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D-branes and K-theory in 2D topological field theory

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This expository paper describes sewing conditions in two-dimensional open/closed topological field theory. We include a description of the G-equivariant case, where G is a finite group. We determine the category of boundary conditions in the case that the closed string algebra is semisimple. In this case we find that sewing constraints – the most primitive form of worldsheet locality – already imply that D-branes are (G-twisted) vector bundles on spacetime. We comment on extensions to cochain-valued theories and various applications. Finally, we give uniform proofs of all relevant sewing theorems using Morse theory.

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1. Introduction and Summary

The theory of D-branes has proven to be of great importance in the development of string theory. In this paper we will focus on certain mathematical structures central to the idea of D-branes. One of the questions which motivated our work was: "Given a closed string background, what is the set of possible D-branes?" This is a rather complicated question. One might at first be tempted to declare that D-branes simply correspond to conformally invariant boundary conditions for the open string. This viewpoint is not very useful because there are too many such boundary conditions, and in general they have no geometrical description. It also neglects important restrictions imposed by sewing consistency conditions.

In this paper we address the above problem in the drastically simpler case of two-dimensional *topological* field theory (TFT), where the whole content of the theory is encoded in a finite-dimensional commutative Frobenius algebra. We shall find that describing the sewing conditions, and their solutions for 2d topological open and closed TFT, is a tractable but not entirely trivial problem. We also extend our results to the equivariant case, where we are given a finite group G, and the worldsheets are surfaces equipped with G-bundles. This is relevant to the classification of D-branes in orbifolds.

Another one of our primary motivations has been the desire to understand the relation between D-branes and K-theory in the simplest possible terms. This relation is often justified by considerations of anomaly cancellation or of brane-antibrane annihilation. Our analysis shows that the relation is, in some sense, more primitive, and follows simply from sewing constraints.

We hope the present work will be of some pedagogical interest in explaining the structure of boundary conformal field theory and its connections to K-theory in the simplest context. There are also, however, some potential applications of our results. One ambitious goal is to classify the boundary conditions in topologically twisted nonlinear sigma models and their allied topological string theories. Here we have some suggestive results, but they are far from a complete theory.

Our main concrete results are the following two theorems. To state the first we must point out that a semisimple Frobenius algebra[1] $\mathcal{C}$ is automatically the algebra of complex-valued functions on a finite set $X = \mathrm{Spec}(\mathcal{C})$ — the "space-time" — which is equipped with a "volume-form" or "dilaton field" θ which assigns a measure θ_x to each point $x \in X$.

[1] For basic material on Frobenius algebras see, for example, [1], ch. 9 or [2].

Theorem A For a semisimple 2-dimensional TFT corresponding to a finite space-time (X, θ) the choice of a maximal category of D-branes fixes a choice of a square-root of θ_x for each point x of X. The category of boundary conditions is equivalent to the category $\mathrm{Vect}(X)$ of finite-dimensional complex vector bundles on X. The correspondence is, however, not canonical, but is arbitrary up to composition with an equivalence $\mathrm{Vect}(X) \to \mathrm{Vect}(X)$ given by tensoring each vector bundle with a fixed line bundle (i.e. one which does not depend on the particular D-brane).

Conversely, if we are given a semisimple Frobenius category $\mathcal{B}$, then it is the category of boundary conditions for a canonical 2-dimensional TFT corresponding to the commutative Frobenius algebra which is the ring of endomorphisms of the identity functor of $\mathcal{B}$.

We shall explain in the next section the sense in which the boundary conditions form a category. The theorem will be proved in section 3. In §3.4 we shall describe an analogue of the theorem for spin theories.

The second theorem relates to "G-equivariant" or "G-gauged" TFTs, where G is a finite group. Turaev has shown that in dimension 2 a semisimple G-equivariant TFT corresponds to a finite space-time X on which the group G acts in a given way, and which is equipped with a G-invariant dilaton field θ and as well as a "B-field" B representing an element of the equivariant cohomology group $H^3_G(X; \mathbb{Z})$.

Theorem B For a semisimple G-equivariant TFT corresponding to a finite space time (X, θ, B) the choice of a maximal category of D-branes fixes a G-invariant choice of square roots $\sqrt{\theta}_x$ as before, and then the category is equivalent to the category of finite-dimensional B-twisted G-vector-bundles on X, up to an overall tensoring with a G-line-bundle.

In this case the category of D-branes is equivalent to that of the "orbifold" theory obtained from the gauged theory by integrating over the gauge fields, and it does not remember the equivariant theory from which the orbifold theory arose. There is, however, a natural enrichment which does remember the equivariant theory.

This will be explained and proved in section 7.

The restriction to the semisimple case in our results seems at first a damaging weakness. We believe, however, that it is this case that reveals the essential structure of the theory. To go beyond it, the appropriate objects of study, in our view, are cochain-complex-valued TFTs rather than non-semisimple TFTs in the usual sense. (This is analogous to the fact that in ordinary algebra the duality theory of non-projective modules is best studied in the derived category.) We have said something about this line of development in sections 2 and 6, explaining how the category of boundary conditions is naturally an A_∞ category in the sense of Fukaya, Kontsevich, and others.

Let us comment on one important conceptual aspect of the results of this paper. In the Matrix theory approach to nonperturbative string theory [3][4] *open* string field theory, or rather its low-energy Yang-Mills theory, is taken to be the fundamental starting point for the formulation of the entire string theory. In particular, spacetime, and the closed strings, are regarded as derived concepts. A similar philosophy lies at the root of the AdS/CFT correspondence. In this paper we begin our discussion with the viewpoint that the spacetime and closed strings are fundamental and then ask what category of boundary conditions is compatible with that background. However, in the semisimple case, we find that one could equally well start with the Frobenius category of boundary conditions and derive the closed strings and the spacetime. Thus, our treatment is in harmony with the philosophy of Matrix theory. Indeed, it is possible to obtain the closed string algebra from the open string algebra by taking the *center* of the open string algebra $Z(\mathcal{O}) \cong \mathcal{C}$. (In general the Cardy condition only shows that $\iota_*(\mathcal{C})$ maps into the center.) A more sophisticated version of this idea is that the closed string algebra is obtained from the category of boundary conditions by considering the endomorphisms of the identity functor. All this is discussed in §3.3, and justifies the important point that there is a converse statement to Theorem A.

A closely related point is that in open string field theory there are different open string algebras $\mathcal{O}_{aa}$ for the different boundary conditions a. For boundary conditions with maximal support, however, they are Morita equivalent via the bimodules $\mathcal{O}_{ab}$. For some purposes it might seem more elegant to start with a single algebra. (Indeed, Witten has suggested in [5] that one should use something analogous to stabilization of C^* algebras, namely one should replace the string field algebra by $\mathcal{O}_{aa} \otimes \mathcal{K}$ where $\mathcal{K}$ is the algebra of compact operators.) In our framework, the single algebra is replaced by the category of boundary conditions. If one believes that a stringy spacetime is a non-commutative space, our framework is in good agreement with Kontsevich's approach to non-commutative geometry, according to which a non-commutative space *is* a linear category — essentially the category of modules for the ring, if the space is defined by a ring. For commutative rings the category of modules determines the ring, but in the non-commutative case the ring is determined only up to Morita equivalence. We discuss this further in section §3.

Finally, we comment briefly on some related literature. There is a rather large literature on 2d TFT and it is impossible to give comprehensive references. Here we just indicate some closely related works. The 2d closed sewing theorem is a very old result implicit in the earliest papers in string theory. The algebraic formulation was perhaps first formulated by Friedan. Accounts have been given in [6][7][8] and in the Stanford lectures by Segal [9]. Sewing constraints in 2D open and closed string theory were first investigated in [10]. Extensions to unorientable worldsheets were described in [11][12][13][14]. Our work– which is primarily intended as a pedagogical exposition – was first described at Strings 2000 [15] and summarized briefly in [16]. It was described more completely in lectures at the KITP in 2001 and at the 2002 Clay School [17]. In [18] one can find alternative (more computational) proofs and examples to those we give below, together with better quality pictures. [2] Some of our work was independently obtained in the papers of C. Lazaroiu [19] although the emphasis in these papers is on applications to disk instanton corrections in low energy supergravity. Regarding G-equivariant theories, there is a very large literature on D-branes and orbifolds not reflected in the above references. In the context of 2D TFT two relevant references are [20][21]. Alternative discussions on the meaning of B-fields in orbifolds (in TFT) can be found in [22][23][24]. Our treatment of cochain-level theories and A_∞ algebras has been developed considerably further by Costello [25].

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[2] Please note that the arrows on some morphisms in figures 4, 35-38, 47-48 have the wrong orientation.

2. The sewing theorem

2.1. Definition of open and closed 2D TFT

Roughly speaking, a quantum field theory is a functor from a geometric category to a linear category. The simplest example is a *topological field theory*, where we choose the geometric category to be the category whose objects are closed, oriented $(d-1)$-manifolds, and whose morphisms are oriented cobordisms (two such cobordisms being identified if they are diffeomorphic by a diffeomorphism which is the identity on the incoming and outgoing boundaries). The linear category in this case is simply the category of complex vector spaces and linear maps, and the only property we require of the functor is that (on objects and morphisms) it takes disjoint unions to tensor products. The case $d = 2$ is of course especially well known and understood.

There are several natural ways to generalize the geometric category. One may, for example, consider manifolds equipped with some structure such as a Riemannian metric. (We shall discuss some examples in the following.) The focus of this paper is on a different kind of generalization where the objects of the geometric category are oriented $(d-1)$-manifolds with boundary, and each boundary component is labelled with an element of a fixed set $\mathcal{B}_0$ called the set of *boundary conditions*. In this case a cobordism from Y_0 to Y_1 means a d-manifold X whose boundary consists of three parts $\partial X = Y_0 \cup Y_1 \cup \partial_{\mathrm{cstr}} X$, where the "constrained boundary" $\partial_{\mathrm{cstr}} X$ is a cobordism from ∂Y_0 to ∂Y_1. Furthermore, we require the connected components of $\partial_{\mathrm{cstr}} X$ to be labelled with elements of $\mathcal{B}_0$ in agreement with the labelling of ∂Y_0 and ∂Y_1.

Thus when $d = 2$ the objects of the geometric category are disjoint unions of circles and oriented intervals with labelled ends. A functor from this category to complex vector spaces which takes disjoint unions to tensor products will be called an *open and closed topological field theory*: such theories will give us a "baby" model of the theory of D-branes. We shall always write $\mathcal{C}$ for the vector space associated to the standard circle S^1, and $\mathcal{O}_{ab}$ for the vector space associated to the interval $[0,1]$ with ends labelled by $a, b \in \mathcal{B}_0$.

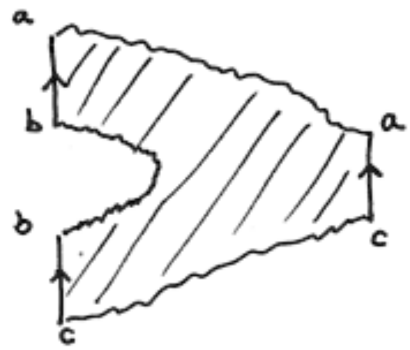

Fig. 1: Basic cobordism on open strings.

