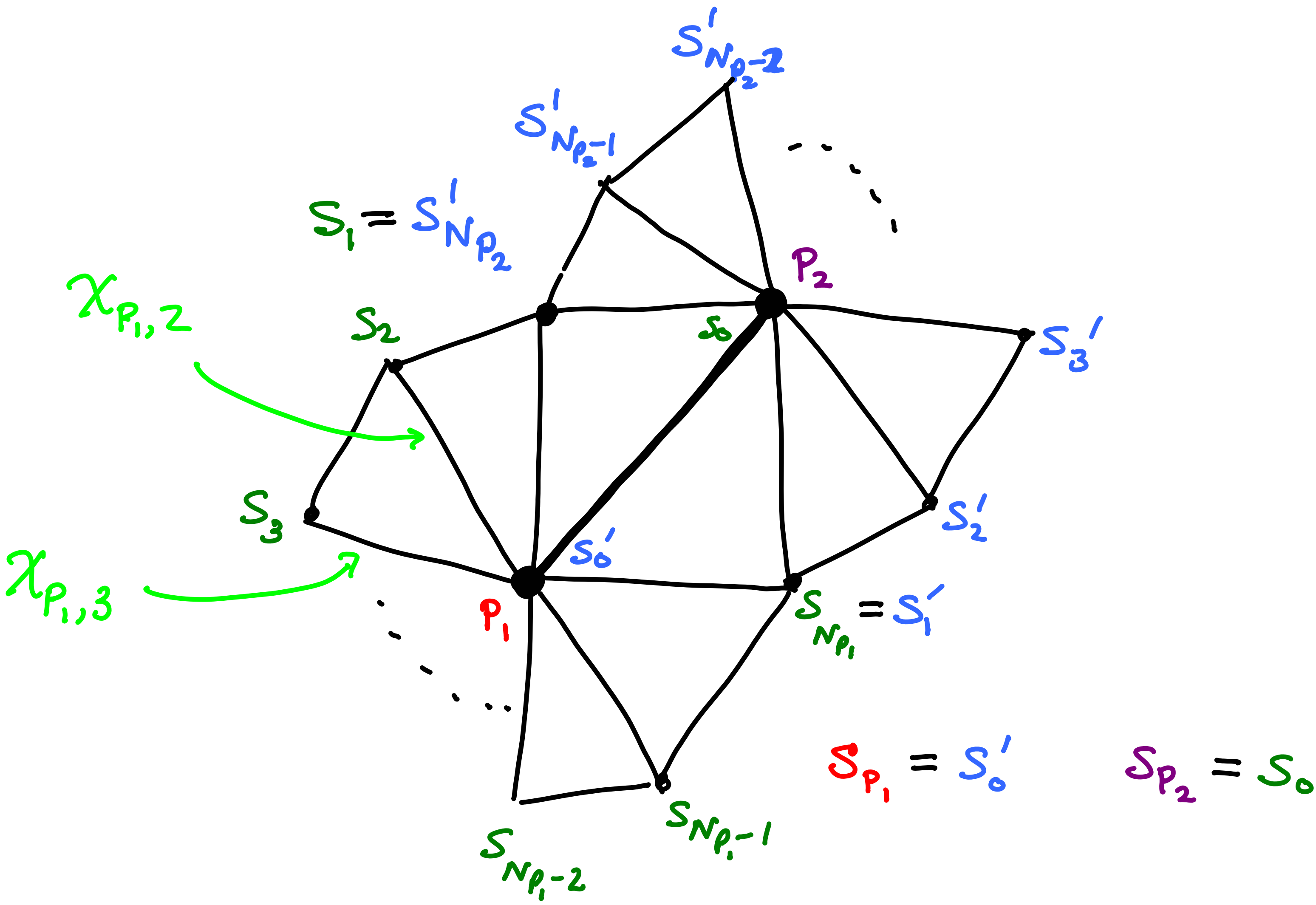
SPECTRUM-GENERATING STOKES MATRIX

WHILE THE CHANGE χ^T_E FOR POPPING ONE VERTEX IS COMPLICATED, IT TURNS OUT THAT POPPING ALL VERTICES LEADS TO A RATHER SIMPLE FORMULA!



$$\tilde{\chi}^{\tilde{T}}_{ac}\,\chi^{T}_{ac} = \frac{(1+A_{ab})(1+A_{cd})}{(1+A_{bc})(1+A_{da})}$$

TO GIVE A FORMULA FOR $A_{p_1 p_2}$:



$$\boxed{\begin{aligned} A_{p_1 p_2} &= \frac{1}{1-\mu_{p_1}^2}\,\frac{1}{1-\mu_{p_2}^2}\cdot\chi_{p_1 p_2}\,\cdot \\ &\quad \left(1+\sum_{k=1}^{N_{p_1}-1}\prod_{j=1}^{k}\chi_{p_1,j}\right)\cdot\left(1+\sum_{k=1}^{N_{p_2}-1}\prod_{j=1}^{k}\chi_{p_2,j}\right)\end{aligned}}$$

TO FIND THE BPS SPECTRUM

THE TRANSFORMATION

$$S : \mathcal{X}_i \longrightarrow \tilde{\mathcal{X}}_i$$

$$\tilde{\mathcal{X}}_i = \mathcal{X}_i \frac{(1+A_{ab}(i))(1+A_{cd}(i))}{(1+A_{bc}(i))(1+A_{da}(i))}$$

HAS A UNIQUE DECOMPOSITION

OF THE FORM:

$$S = \prod_{\vartheta < -\arg Z < \vartheta + \pi} K_\gamma^{\Omega(\gamma,\lambda)}$$

$$\boxed{\text{THIS DETERMINES THE } \Omega(\gamma, u)}$$

CONCLUSION: FUTURE DIRECTIONS

1. WE HAVE SOME IDEAS ABOUT HOW TO GO TO RANK $k > 2$.

2. RELATION TO INTEGRABLE SYSTEMS (e.g. THE INTEGRAL EQUATION FOR $\mathcal{X}_\gamma$ IS A VERSION OF THE TBA.)

3. SUPERGRAVITY

4. NEW MODULAR FUNCTORS