The cobordism fig. 1 gives us a linear map $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$, or equivalently a bilinear map

$$\mathcal{O}_{ab} \times \mathcal{O}_{bc} \to \mathcal{O}_{ac}, \tag{2.1}$$

which we think of as a composition law. In fact we have a $\mathbb{C}$-linear *category* $\mathcal{B}$ whose objects are the elements of $\mathcal{B}_0$, and whose set of morphisms from b to a is the vector space $\mathcal{O}_{ab}$, with composition of morphisms given by (2.1) . (To say that $\mathcal{B}$ is a category means no more than that the composition (2.1) is associative in the obvious sense, and that there is an identity element $1_a \in \mathcal{O}_{aa}$ for each $a \in \mathcal{B}_0$; we shall explain presently why these properties hold.)

For any open and closed TFT we have a map $e : \mathcal{C} \to \mathcal{C}$ defined by the cylindrical cobordism $S^1 \times [0,1]$, and a map $e_{ab} : \mathcal{O}_{ab} \to \mathcal{O}_{ab}$ defined by the square $[0,1] \times [0,1]$. Clearly $e^2 = e$ and $e_{ab}^2 = e_{ab}$. If all these maps are identity maps we say the theory is *reduced*. There is no loss in restricting ourselves to reduced theories, and we shall do so from now on.

2.2. Algebraic characterization

The most general 2D open and closed TFT, formulated as in the previous section, is given by the following algebraic data:

1. $(\mathcal{C}, \theta_{\mathcal{C}}, 1_{\mathcal{C}})$ is a commutative Frobenius algebra.

2a. $\mathcal{O}_{ab}$ is a collection of vector spaces for $a, b \in \mathcal{B}_0$ with an associative bilinear product

$$\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac} \tag{2.2}$$

2b. The $\mathcal{O}_{aa}$ have nondegenerate traces

$$\theta_a : \mathcal{O}_{aa} \to \mathbb{C} \tag{2.3}$$

In particular, each $\mathcal{O}_{aa}$ is a not-necessarily commutative Frobenius algebra.

2c. Moreover,

$$\begin{aligned} \mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathcal{O}_{aa} \stackrel{\theta_a}{\to} \mathbb{C} \\ \mathcal{O}_{ba} \otimes \mathcal{O}_{ab} \to \mathcal{O}_{bb} \stackrel{\theta_b}{\to} \mathbb{C} \end{aligned} \tag{2.4}$$

are perfect pairings with

$$\theta_a(\psi_1\psi_2) = \theta_b(\psi_2\psi_1) \tag{2.5}$$

for $\psi_1 \in \mathcal{O}_{ab}, \psi_2 \in \mathcal{O}_{ba}$.

3. There are linear maps:

$$\begin{aligned} \iota_a : \mathcal{C} \to \mathcal{O}_{aa} \\ \iota^a : \mathcal{O}_{aa} \to \mathcal{C} \end{aligned} \tag{2.6}$$

such that

3a. ι_a is an algebra homomorphism

$$\iota_a(\phi_1\phi_2) = \iota_a(\phi_1)\iota_a(\phi_2) \tag{2.7}$$

3b. The identity is preserved

$$\iota_a(1_{\mathcal{C}}) = 1_a \tag{2.8}$$

3c. Moreover, ι_a is central in the sense that

$$\iota_a(\phi)\psi = \psi\iota_b(\phi) \tag{2.9}$$

for all $\phi \in \mathcal{C}$ and $\psi \in \mathcal{O}_{ab}$

3d. ι_a and ι^a are adjoints:

$$\theta_{\mathcal{C}}(\iota^a(\psi)\phi) = \theta_a(\psi\iota_a(\phi)) \tag{2.10}$$

for all $\psi \in \mathcal{O}_{aa}$.

3e. The "Cardy conditions."[3] Define $\pi_b{}^a : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$ as follows. Since $\mathcal{O}_{ab}$ and $\mathcal{O}_{ba}$ are in duality (using θ_a or θ_b), if we let ψ_μ be a basis for $\mathcal{O}_{ba}$ then there is a dual basis ψ^μ for $\mathcal{O}_{ab}$. Then we define

$$\pi_b{}^a(\psi) = \sum_\mu \psi_\mu \psi \psi^\mu, \tag{2.11}$$

and we have the "Cardy condition":

$$\pi_b{}^a = \iota_b \circ \iota^a. \tag{2.12}$$

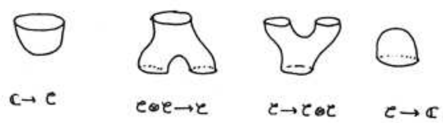

Fig. 2: Four diagrams defining the Frobenius structure in a closed 2d TFT. It is often more convenient to represent the morphisms by the planar diagrams. In this case our convention is that a circle oriented so that the right hand points into the surface is an ingoing circle.

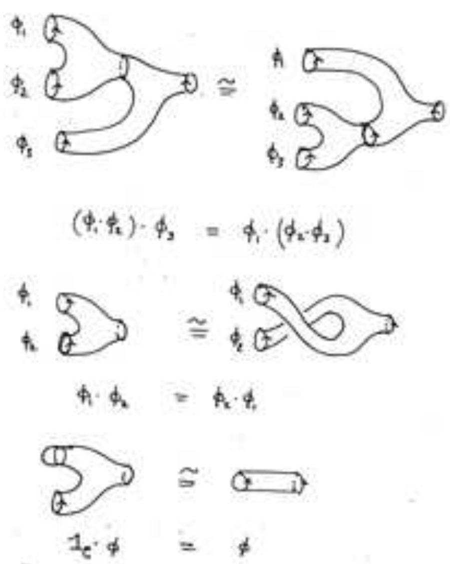

Fig. 3: Associativity, commutativity, and unit constraints in the closed case. The unit constraint requires the natural assumption that the cylinder correspond to the identity map $\mathcal{C} \to \mathcal{C}$.

[3] These are actually generalization of the conditions stated by Cardy. One recovers his conditions by taking the trace. Of course, the factorization of the double twist diagram in the closed string channel is an observation going back to the earliest days of string theory.

2.3. Pictorial representation

Let us explain the pictorial basis for these algebraic conditions. The case of a closed 2d TFT is very well-known. The data of the Frobenius structure is provided by the diagrams in fig. 2. The consistency conditions follow from fig. 3.

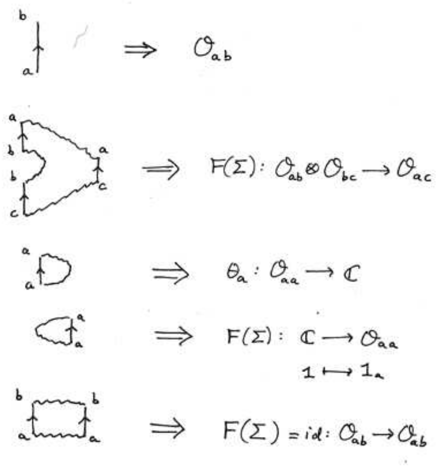

Fig. 4: Basic data for the open theory. Constrained boundaries are denoted with wiggly lines, and carry a boundary condition $a, b, c, \ldots \in \mathcal{B}_0$..

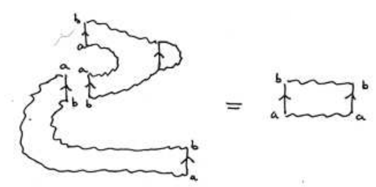

Fig. 5: Assuming that the strip corresponds to the identity morphism we must have perfect pairings in (2.4).

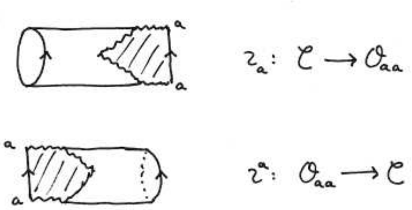

Fig. 6: Two ways of representing open to closed and closed to open transitions.

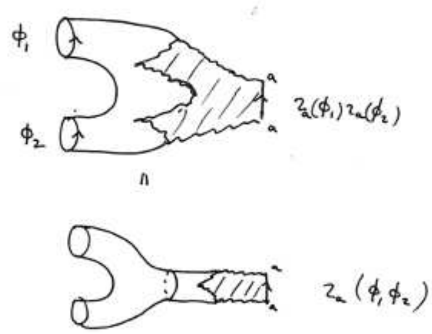

Fig. 7: ι_a is a homomorphism.

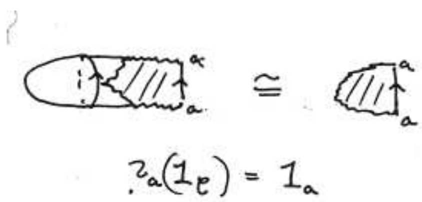

Fig. 8: ι_a preserves the identity.

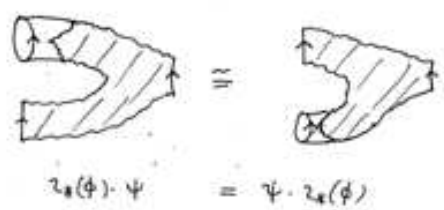

Fig. 9: ι_a maps into the center of $\mathcal{O}_{aa}$.

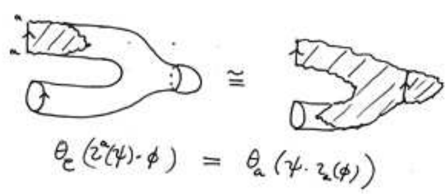

Fig. 10: ι^a is the adjoint of ι_a.

In the open case, entirely analogous considerations lead to the construction of a non-necessarily commutative Frobenius algebra in the open sector. The basic data are summarized in fig. 4. The fact that (2.4) are dual pairings follows from fig. 5. The essential new

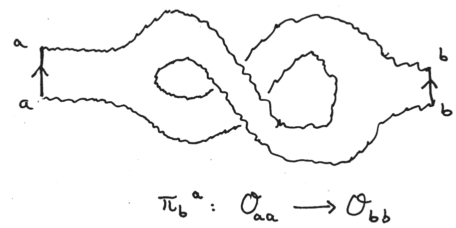

Fig. 11: The double-twist diagram defines the map ${\pi_b}^a : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$.

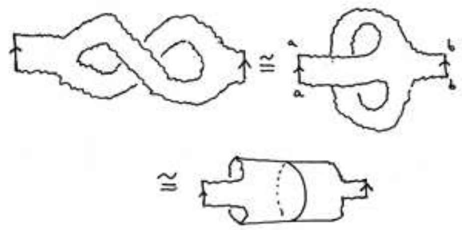

Fig. 12: The (generalized) Cardy-condition expressing factorization of the double-twist diagram in the closed string channel.

ingredient in the open/closed theory are the open to closed and closed to open transitions. In 2d TFT these are the maps ι_a, ι^a. They are represented by fig. 6. There are five new consistency conditions associated with the open/closed transitions. These are illustrated in fig. 7 to fig. 12.

2.4. Sewing theorem

Geometrically, any oriented surface can be decomposed into a composition of morphisms corresponding to the basic data defining the Frobenius structure. However, a given surface can be decomposed in many different ways. The above sewing axioms follow from consistency of these decompositions. The sewing theorem guarantees that there are no further relations on the algebraic data imposed by consistency of sewing.

Theorem 1 Conditions 1,2,3 above are the only conditions on the algebraic data coming from cutting the morphisms in all possible ways.

The proof is in appendix A.

2.5. The category of boundary conditions

The category $\mathcal{B}$ of boundary conditions of an open and closed TFT is an additive category. We can always adjoin new objects to it in various ways. For example, we may as well assume that it possesses direct sums, as we can define for any two objects a and b a new object $a \oplus b$ by

$$\mathcal{O}_{a\oplus b,c} := \mathcal{O}_{ac} \oplus \mathcal{O}_{bc} \tag{2.13}$$

$$\mathcal{O}_{c,a\oplus b} := \mathcal{O}_{ca} \oplus \mathcal{O}_{cb}, \tag{2.14}$$

and hence

$$\mathcal{O}_{a\oplus b,a\oplus b} := \begin{pmatrix} \mathcal{O}_{aa} & \mathcal{O}_{ab} \\ \mathcal{O}_{ba} & \mathcal{O}_{bb} \end{pmatrix}, \tag{2.15}$$

with the obvious composition laws, and

$$\theta_{a\oplus b} : \mathcal{O}_{a\oplus b,a\oplus b} \to \mathbb{C} \tag{2.16}$$

given by

$$\theta_{a\oplus b} \begin{pmatrix} \psi_{aa} & \psi_{ab} \\ \psi_{ba} & \psi_{bb} \end{pmatrix} = \theta_a(\psi_{aa}) + \theta_b(\psi_{bb}). \tag{2.17}$$

The new object is the direct sum of a and b in the enlarged category of boundary conditions. If there was already a direct sum of a and b in the category $\mathcal{B}$ then the new object will be canonically isomorphic to it. In the opposite direction, if we have a boundary condition a and a projection $p \in \mathcal{O}_{aa}$ (i.e. an element such that $p^2 = p$) then we may as well assume there is a boundary condition $b = \mathrm{image}(p)$ such that for any c we have $\mathcal{O}_{cb} = \{f \in \mathcal{O}_{ab} : pf = f\}$ and $\mathcal{O}_{bc} = \{f \in \mathcal{O}_{ba} : fp = f\}$. Then we shall have $a \cong \mathrm{image}(p) \oplus \mathrm{image}(1-p)$.[4]

One very special property that the category $\mathcal{B}$ possesses is that for any two objects a and b the space $\mathcal{O}_{ab}$ of morphisms is canonically dual to $\mathcal{O}_{ba}$, by a pairing which factorizes through the composition in either order. It is natural to call a category with this property a *Frobenius* category, or perhaps a *Calabi-Yau* category.[5] It is a strong restriction on the

[4] A linear category in which idempotents split in this way is often called *Karoubian.*

[5] The latter terminology comes from the case of coherent sheaves on a compact Kähler manifold, where for two sheaves E and F the dual of the morphism space $\mathrm{Ext}(E;F)$ is in general $\mathrm{Ext}(F;E\otimes\omega)$, where ω is the canonical bundle. This coincides with $\mathrm{Ext}(F;E)$ only when ω is trivial, i.e. in the Calabi-Yau case. We shall discuss this example further in §6.

category: for example the category of finitely generated modules over a finite dimensional algebra does not have the property unless the algebra is semisimple.

Example Probably the simplest example of an open and closed theory of the type we are studying is one associated to a finite group G. The category $\mathcal{B}$ is the category of finite dimensional complex representations M of G, and the trace $\theta_M : \mathcal{O}_{MM} = \mathrm{End}(M) \to \mathbb{C}$ takes $\psi : M \to M$ to $1/|G|$ trace(ψ). The closed algebra $\mathcal{C}$ is the center of the group-algebra $\mathbb{C}[G]$, which maps to each $\mathrm{End}(M)$ in the obvious way. The trace $\theta_{\mathcal{C}} : \mathcal{C} \to \mathbb{C}$ takes a central element $\sum \lambda_g g$ of the group-algebra to $\lambda_1/|G|$.

In this example the partition function of the theory on a surface Σ with constrained boundary circles $C_1, \ldots, C_k$ labelled $M_1, \ldots, M_k$ is the weighted sum over the isomorphism classes of principal G-bundles P on Σ of

$$\chi_{M_1}(h_P(C_1)) \ldots \chi_{M_k}(h_P(C_k)),$$

where $\chi_M : G \to \mathbb{C}$ is the character of a representation M, and $h_P(C)$ denotes the holonomy of P around a boundary circle C. Each bundle P is weighted by the reciprocal of the order of its group of automorphisms.

Returning to the general theory, we can now ask three basic questions.

(i) If we are given a "closed" TFT, can we enlarge it to an open and closed theory, and, if so, is the enlargement unique?

(ii) If we are given the category $\mathcal{B}$ of boundary conditions of an open and closed theory, together with the linear maps $\theta_a : \mathcal{O}_{aa} \to \mathbb{C}$ which define the Frobenius structure, can we reconstruct the whole theory, i.e. can we find the closed Frobenius algebra $\mathcal{C}$?

(iii) Is an arbitrary Frobenius category the category of boundary conditions for some closed theory?

For the first question to be well-posed, we should assume that the category of boundary conditions is *maximal*, in the sense that if $\mathcal{B}'$ is an enlargement of it then any object of $\mathcal{B}'$ is isomorphic to an object of $\mathcal{B}$. Even so, we shall see that there are subtleties which prevent any of these question from having a simple affirmative answer.

We can obtain many interesting generalizations of the above structure by modifying either the geometrical or the linear category.

The most general target category we can consider is a symmetric tensor category: clearly we need a tensor product, and the axiom $\mathcal{H}_{Y_1 \sqcup Y_2} \cong \mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2}$ only makes sense if there is an involutory canonical isomorphism $\mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2} \cong \mathcal{H}_{Y_2} \otimes \mathcal{H}_{Y_1}$

A very common choice in physics is the category of *super vector spaces*, i.e. vector spaces V with a mod 2 grading $V = V^0 \oplus V^1$, where the canonical isomorphism $V \otimes W \cong W \otimes V$ is $v \otimes w \mapsto (-1)^{\deg v \deg w} w \otimes v$. One can also consider the category of $\mathbb{Z}$-graded vector spaces, with the same sign convention for the tensor product.

In either case the closed string algebra is a graded-commutative algebra $\mathcal{C}$ with a trace $\theta : \mathcal{C} \to \mathbb{C}$. In principle the trace should have degree zero, but in fact the commonly encountered theories have a *grading anomaly* which makes the trace have degree $-n$ for some integer n.[6] The formulae (2.5), (2.9), and (2.11) must be replaced by their graded-commutative analogues. In particular if we choose a basis ψ_μ and its dual ψ^μ so that

$$\theta_{\mathcal{C}}(\psi^\mu \psi_{\ \nu}) = \delta^\mu_{\ \nu} \tag{2.18}$$

then

$$\pi_b^{\ a}(\psi) = \sum_\mu (-1)^{\deg \psi_\mu \deg \psi} \psi_\mu \psi \psi^\mu \tag{2.19}$$

We can also obtain interesting structures by changing the geometrical category of manifolds and cobordisms by equipping them with extra stucture.

Example 1 We define *topological-spin theories* by replacing "manifolds" with "manifolds with spin-structure."

A *spin structure* on a surface means a double covering of its space of non-zero tangent vectors which is non-trivial on each individual tangent space. On an oriented 1-dimensional manifold S it means a double covering of the space of positively-oriented tangent vectors. For purposes of gluing it is useful to note that this is the same thing as a spin structure on a ribbon neighbourhood of S in an orientable surface. Each spin structure has an

[6] It is easy to see that, up to an overall translation of the grading, the most general anomaly assigns an operator of degree $\frac{1}{2}n(i - o - \chi)$ to a cobordism with Euler number χ and i incoming and o outgoing boundary circles.

automorphism which interchanges its sheets, and this will induce an involution T on any vector space which is naturally associated to a 1-manifold with spin structure, giving the vector space a mod 2 grading by its ± 1-eigenspaces. We define a topological-spin theory as a functor from the cobordism category of manifolds with spin structures to the category of super vector spaces with its graded tensor structure. The functor is required to take disjoint unions to super tensor products, and we also require the automorphism of the spin structure of a 1-manifold to induce the grading automorphism $T = (-1)^{\mathrm{degree}}$ of the super vector space. We shall see presently that this choice of the supersymmetry of the tensor product rather than the naive symmetry which ignores the grading is forced on us by the geometry of spin structures if we want to allow the possibility of a semisimple category of boundary conditions. There are two non-isomorphic circles with spin structure: S^1_{ns}, with the Möbius or "Neveu-Schwarz" structure, and S^1_r, with the trivial or "Ramond" structure. A topological-spin theory gives us state-spaces $\mathcal{C}_{ns}$, respectively $\mathcal{C}_r$ corresponding to S^1_{ns}, S^1_r.

There are four annuli with spin structures, for, alongside the cylinders $A^+_{ns,r} = S^1_{ns,r} \times [0,1]$ which induce the identity maps of $\mathcal{C}_{ns,r}$ there are also cylinders $A^-_{ns,r}$ which connect $S^1_{ns,r}$ to itself while interchanging the sheets. These cylinders $A^-_{ns,r}$ induce the grading automorphism on the state spaces. But because $A^-_{ns} \cong A^+_{ns}$ by an isomorphism which is the identity on the boundary circles — the Dehn twist which "rotates one end of the cylinder by 2π" — the grading on $\mathcal{C}_{ns}$ must be purely even. The space $\mathcal{C}_r$ can have both even and odd components. The situation is a little more complicated for "U-shaped" cobordisms, i.e. cylinders with two incoming or two outgoing boundary circles. If the boundaries are S^1_{ns} there is only one possibility, but if the boundaries are S^1_r there are two, corresponding to $A^{ns,r}_-$. The complication is that there seems no special reason to prefer either of the spin structures as "positive". We shall simply *choose* one — let us call it P — with incoming boundary $S^1_r \sqcup S^1_r$, and use P to define a pairing $\mathcal{C}_r \otimes \mathcal{C}_r \to \mathbb{C}$. We then choose a preferred cobordism Q in the other direction so that when we sew its right-hand outgoing S^1_r to the left-hand incoming one of P the resulting S-bend is the "trivial" cylinder A^+_r. We shall need to know, however, that the closed torus formed by the composition $P \circ Q$ has an *even* spin structure. Note that Frobenius structure θ on $\mathcal{C}$ restricts to 0 on $\mathcal{C}_r$.

There is a unique spin structure on the pair-of-pants cobordism of fig.2 which restricts to S^1_{ns} on each boundary circle, and it makes $\mathcal{C}_{ns}$ into a commutative Frobenius algebra in the usual way. If one incoming circle is S^1_{ns} and the other is S^1_r then the outgoing circle is S^1_r, and there are two possible spin structures, but the one obtained by removing a disc from the cylinder A^+_r is preferred: it makes $\mathcal{C}_r$ into a graded module over $\mathcal{C}_{ns}$. The chosen U-shaped cobordism P, with two incoming circles S^1_r, can be punctured to give us a pair of pants with an outgoing S^1_{ns}, and it induces a graded bilinear map $\mathcal{C}_r \times \mathcal{C}_r \to \mathcal{C}_{ns}$ which, composing with the trace on $\mathcal{C}_{ns}$, gives a non-degenerate inner product on $\mathcal{C}_r$. At this point the choice of symmetry of the tensor product becomes important. For the diffeomorphism of the pair of pants which shows us in the usual case that the Frobenius algebra is commutative, when we lift it to the spin structure, induces the identity on one incoming circle but reverses the sheets over the other incoming circle, and this proves that the cobordsism must have the same output when we change the input from $S(\phi_1 \otimes \phi_2)$ to $T(\phi_1) \otimes \phi_2$, where T is the grading involution and $S : \mathcal{C}_r \otimes \mathcal{C}_r \to \mathcal{C}_r \otimes \mathcal{C}_r$ is the symmetry of the tensor category. If we take S to be the identity, this shows that the product on the graded vector space $\mathcal{C}_r$ is graded-symmetric with the usual sign; but if S is the graded symmetry then we see that *the product on $\mathcal{C}_r$ is symmetric in the naive sense.* (We must bear in mind here that if ψ_1 and ψ_2 do not have the same parity then their product is in any case zero, as we have seen that $\mathcal{C}_+$ is purely even.)

There is an analogue for spin theories of the theorem which tells us that a two-dimensional topological field theory "is" a commutative Frobenius algebra. It asserts that a spin-topological theory "is" a Frobenius algebra $\mathcal{C} = (\mathcal{C}_{ns} \oplus \mathcal{C}_r, \theta_{\mathcal{C}})$ with the properties just mentioned, and with the following additional property. Let $\{\phi_k\}$ be a basis for $\mathcal{C}_{ns}$, with dual basis $\{\phi^k\}$ such that $\theta_{\mathcal{C}}(\phi_k \phi^m) = \delta^m_k$, and let β_k and β^k be similar dual bases for $\mathcal{C}_r$. Then the Euler elements $\chi_{ns} := \sum \phi_k \phi^k$ and $\chi_r = \sum \beta_k \beta^k$ are independent of the choices of bases, and the condition we need on the algebra $\mathcal{C}$ is that $\chi_{ns} = \chi_r$. In particular, this condition implies that the vector spaces $\mathcal{C}_{ns}$ and $\mathcal{C}_r$ have the same dimension. [7] In fact, the Euler elements can be obtained from cutting a hole out of the torus. There are actually four spin structures on the torus. The output state is necessarily in $\mathcal{C}_{ns}$. The Euler elements for the three even spin structures are equal to $\chi_e = \chi_{ns} = \chi_r$. There is in addition an Euler element χ_o corresponding to the odd spin structure, it is given by $\chi_o = \sum (-1)^{\deg \beta_k} \beta_k \beta^k$.

We shall omit the proof that the general spin theory is what we have just described, but it is almost identical with the proof we shall give in the appendix of the theorem of

[7] Thus, in a sense, the theory has "spacetime supersymmetry."

Turaev about G-equivariant theories in the simple case when the group G is $\mathbb{Z}/2$. Indeed a spin theory is very similar to — but not the same as — a $\mathbb{Z}/2$-equivariant theory, which is the structure obtained when the surfaces are equipped with principal $\mathbb{Z}/2$-bundles (i.e. double coverings) rather than spin structures. We shall discuss equivariant theories in §7. (One difference is that in the equivariant case the $\mathbb{Z}/2$ action is nontrivial in the sector $\mathcal{C}_1$ and trivial in $\mathcal{C}_g$, precisely the opposite of what we have found in the spin case.) Comparing with the equivariant theory, the surprising result that the product on $\mathcal{C}_r$ is naive-symmetric can be understood as twisted-anticommutativity.

It seems reasonable to call a spin theory *semisimple* if the algebra $\mathcal{C}_{ns}$ is semisimple, i.e. is the algebra of functions on a finite set X. Then $\mathcal{C}_r$ is the space of sections of a vector bundle E on X, and it follows from the condition $\chi_{ns} = \chi_r$ that the fibre at each point must have dimension 1. Thus the whole structure is determined by the Frobenius algebra $\mathcal{C}_{ns}$ together with the binary choice of the grading of the fibre of the line bundle E at each point.

We can now see that if we had used the graded symmetry in defining the tensor category we should have forced the grading of $\mathcal{C}_r$ to be purely even. For on the odd part the inner product would have had to be skew, and that is impossible on a 1-dimensional space. And if both $\mathcal{C}_{ns}$ and $\mathcal{C}_r$ are purely even then the theory is in fact completely independent of the spin structures on the surfaces.

A concrete example of a two-dimensional topological-spin theory is given by $\mathcal{C} = \mathbb{C} \oplus \mathbb{C}\eta$ where $\eta^2 = 1$ and η is odd. The Euler elements are $\chi_e = 1$ and $\chi_o = -1$. It follows that the partition function of a closed surface with spin structure is ± 1 according as the spin structure is even or odd. (To prove this it is useful to compute the Arf invariant of the quadratic refinement of the intersection product associated to the spin structure and to note that it is multiplicative for adding handles.)

The most common theories defined on surfaces with spin structure are not topological: they are 2-dimensional conformal field theories with $\mathcal{N} = 1$ supersymmetry. The general features of the structure are still as we have described, but it should be noticed that if the theory is not topological one does not expect the grading on $\mathcal{C}_{ns}$ to be purely even: states can change sign on rotation by 2π. If a surface Σ has a conformal structure then a double covering of the non-zero tangent vectors is the complement of the zero-section in a two-dimensional real vector bundle L on Σ which is called the *spin bundle*. The covering map then extends to a symmetric pairing of vector bundles $L \otimes L \to T\Sigma$, which if we regard L and $T\Sigma$ as complex line bundles in the natural way, induces an isomorphism $L \otimes_{\mathbb{C}} L \cong T\Sigma$. An $\mathcal{N} = 1$ superconformal field theory is a conformal-spin theory with an additional map

$$\Gamma(S;L) \otimes \mathcal{H}_{S,L} \to \mathcal{H}_{S,L} \tag{2.20}$$

$$(\sigma, \psi) \mapsto G_\sigma \psi \tag{2.21}$$

such that G_σ is real-linear in the section σ of L and satisfies $G_\sigma^2 = D_{\sigma^2}$, where D_{σ^2} is the Virasoro action of the vector field σ^2. Furthermore, when we have a cobordism (Σ, L) from (S_0, L_0) to (S_1, L_1) and a holomorphic section σ of L which restricts to σ_i on S_i we have the intertwining property

$$G_{\sigma_1} \circ U_{\Sigma,L} = U_{\Sigma,L} \circ G_{\sigma_0}. \tag{2.22}$$

Example 2 We define *topological-spin*^{c} *theories*, which model 2d theories with $\mathcal{N} = 2$ supersymmetry, by replacing "manifolds" with "manifolds with spin^c-structure".

A *spin*^{c}*-structure* on a surface with a conformal structure is a pair of holomorphic line bundles L_1, L_2 with an isomorphism $L_1 \otimes L_2 \cong T\Sigma$ of holomorphic line bundles. A spin structure is the particular case when $L_1 = L_2$. An $\mathcal{N} = 2$ superconformal theory assigns a vector space $\mathcal{H}_{S;L_1,L_2}$ to each 1-manifold S with spin^c-structure, and an operator

$$U_{S_0;L_1,L_2} : \mathcal{H}_{S_0;L_1,L_2} \to \mathcal{H}_{S_1;L_1,L_2} \tag{2.23}$$

to each spin^c-cobordism from S_0 to S_1. To explain the rest of the structure we need to define the $\mathcal{N} = 2$ Lie superalgebra associated to a spin^c 1-manifold $(S; L_1, L_2)$. Let $\mathcal{G} = \mathrm{Aut}(L_1)$ denote the group of bundle isomorphisms $L_1 \to L_1$ which cover diffeomorphisms of S. (We can identify this group with $\mathrm{Aut}(L_2)$.) Its Lie algebra $\mathrm{Lie}(\mathcal{G})$ is an extension of $\mathrm{Vect}(S)$ by $\Omega^0(S)$. Let $\Lambda^0_{S;L_1,L_2}$ denote the complex Lie algebra obtained from $\mathrm{Lie}(\mathcal{G})$ by complexifying $\mathrm{Vect}(S)$. This is the even part of a Lie superalgebra whose odd part is $\Lambda^1_{S;L_1,L_2} = \Gamma(L_1) \oplus \Gamma(L_2)$. The bracket $\Lambda^1 \otimes \Lambda^1 \to \Lambda^0$ is completely determined by the property that elements of $\Gamma(L_1)$ and of $\Gamma(L_2)$ anticommute among themselves, while the composite

$$\Gamma(L_1) \otimes \Gamma(L_2) \to \Lambda^1 \to \mathrm{Vect}_{\mathbb{C}}(S) \tag{2.24}$$

takes (λ_1, λ_2) to $\lambda_1 \lambda_2 \in \Gamma(TS)$.

In an $\mathcal{N} = 2$ theory we require the superalgebra $\Lambda(S; L_1, L_2)$ to act on the vector space $\mathcal{H}_{S;L_1,L_2}$, compatibly with the action of the group $\mathcal{G}$, and with a similar intertwining property with the cobordism operators to that of the $\mathcal{N} = 1$ case. For an $\mathcal{N} = 2$ theory the state space always has an action of the circle group coming from its embedding in $\mathcal{G}$ as the group of fibrewise multiplications on L_1 and L_2. Equivalently, the state space is always $\mathbb{Z}$-graded.

An $\mathcal{N} = 2$ theory always gives rise to two ordinary conformal field theories by equipping a surface Σ with the spin^c structures $(\mathbb{C}, T\Sigma)$ and $(T\Sigma, \mathbb{C})$. These are called the "A-model" and the "B-model" associated to the $\mathcal{N} = 2$ theory. In each case the state spaces are cochain complexes in which the differential is the action of the constant section 1 of the trivial component of the spin^c-structure.

Cochain level theories

The most important "generalization," however, of the open and closed topological field theory we have described is the one of which it is intended to be a toy model. In closed string theory the central object is the vector space $\mathcal{C} = \mathcal{C}_{S^1}$ of states of a single parametrized string. This has an integer grading by the "ghost number", and an operator $Q : \mathcal{C} \to \mathcal{C}$ called the "BRST operator" which raises the ghost number by 1 and satisfies $Q^2 = 0$. In other words, $\mathcal{C}$ is a *cochain complex*. If we think of the string as moving in a space-time M then $\mathcal{C}$ is roughly the space of differential forms defined along the orbits of the action of the reparametrization group $\mathrm{Diff}^+(S^1)$ on the free loop space $\mathcal{L}M$. (More precisely, square-integrable forms of semi-infinite degree.) Similarly, the space $\mathcal{C}$ of a topologically-twisted $N = 2$ supersymmetric theory, as just described, is a cochain complex which models the space of semi-infinite differential forms on the loop space of a Kähler manifold — in this case, *all* square-integrable differential forms, not just those along the orbits of $\mathrm{Diff}^+(S^1)$. In both kinds of example, a cobordism Σ from p circles to q circles gives an operator $U_{\Sigma,\mu} : \mathcal{C}^{\otimes p} \to \mathcal{C}^{\otimes q}$ which depends on a conformal structure μ on Σ. This operator is a cochain map, but its crucial feature is that changing the conformal structure μ on Σ changes the operator $U_{\Sigma,\mu}$ only by a cochain-homotopy. The cohomology $H(\mathcal{C}) = \ker(Q)/\mathrm{im}(Q)$ — the "space of physical states" in conventional string theory — is therefore the state space of a topological field theory. (In the usual string theory situation the topological field theory we obtain is not very interesting, for the BRST cohomology is concentrated in one or two degrees, and there is a "grading anomaly" which means that the operator associated to a cobordism Σ changes the degree by a multiple of the Euler number $\chi(\Sigma)$. In the case of the $N = 2$ supersymmetric models, however, there is no grading anomaly, and the full structure is visible.)

A good way to describe how the operator $U_{\Sigma,\mu}$ varies with μ is as follows.

If $\mathcal{M}_\Sigma$ is the moduli space of conformal structures on the cobordism Σ, modulo diffeomorphisms of Σ which are the identity on the boundary circles, then we have a cochain map

$$U_\Sigma : \mathcal{C}^{\otimes p} \to \Omega(\mathcal{M}_\Sigma; \mathcal{C}^{\otimes q}) \tag{2.25}$$

where the right-hand side is the de Rham complex of forms on $\mathcal{M}_\Sigma$ with values in $\mathcal{C}^{\otimes q}$. The operator $U_{\Sigma,\mu}$ is obtained from U_Σ by restricting from $\mathcal{M}_\Sigma$ to $\{\mu\}$. The composition property when two cobordisms Σ_1 and Σ_2 are concatenated is that the diagram

$$\begin{array}{ccc} \mathcal{C}^{\otimes p} & \longrightarrow & \Omega(\mathcal{M}_{\Sigma_1}; \mathcal{C}^{\otimes q}) \\ \downarrow & & \downarrow \\ \Omega(\mathcal{M}_{\Sigma_2 \circ \Sigma_1}; \mathcal{C}^{\otimes r}) & \longrightarrow & \Omega(\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}; \mathcal{C}^{\otimes r}) = \Omega(\mathcal{M}_{\Sigma_1}; \Omega(\mathcal{M}_{\Sigma_2}; \mathcal{C}^{\otimes r})) \end{array} \tag{2.26}$$

commutes, where the lower horizontal arrow in induced by the map $\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2} \to \mathcal{M}_{\Sigma_2 \circ \Sigma_1}$ which expresses concatenation of the conformal structures.

Many variants of this formulation are possible. For example, we might prefer to give a cochain map

$$U_\Sigma : C.(\mathcal{M}_\Sigma) \to (\mathcal{C}^{\otimes p})^* \otimes \mathcal{C}^{\otimes q}, \tag{2.27}$$

where $C.(\mathcal{M}_\Sigma)$ is, say, the complex of smooth singular chains of $\mathcal{M}_\Sigma$. We may also prefer to use the moduli spaces of Riemannian structures instead of conformal structures.

There is no difficulty in passing from the closed-string picture just presented to an open and closed theory. We shall not discuss these cochain-level theories in any depth in this work, but it is important to realize that they are the real objective. We shall now point out a few basic things about them. A much fuller discussion can be found in Costello [25].

For each pair a, b of boundary conditions we shall still have a vector space — indeed a cochain complex — $\mathcal{O}_{ab}$, but it is no longer the space of morphisms from b to a in a category. Rather, what we have is, in the terminology of Fukaya, Kontsevich, and others, an A_∞-*category*. This means that instead of a composition law $\mathcal{O}_{ab} \times \mathcal{O}_{bc} \to \mathcal{O}_{ac}$ we have a *family* of ways of composing, parametrized by the contractible space of conformal structures on the surface of fig. 1. In particular, any two choices of a composition law from the family are cochain-homotopic. Composition is associative in the sense that we have a contractible family of triple compositions $\mathcal{O}_{ab} \times \mathcal{O}_{bc} \times \mathcal{O}_{cd} \to \mathcal{O}_{ad}$, which contains all the maps obtained by choosing a binary composition law from the given family and bracketing the triple in either of the two possible ways.

Note This is not the usual way of defining an A_∞-structure. According to Stasheff's original definition, an A_∞-structure on a space X consists of a sequence of choices: first, a composition law $m_2 : X \times X \to X$; then, a choice of a map

$$m_3 : [0,1] \times X \times X \times X \to X$$

which is a homotopy between $(x,y,z) \mapsto m_2(m_2(x,y),z)$ and $(x,y,z) \mapsto m_2(x, m_2(y,z))$; then, a choice of a map

$$m_4 : C_2 \times X^4 \to X,$$

where C_2 is a convex plane polygon whose vertices are indexed by the five ways of bracketing a 4-fold product, and $m_4 | ((\partial C_2) \times X^4)$ is determined by m_3; and so on.

There is an analogous definition — in fact slightly simpler — applying to cochain complexes rather than spaces. These definitions, however, are essentially equivalent to the one above coming from 2-dimensional field theory: the only important point is to have a *contractible* family of k-fold compositions for each k. (A discussion of the relation between the definitions can be found in [26].)

Apart from the composition law, the essential algebraic properties we have found in our theories are the non-degenerate inner product, and the commutativity of the closed algebra $\mathcal{C}$. Concerning the latter, when we pass to cochain theories the multiplication in $\mathcal{C}$ will of course be commutative up to cochain homotopy, but, unlike what happened with the open-string composition, the moduli space $\mathcal{M}_\Sigma$ of closed-string multiplications, i.e. the moduli space of conformal structures on a pair of pants Σ, modulo diffeomorphisms of Σ which are the identity on the boundary circles, is not contractible: it contains a natural *circle* of multiplications, and there are two *different* natural homotopies between the multiplication and the reversed multiplication. This might be a clue to an important difference between stringy and classical space-times. The closed string cochain complex $\mathcal{C}$ is the string-theory substitute for the de Rham complex of space-time, an algebra whose multiplication is associative and (graded-)commutative on the nose. Over the rationals or the real or complex numbers, such cochain algebras are known by the work of Sullivan [27] and Quillen [28] to model[8] the category of topological spaces up to homotopy, in the sense that to each such algebra $\mathcal{C}$ we can associate a space $X_{\mathcal{C}}$ and a homomorphism of cochain algebras from $\mathcal{C}$ to the de Rham complex of $X_{\mathcal{C}}$ which is a cochain homotopy equivalence. If we do not want to ignore torsion in the homology of spaces we can no longer encode the homotopy type in a strictly commutative cochain algebra. Instead, we must replace commutative algebras with so-called E_∞-algebras, i.e., roughly, cochain complexes $\mathcal{C}$ over the integers equipped with a multiplication which is associative and commutative up to given arbitrarily high-order homotopies. An arbitrary space X has an E_∞-algebra $\mathcal{C}_X$ of cochains, and conversely one can associate a space $X_{\mathcal{C}}$ to each E_∞-algebra $\mathcal{C}$. Thus we have a pair of adjoint functors, just as in rational homotopy theory. A long evolution in algebraic topology has culminated in recent theorems of Mandell [29] which show that the actual homotopy category of topological spaces is more or less equivalent to the category of E_∞-algebras. The cochain algebras of closed string theory have less higher commutativity than do E_∞-algebras, and this may be an indication that we are dealing with non-commutative spaces in Connes's sense: that fits in well with the interpretation of the B-field of a string background as corresponding to a bundle of matrix algebras on space-time. At the same time, the nondegenerate inner product on $\mathcal{C}$ — corresponding to Poincaré duality — seems to show we are concerned with *manifolds*, rather than more singular spaces.

For readers not accustomed to working with cochain complexes it may be worth saying a few words about what one gains by doing so. To take the simplest example, let us consider the category $\mathcal{K}$ of cochain complexes of finitely generated *free* abelian groups and cochain-homotopy classes of cochain maps. This is called the *derived category* of the category of finitely generated abelian groups. Passing to cohomology gives us a functor from $\mathcal{K}$ to the category of $\mathbb{Z}$-graded finitely generated abelian groups. In fact the subcategory $\mathcal{K}_0$ of $\mathcal{K}$ consisting of complexes whose cohomology vanishes except in degree 0 is actually equivalent

[8] In this and the following sentence we are overlooking subtleties related to the fundamental group.

to the category of finitely generated abelian groups.[9] But the category $\mathcal{K}$ inherits from the category of finitely generated free abelian groups a duality functor with properties as ideal as one could wish: each object is isomorphic to its double dual, and dualizing preserves exact sequences. (The dual C^* of a complex C is defined by $(C^*)^i = \mathrm{Hom}(C^{-i};\mathbb{Z})$.) There is no such nice duality in the category of finitely generated abelian groups. Indeed, the subcategory $\mathcal{K}_0$ is not closed under duality, for the dual of the complex C_A corresponding to a group A has in general *two* non-vanishing cohomology groups: $\mathrm{Hom}(A;\mathbb{Z})$ in degree 0, and in degree $+1$ the finite group $\mathrm{Ext}(A;\mathbb{Z})$ Pontrjagin-dual to the torsion subgroup of A. This follows from the exact sequence (not to be confused with the cochain complex):

$$0 \to \mathrm{Hom}(A,\mathbb{Z}) \to \mathrm{Hom}(F_A,\mathbb{Z}) \to \mathrm{Hom}(R_A,\mathbb{Z}) \to \mathrm{Ext}(A,\mathbb{Z}) \to 0 \qquad (2.28)$$

The category $\mathcal{K}$ also has a tensor product with better properties than the tensor product of abelian groups (which does not preserve exact sequences), and, better still, there is a canonical cochain functor from (locally well-behaved) compact spaces to $\mathcal{K}$ which takes Cartesian products to tensor products. (The simplicial, Čech, and other candidates for the cochain complex of a space are *canonically* isomorphic in $\mathcal{K}$.)

We shall return to this discussion in §6.

3. Solutions of the algebraic conditions: the semisimple case

3.1. Classification theorem

We now turn to the question : given a closed string theory $\mathcal{C}$, what is the corresponding category of boundary conditions? In our formulation this becomes the question: given a commutative Frobenius algebra $\mathcal{C}$, what are the possible $\mathcal{O}_{ab}$'s?

[9] To an abelian group A one can associate the cochain complex

$$C_A = (\ \cdots\ \to 0 \to R_A \to F_A \to 0 \to\ \cdots\ ),$$

where F_A is a free abelian group (in degree 0) with a surjective map $F_A \to A$, and R_A is the kernel of $F_A \to A$. The choice of F_A is far from unique, but nevertheless the different choices of C_A are *canonically* isomorphic objects of $\mathcal{K}$.

We can answer this question in the case when $\mathcal{C}$ is semisimple. We will take $\mathcal{C}$ to be an algebra over the complex numbers, and in this case the most useful characterization of semisimplicity is that the "fusion rules"

$$\phi_\mu \phi_\nu = N_{\mu\nu}^{\ \ \lambda} \phi_\lambda \tag{3.1}$$

are diagonalizable.[10] That is, the matrices $L(\phi_\mu)$ of the left-regular representation, with matrix elements $N_{\mu\nu}^{\ \ \lambda}$, are simultaneously diagonalizable.

Equivalently, there is a set of basic idempotents ε_x such that

$$\begin{aligned} \mathcal{C} &= \oplus_x \mathbb{C}\varepsilon_x \\ \varepsilon_x \varepsilon_y &= \delta_{xy}\varepsilon_y \end{aligned} \tag{3.2}$$

Equivalently, yet again, $\mathcal{C}$ is the algebra of complex-valued functions on the finite set $X = \mathrm{Spec}(\mathcal{C})$ of characters of $\mathcal{C}$.

The trace $\theta_{\mathcal{C}} : \mathcal{C} \to \mathbb{C}$, which should be thought of as a "dilaton field" on the finite space-time $\mathrm{Spec}(\mathcal{C})$, is completely described by the unordered set of non-zero complex numbers

$$\theta_x := \theta_{\mathcal{C}}(\varepsilon_x) \tag{3.3}$$

which is the only invariant of a finite dimensional commutative semisimple Frobenius algebra.

It should be mentioned that the most general finite dimensional commutative algebra over the complex numbers is of the form $\mathcal{C} = \oplus \mathcal{C}_x$, where x runs through the set $\mathrm{Spec}(\mathcal{C})$, and $\mathcal{C}_x$ is a local ring, i.e. $\mathcal{C}_x = \mathbb{C}\varepsilon_x \oplus m_x$, with ε_x as in (3.2) , and m_x a nilpotent ideal. If $\mathcal{C}$ is a Frobenius algebra, then so is each $\mathcal{C}_x$, and there is some ν_x for which $\theta_{\mathcal{C}} : m_x^{\nu_x} \to \mathbb{C}$ is an isomorphism, while $m_x^{\nu_x+1} = 0$. Let us write $\omega_x \in m_x^{\nu_x}$ for the element such that $\theta_{\mathcal{C}}(\omega_x) = 1$. The element ω of $\mathcal{C}$ with components ω_x can be regarded as a "volume form" on space-time. (A typical example of such a local Frobenius algebra $\mathcal{C}_x$ is the cohomology ring — with complex coefficients — of complex projective space $\mathbb{P}^n$ of dimension n. The cohomology ring is generated by a single 2-dimensional class t which satisfies $t^{n+1} = 0$. The trace is given by integration over $\mathbb{P}^n$, and takes t^k to 1 if $k = n$, and to 0 otherwise. Thus $\omega_x = t^n$ here.)

[10] The structure constants $N_{\mu\nu}^{\ \ \lambda}$ need not be integral, though in many interesting examples there is a basis for the algebra in which they are integral.

A useful technical fact about Frobenius algebras — not necessarily commutative — is that, in the notation of (2.11) , the "Euler" element $\chi = \sum \psi_\mu \psi^\mu$ is invertible if and only if the algebra is semisimple [11], which in the general case means that the algebra is isomorphic to a sum of full matrix algebras. The element χ always belongs to the centre of the algebra; in the commutative case it has components $\dim(\mathcal{C}_x)\omega_x$.

In the semisimple case we have the following complete characterization of the possible open algebras $\mathcal{O}_{aa}$ compatible with a fixed closed algebra $\mathcal{C}$. Unfortunately, though, the arguments we use do not work for graded Frobenius algebras.

Theorem 2: If $\mathcal{C}$ is semisimple then $\mathcal{O} = \mathcal{O}_{aa}$ is semisimple for each a and necessarily of the form $\mathcal{O} = \mathrm{End}_{\mathcal{C}}(W)$ for some finite dimensional representation W of $\mathcal{C}$.

Proof: The images $\iota_a(\varepsilon_x) = P_x$ are central simple idempotents. Therefore $\mathcal{O}_x = P_x\mathcal{O} = P_x\mathcal{O}P_x$ is an algebra over the Frobenius algebra $\mathcal{C}_x = \varepsilon_x\mathcal{C} \cong \mathbb{C}$, and so it suffices to work over a single space-time point. Then $\iota^a(1_{\mathcal{O}_x}) = \alpha 1_{\mathcal{C}_x}$ for some element $\alpha \in \mathbb{C}$. By the Cardy condition

$$\alpha 1_{\mathcal{O}_x} = \chi_{\mathcal{O}_x} = \sum \psi_\mu \psi^\mu \tag{3.4}$$

Applying θ we find $\alpha = \dim \mathcal{O}_x$, and hence $\chi_{\mathcal{O}_x}$ is invertible if $\mathcal{O}_x \neq 0$. It follows that $\mathcal{O}_x$ is semisimple at each point x, i.e. a sum of matrix algebras $\oplus_i \mathrm{End}(W_i)$. In fact, the Cardy condition shows that there can be at most one summand W_i at each point, i.e. the algebra is simple. For the map $\pi : \mathcal{O}_x \to \mathcal{O}_x$ must take each summand $\mathrm{End}(W_i)$ into itself, and cannot factor through the 1-dimensional $\mathcal{C}_x$ if more than one W_i is non-zero. ♠

According to Theorem 2 the most general $\mathcal{O}_{aa}$ is obtained by choosing a vector space $W_{x,a}$ for each basic idempotent ε_x, i.e. a *vector bundle* on the finite space-time $X = \mathrm{Spec}(\mathcal{C})$, and forming:

$$\mathcal{O}_{aa} = \oplus_x \mathrm{End}(W_{x,a}). \tag{3.5}$$

[11] To see this, one observes that for any element ψ of the algebra we have $\theta(\psi\chi) = \mathrm{tr}(\psi)$, where $\mathrm{tr}(\psi)$ denotes the trace of ψ in the regular representation. As the pairing $(\psi_1, \psi_2) \mapsto \theta(\psi_1\psi_2)$ is nondegenerate, it follows that the trace-form $(\psi_1, \psi_2) \mapsto \mathrm{tr}(\psi_1\psi_2)$ is nondegenerate if and only if χ is invertible, and non-degeneracy of the trace-form is well-known to be a criterion for a finite dimensional algebra to be semisimple. There are several definitions of semisimplicity, and their equivalence amounts to the classical theorem of Wedderburn. For our purposes, a semisimple algebra is just a sum of full matrix algebras.

But let us notice that when we have an algebra of the form $\mathrm{End}(W)$ the vector space W is determined by the algebra only up to tensoring with an arbitrary complex line: any irreducible representation of the algebra will do for W.

Elements $\psi \in \mathcal{O}_{aa}$ will be denoted $\psi = \oplus \psi_x$. Let P_x be the projection operator onto the x^{th} summand. From the equation

$$\iota_a(\varepsilon_x) = P_x \tag{3.6}$$

the adjoint relation and the Cardy condition determine the relations:

$$\begin{aligned} \theta_a(\psi) &= \sum_x \sqrt{\theta_x} \mathrm{Tr}(\psi_x) \\ \iota^a(\psi) &= \oplus_x \mathrm{Tr}(\psi_x) \frac{\varepsilon_x}{\sqrt{\theta_x}} \\ \pi_b{}^a(\psi_{aa}) &= \oplus_x \frac{1}{\sqrt{\theta_x}} \mathrm{Tr}_{W_{x,a}}(\psi_{x,aa}) P_{x,b} \end{aligned} \tag{3.7}$$

(one must use the same square-root in the formula for $\theta_{\mathcal{O}}$ and ι^a.) Note that $\theta_{\mathcal{C}}(\frac{\varepsilon_x}{\sqrt{\theta_x}} \frac{\varepsilon_y}{\sqrt{\theta_y}}) = \delta_{x,y}$, i.e. the elements $\frac{\varepsilon_x}{\sqrt{\theta_x}}$ form a natural orthonormal basis for $\mathcal{C}$. Thus, a boundary condition a gives us a tuple of *positive integers* $w_x = \dim W_x$, one for each basic idempotent, as well as a choice of the square-root $\sqrt{\theta_x}$. The relation (2.5), however, shows that these square-roots are an intrinsic property of the Frobenius category $\mathcal{B}$, and do not depend on which particular object in it we are considering.

Let us now determine the $\mathcal{O}_{aa} \times \mathcal{O}_{bb}$ bimodules $\mathcal{O}_{ab}$ associated to a pair of boundary conditions a, b. These are again fixed by the Cardy condition.

Lemma: When $\mathcal{C}$ is semisimple we have

$$\mathcal{O}_{ab} \cong \oplus_x \mathrm{Hom}(W_{x,a}; W_{x,b}) \tag{3.8}$$

Proof: Restricting to each $\mathcal{O}_{aa}$ we can invoke Theorem 2. Then the $\iota_a(\varepsilon_x)\mathcal{O}_{ab} = \mathcal{O}_{ab}\iota_b(\varepsilon_x)$ are bimodules for the simple algebras $\mathcal{O}_{x,aa}$ and $\mathcal{O}_{x,bb}$. We restrict to a single idempotent and drop the x, that is, we take $\mathcal{C} = \mathbb{C}$. The only irreducible representation of $\mathcal{O}_{aa} = \mathrm{End}(W_a)$ is W_a itself, and the only $\mathcal{O}_{aa} \times \mathcal{O}_{bb}$-bimodule is $W_a^* \otimes W_b$. Therefore, $\mathcal{O}_{ab} \cong n_{ab} W_a^* \otimes W_b$, where n_{ab} is a nonnegative integer. Let us work out the Cardy condition. If v_m is a basis for W_a and w_n is a basis for W_b then a basis for $\mathcal{O}_{ab}$ is $v_{m,\alpha}^* \otimes w_{n,\alpha}$ where $\alpha = 1, \ldots, n_{ab}$. Then $\pi(\psi) = n_{ab} \mathrm{tr}_{W_a}(\psi) P_b$. Comparing to $\iota_b \iota^a(\psi)$ we get $n_{ab} = 1$. ♠

We can now describe the maximal category $\mathcal{B}$ of boundary conditions. We first observe that if $p \in \mathcal{O}_{aa}$ is a projection —i.e. $p^2 = p$ —we can assume that $a = b \oplus c$ in $\mathcal{B}$, where b is the image of p. For we can adjoin images of projections to any additive category in much the same way as we adjoined direct sums. If the closed algebra $\mathcal{C}$ is semisimple we can therefore choose an object a_x of $\mathcal{B}$ for each space-time point x so that a_x is supported at x — i.e. $\iota_{a_x}(\varepsilon_x)\mathcal{O}_{a_x a_x} = \mathcal{O}_{a_x a_x}$ — and is simple, i.e. $\mathcal{O}_{a_x a_x} = \mathbb{C}$. For any object b of $\mathcal{B}$ we then have a canonical morphism

$$\oplus_x \mathcal{O}_{ba_x} \otimes a_x \to b, \tag{3.9}$$

where on the left we have used the possibility of tensoring any object of a linear category by a finite dimensional vector space. Furthermore, it follows from the lemma that the morphism (3.9) is an isomorphism, for both sides have the same space of morphisms into any other object c. Finally, notice that a_x is unique up to tensoring with a line L_x, for if a'_x is another choice then $a'_x \cong a_x \otimes L_x$, where $L_x = \mathcal{O}_{a_x a'_x}$.

Theorem 3

(i) If $\mathcal{C}$ is semisimple, corresponding to a space-time X, then the category $\mathcal{B}$ of boundary conditions is equivalent to the category $\mathrm{Vect}(X)$ of vector bundles on X, by the inverse functors

$$\{W_x\} \mapsto \oplus W_x \otimes a_x, \tag{3.10}$$

$$a \mapsto \{\mathcal{O}_{a_x a}\}. \tag{3.11}$$

(ii) The equivalence of $\mathcal{B}$ with $\mathrm{Vect}(X)$ is unique up to transformations $\mathrm{Vect}(X) \to \mathrm{Vect}(X)$ given by tensoring with a line bundle $L = \{L_x\}$ on X.

(iii) The Frobenius structure on $\mathcal{B}$ is determined by choosing a square-root $\{\sqrt{\theta}_x\}$ of the dilaton field. It is therefore unique up to multiplication by an element $\sigma \in \mathcal{C}$ such that $\sigma^2 = 1$.

Remarks

1. A boundary condition a has a *support*

$$\mathrm{supp}(a) = \{x \in X : W_x \neq 0\} \tag{3.12}$$

contained in $X = \mathrm{spec}(\mathcal{C})$. If two boundary conditions a and b have the same support then $\mathcal{O}_{ab}$ is a Morita equivalence bimodule between $\mathcal{O}_{aa}$ and $\mathcal{O}_{bb}$. The reader might wish to compare this discussion to section 6.4 of [30]. Note that it is necessary to invoke the Cardy condition to draw this conclusion.

2. Examples of semisimple Frobenius algebras in physics include:

a) The fusion rule algebra (Verlinde algebra) of a RCFT.

b) The chiral ring of an $\mathcal{N} = 2$ Landau-Ginzburg theory for generic superpotential W (that is, as long as all the critical points of W are Morse critical points). This is the case when the IR theory is massive.

c) Generic quantum cohomology of manifolds.

3.2. Comment on B-fields

We can see from this discussion just where the idea of a B-field would appear, though in fact on a 0-dimensional space-time any B-field must be trivial. We showed that there is a category of boundary conditions associated to each point of space time, and that it is isomorphic to the category of finite dimensional vector spaces, though not canonically. More precisely, it contains minimal — i.e. irreducible — objects from which any other object can be obtained by tensoring with a finite dimensional vector space.

Now a B-field is in essence a bundle of *categories* on space-time in which the fibre-categories are all isomorphic but not canonically. We can suppose that each fibre is isomorphic to the category of finite dimensional vector spaces. The crucial feature is that the ambiguity in identifying each fibre with the standard fibre is a “group” — in this case actually a category — of equivalences whose elements are complex lines and in which composition is given by the tensor product. Our category of boundary conditions is precisely the category of “sections” of a bundle of categories with this structural group.

It may be helpful to think of this in the following way. An electromagnetic field is a line bundle with connection on space-time. It is something we can think of as part of the structure of space-time, and makes sense in the absence of fermions. But in a theory with fermions there is a spinor-space at each point of space-time, and the electromagnetic field is “really” the information about how the spinor spaces are connected together from point to point of space-time. In this sense the electromagmetic field “is” the spinor-bundle with its connection. A B-field similarly “is” the bundle of boundary conditions.

On a general topological space X the classes of B-fields are classified by the elements of the cohomology group $H^3(X;\mathbb{Z})$, which can be understood as $H^1(X;\mathcal{G})$, where $\mathcal{G}$ is the "group" of line bundles under tensor product, which in algebraic topology is an Eilenberg-Maclane object of type $K(\mathbb{Z},2)$. We shall return to this topic in §7.

3.3. Reconstructing the closed algebra

When we have an open and closed TFT each element ξ of the closed algebra $\mathcal{C}$ defines an endomorphism $\xi_a = i_a(\xi) \in \mathcal{O}_{aa}$ of each object a of $\mathcal{B}$, and $\eta \circ \xi_a = \xi_b \circ \eta$ for each morphism $\eta \in \mathcal{O}_{ba}$ from a to b. The family $\{\xi_a\}$ thus constitutes a natural transformation from the identity functor $1_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ to itself.

For any $\mathbb{C}$-linear category $\mathcal{B}$ we can consider the ring $\mathcal{E}$ of natural transformations of $1_{\mathcal{B}}$. It is automatically commutative, for if $\{\xi_a\}, \{\eta_a\} \in \mathcal{E}$ then $\xi_a \circ \eta_a = \eta_a \circ \xi_a$ by the definition of naturality. If $\mathcal{B}$ is a Frobenius category then there is a map ${\pi_a}^b : \mathcal{O}_{bb} \to \mathcal{O}_{aa}$ for each pair of objects a, b, and we can define $j^b : \mathcal{O}_{bb} \to \mathcal{E}$ by $j^b(\eta)_a = {\pi_a}^b(\eta)$ for $\eta \in \mathcal{O}_{bb}$. In other words, j^b is defined so that the Cardy condition $\iota_a \circ j^b = {\pi_a}^b$ holds. But the question arises whether we can define a trace $\theta : \mathcal{E} \to \mathbb{C}$ to make $\mathcal{E}$ into a Frobenius algebra, and with the property that

$$\theta_a(\iota_a(\xi).\eta) = \theta(\xi.j^a(\eta)) \tag{3.13}$$

for all $\xi \in \mathcal{E}$ and $\eta \in \mathcal{O}_{aa}$. This is certainly true if $\mathcal{B}$ is a semisimple Frobenius category with finitely many simple objects, for then $\mathcal{E}$ is just the ring of complex-valued functions on the set of classes of these simple elements, and we can readily define $\theta : \mathcal{E} \to \mathbb{C}$ by $\theta(\varepsilon_a) = \theta_a(1_a)^2$, where a is an irreducible object, and $\varepsilon_a \in \mathcal{E}$ is the characteristic function of the point a in the spectrum of $\mathcal{E}$. Nevertheless, a Frobenius category need not be semisimple, and we cannot, unfortunately, take $\mathcal{E}$ as the closed string algebra in the general case. If, for example, $\mathcal{B}$ has just one object a, and $\mathcal{O}_{aa}$ is a commutative local ring of dimension greater than 1, then $\mathcal{E} = \mathcal{O}_{aa}$, and so $\iota_a : \mathcal{E} \to \mathcal{O}_{aa}$ is an isomorphism, and its adjoint map j^a ought to be an isomorphism too. But that contradicts the Cardy condition, as ${\pi_a}^a$ is multiplication by $\sum \psi_i \psi^i$, which must be nilpotent. In §6 we shall give an example of two *distinct* closed string Frobenius algebras which admit the same open string algebra $\mathcal{O}_{aa}$.

The commutative algebra $\mathcal{E}$ of natural endomorphisms of the identity functor of a linear category $\mathcal{B}$ is called the *Hochschild cohomology* $HH^0(\mathcal{B})$ of $\mathcal{B}$ in degree 0. The groups $HH^p(\mathcal{B})$ for $p > 0$, whose definition will be given in a moment, vanish if $\mathcal{B}$ is semisimple, but in the general case they appear to be relevant to the construction of a closed string algebra from $\mathcal{B}$. Let us notice meanwhile that for any Frobenius category $\mathcal{B}$ there is a natural homomorphism $K(\mathcal{B}) \to HH^0(\mathcal{B})$ from the Grothendieck group[12] of $\mathcal{B}$,

[12] I.e. the group formed from the semigroup of isomorphism classes of objects of $\mathcal{B}$ under $\oplus$.

which assigns to an object a the transformation whose value on b is $\pi_b{}^a(1_a) \in \mathcal{O}_{bb}$. In the semisimple case this homomorphism induces an isomorphism $K(\mathcal{B}) \otimes \mathbb{C} \to HH^0(\mathcal{B})$.

For any additive category $\mathcal{B}$ the Hochschild cohomology is defined as the cohomology of the cochain complex in which a k-cochain F is a rule that to each composable k-tuple of morphisms

$$Y_0 \xrightarrow{\phi_1} Y_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_k} Y_k \tag{3.14}$$

assigns $F(\phi_1, \ldots, \phi_k) \in \mathrm{Hom}(Y_0; Y_k)$. The differential in the complex is defined by

$$\begin{aligned}(dF)(\phi_1, \ldots, \phi_{k+1}) = & F(\phi_2, \ldots, \phi_{k+1}) \circ \phi_1 + \\ & + \sum_{i=1}^{k} (-1)^i F(\phi_1, \ldots, \phi_{i+1} \circ \phi_i, \ldots, \phi_{k+1}) \\ & + (-1)^{k+1} \phi_{k+1} \circ F(\phi_1, \ldots, \phi_k).\end{aligned} \tag{3.15}$$

(Notice, in particular, that a 0-cochain assigns an endomorphism F_Y to each object Y, and is a cocycle if the endomorphisms form a natural transformation. Similarly, a 2-cochain F gives a possible infinitesimal deformation $F(\phi_1, \phi_2)$ of the composition-law $(\phi_1, \phi_2) \mapsto \phi_2 \circ \phi_1$ of the category, and the deformation preserves the associativity of composition if and only if F is a cocycle.)

In the case of a category $\mathcal{B}$ with a single object whose algebra of endomorphisms is $\mathcal{O}$ the cohomology just described is usually called the Hochschild cohomology of the algebra $\mathcal{O}$ with coefficients in $\mathcal{O}$ regarded as a $\mathcal{O}$-bimodule. This must be carefully distinguished from the Hochschild cohomology with coefficients in the dual $\mathcal{O}$-bimodule $\mathcal{O}^*$. But if $\mathcal{O}$ is a Frobenius algebra it is isomorphic as a bimodule to $\mathcal{O}^*$, and the two notions of Hochschild cohomology need not be distinguished. The same applies to a Frobenius category $\mathcal{B}$: because $\mathrm{Hom}(Y_k; Y_0)$ is the dual space of $\mathrm{Hom}(Y_0; Y_k)$ we can think of a k-cochain as a rule which associates to each composable k-tuple (3.14) of morphisms a linear function of an element $\phi_0 \in \mathrm{Hom}(Y_k; Y_0)$. In other words, a k-cochain is a rule which to each "circle" of $k+1$ morphisms

$$\cdots \xrightarrow{\phi_0} Y_0 \xrightarrow{\phi_1} Y_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_k} Y_k \xrightarrow{\phi_0} \cdots \tag{3.16}$$

assigns a complex number $F(\phi_0, \phi_1, \ldots, \phi_k)$.

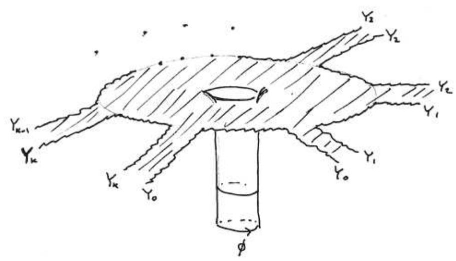

Fig. 13: A cyclic pairing of a closed string state ϕ with $k+1$ open string states.

If in this description we restrict ourselves to cochains which are cyclically invariant under rotating the circle of morphisms $(\phi_0, \phi_1, \ldots, \phi_k)$ then we obtain a sub-cochain-complex of the Hochschild complex whose cohomology is called the *cyclic cohomology* $HC^*(\mathcal{B})$ of the category $\mathcal{B}$. The cyclic cohomology — which evidently maps to the Hochschild cohomology — is a more natural candidate for the closed string algebra associated to $\mathcal{B}$ than is the Hochschild cohomology (for a state represented by the vector (3.16) pairs in a cyclically invariant way with a closed string state to give a number, in virtue of fig. 13). In our baby examples the cyclic and Hochschild cohomology are indistinguishable, but it is worth pointing out[13] that while $HH^2(\mathcal{B})$ is, as indicated above, the space of infinitesimal deformations of $\mathcal{B}$ as a category, the group $HC^2(\mathcal{B})$ is its space of infinitesimal deformations as a Frobenius category.

A very natural Frobenius category on which to test these ideas is the category of holomorphic vector bundles on a compact Calabi-Yau manifold: that example will be discussed in §6.

3.4. Spin theories and mod 2 graded categories

Let us give a brief outline, without proofs, of the modifications of the preceding discussion which are needed to describe the category of boundary conditions for a topological-spin theory as defined in §2.6.

There is just one spin structure on an interval, and its automorphism group is (± 1), so for each pair of boundary conditions a, b the vector space $\mathcal{O}_{ab}$ will have an involution, i.e. a mod 2 grading. The bilinear composition $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$ will preserve the grading. There is a non-degenerate trace $\theta_a : \mathcal{O}_{aa} \to \mathbb{C}$ which satisfies the commutativity condition (2.5) (without signs).

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[13] As we learnt from Kontsevich

If the closed theory is described by a Frobenius algebra $\mathcal{C} = \mathcal{C}_{ns} \oplus \mathcal{C}_r$, as in §2.6, there will be adjoint maps

$$\begin{aligned} \iota_a^{ns} &: \mathcal{C}_{ns} \to \mathcal{O}_{aa} \\ \iota_{ns}^a &: \mathcal{O}_{aa} \to \mathcal{C}_{ns} \\ \iota_a^r &: \mathcal{C}_r \to \mathcal{O}_{aa} \\ \iota_r^a &: \mathcal{C}_{aa} \to \mathcal{C}_r \end{aligned} \tag{3.17}$$

which preserve the grading. Moreover and ι_a^{ns} and ι_a^r fit together to define a homomorphism of algebras $\mathcal{C} \to \mathcal{O}_{aa}$. The centrality condition becomes

$$\begin{aligned} \iota_a^{ns}(\phi)\psi &= \psi\iota_a^{ns}(\phi) \\ \iota_a^r(\phi)\psi &= (-1)^{\deg\phi\deg\psi+\deg\psi}\psi\iota_a^r(\phi) \end{aligned} \tag{3.18}$$

Thus, ι^{ns} maps into the naive center of the algebra $\mathcal{O}_{aa}$. The reason we get the naive centre here, rather than the graded-algebra centre, and also the reason that the trace is naively commutative, is the same as that given in §2.6 for the naive commutativity of the algebra $\mathcal{C}$. The sign for ι^r is obtained by carefully following the choices of sections of the spin bundle one chooses under the diffeomorphism in figure 8.

There are two Cardy conditions

$$\begin{aligned} \iota_a^{ns}\iota_{ns}^b(\psi) = \pi_b^a(\psi) &:= \sum(-1)^{\deg\psi_\mu\deg\psi}\psi_\mu\psi\psi^\mu \\ \iota_a^r\iota_r^b(\psi) = \tilde{\pi}_b^a(\psi) &:= \sum(-1)^{\deg\psi_\mu(\deg\psi+1)}\psi_\mu\psi\psi^\mu. \end{aligned} \tag{3.19}$$

If we assume the closed algebra is semisimple then, just as before, we can assume that $\mathcal{C}_{ns}$ is the algebra of functions on a finite set X, and we can determine the category of boundary conditions point-by-point. In other words, we can assume that $\mathcal{C} = \mathbb{C}[\eta]$, where the generator η of $\mathcal{C}_r$ satisfies $\eta^2 = 1$, but may have either even or odd degree. In either case, the argument we have already used shows (by means of the first Cardy formula) that the algebra $\mathcal{O}_{aa}$ is the full matrix algebra of a vector space W. If the degree of η is even then $\iota^r(\eta) = P$ with P even, $P^2 = 1$, and $P\psi P = (-1)^{\deg\psi}\psi$. In this case the category of boundary conditions at the point is equivalent to the category of mod 2 graded vector spaces. If, on the other hand, the degree of η is odd, then P is odd, $P^2 = 1$ and P is (naive) central. The involution of the algebra $\mathcal{O}_{aa}$ corresponds to an involution of the module W, and the action of P is an isomorphism between the two halves of the grading. The even subalgebra of $\mathcal{O}_{aa}$ is a full matrix algebra. Thus the category of boundary conditions is the equivalent to the category of graded representations of the superalgebra $\mathbb{C}[\eta]$, which in turn is equivalent simply to the category of ungraded vector spaces. The Frobenius structure of the open algebra determines that of the closed algebra by taking the square, as in the ungraded case. The two cases $\deg \eta = 0$ and $\deg \eta = 1$ are roughly analogous to the distinction between the even and odd degree Clifford algebras over the complex numbers.

Suppose, conversely, that we have an arbitrary semisimple mod 2 graded category $\mathcal{B}$, i.e. a linear category equipped with an involutory functor S which one thinks of as the flip of the grading. Such a category has two kinds of simple object P: those such that $S(P) \cong P$, and those for which this is not true. The first kind of object generates a subcategory of $\mathcal{B}$ isomorphic to the category of vector spaces, and the second kind generates a subcategory isomorphic to the category of graded vector spaces. Thus any semisimple graded category $\mathcal{B}$ is the category of boundary conditions for a unique topological-spin theory.

4. Vector bundles, K-theory, and "boundary states"

In the semisimple case there is a nice geometrical interpretation of the category $\mathcal{B}$ of boundary conditions: the possible objects correspond to the vector bundles over the "space-time" $X = \mathrm{Spec}(\mathcal{C})$ associated to $\mathcal{C}$, which is just a finite set of points. The fiber above a point x is just the vector space W_x.

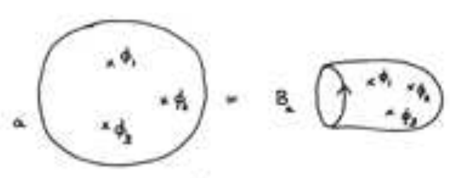

Fig. 14: Correlations on the upper half plane with boundary condition a are the same as the closed string amplitude for an insertion of a boundary state B_a.

Let us now make some comments on "boundary states". In the conformal field theory literature one associates to a boundary condition a a corresponding "state" B_a in the closed string state space. (Strictly, B_a is an element of the algebraic dual.) Translated to the present context $B_a \in \mathcal{C}$. The defining property of the boundary state is that the correlation functions of operators on a disk with the boundary condition a are equal to the correlation functions of the closed theory on the sphere obtained by capping off the disk with another disk and inserting the state B_a at the centre of the cap. This is illustrated in fig. 14.

In equations,

$$\theta_a\big(\iota_a(\phi_1)\cdots\iota_a(\phi_n)\big) = \theta_{\mathcal{C}}\big(B_a\phi_1\cdots\phi_n\big) \tag{4.1}$$

for all $\phi_1,\ldots,\phi_n$. Using the adjoint relation and nondegeneracy of the trace we find that

$$B_a = \iota^a(1_{\mathcal{O}_{aa}}) \tag{4.2}$$

The map $a \mapsto B_a$ is a natural homomorphism

$$K(\mathcal{B}) \to \mathcal{C}. \tag{4.3}$$

The operator which adds g handles and $h = \sum h_a$ holes, where h_a of the holes have the boundary condition a, is just $\chi^g \prod (B_a)^{h_a}$, where χ is the Euler element of $\mathcal{C}$.

Let us record one simple property of these boundary states. First, using the Cardy condition we have

$$\begin{aligned}\theta_{\mathcal{C}}(B_aB_b) &= \theta_a(\iota_a(B_b))\\ &= \theta_a(\iota_a\iota^b(1_b))\\ &= \theta_a({\pi_a}^b(1_b))\\ &= \dim \mathcal{O}_{ab}\end{aligned} \tag{4.4}$$

In the semisimple case we can give an explicit formula for the the "boundary state" in terms of the basic idempotents:

$$B_{\mathcal{O}} = \iota^*(1_{\mathcal{O}}) = \sum_x (\dim W_x)\frac{\varepsilon_x}{\sqrt{\theta_x}} \tag{4.5}$$

The formula shows that the boundary states form a positive cone in the unimodular lattice $\mathcal{L}_B$ spanned by the orthonormal basis $\frac{\varepsilon_x}{\sqrt{\theta_x}}$ in the closed algebra $\mathcal{C}$. In particular it follows from (4.5) that boundary states can only be added with positive integral coefficients.

They are therefore *not* like quantum mechanical states of branes. The fundamental integral structure is a result of the Cardy condition.

It is natural to speculate whether there should be an operation of "multiplication" of boundary conditions. There are arguments both for and against. The original perspective on D-branes, according to which they are viewed as "cycles" in space-time on which open strings can begin and end, suggests that there should be a multiplication, corresponding to the intersection of cycles. As no multiplication seems to emerge from the toy structure we have developed in this paper one may wonder whether an important ingredient has been omitted. Against this there are the following considerations. Our boundary conditions seem to correspond more closely to vector bundles — i.e. to K-theory classes — on space-time than to homology cycles: that will be plainer when we consider the equivariant situation in §7. Now the K-theory classes of a ring have a product, coming from the tensor product of modules, only when the ring is commutative; and we have already remarked that the B-fields which are part of the closed string model of space-time seem to encode a degree of noncommutativity. More precisely, D-branes seem to define classes in the *twisted* K-theory of space-time, twisted by the B-field, and the twisted K-theory of a space does not form a ring: the product of two twisted classes is a twisted class corresponding to the sum of the twistings of the factors. But in string theory there is no concept of "turning off" the B-field to find an underlying untwisted space-time. For example, the conformal field theory corresponding to a torus with a non-zero B-field can be isomorphic by "T-duality" to a theory coming from another torus with no B-field.

Another reason for not expecting a multiplication operation on D-branes also comes from T-duality in conformal field theory. There the closed string theories defined by a Riemannian torus T and its dual T^* are isomorphic, and we do indeed have a K-theory isomorphism $K(T) \cong K(T^*)$, but it is not compatible with the multiplication in K-theory. Furthermore, in some examples of TFTs coming from $N = 2$ supersymmetric sigma models the category of boundary conditions does seem to be a tensor category.

The formula (4.5) for the boundary state shows that the lattice $\mathcal{L}_B$, which is picked out inside $\mathcal{C}$ by the dilaton field θ, is not closed under multiplication in $\mathcal{C}$ unless $\theta_x = 1$ for all points x; but the lattices corresponding to different dilaton fields multiply into each other just as happens with twisted K-classes. Nevertheless, in the semisimple case, if we define an element $S := \sum_x \sqrt{\theta_x}\varepsilon_x$, then the operation

$$(B_1, B_2) \mapsto SB_1B_2 \tag{4.6}$$

does define a multiplication on boundary states, though its significance is unclear.

4.1. "Cardy states" vs. "Ishibashi states"

The formula (4.5) for the boundary state is reminiscent of what is what is known as a "Cardy state" in the construction of conformal field theories with boundary. That leaves the question: what are the "Ishibashi" or "character states"? This question can be nicely addressed in the topological framework of this exposition. In the basis of primary fields in closed (or chiral) conformal field theory *the fusion rules are in general not diagonal.* Thus the usual basis ϕ_μ, $\mu = 1, \ldots, N$ for $\mathcal{C}$ is different from the basis ε_i, $i = 1, \ldots, N$ used above. The fusion rules are diagonal in the basis ε_i.

The analogy with CFT is the following. In boundary conformal field theory one associates an "Ishibashi" or character state to every primary field of the chiral algebra. Formally these states are solutions to $(T - \bar{T})|B\rangle = 0$, or the generalization of this to the case of other chiral algebras. They are best thought of as intertwiners between left and right chiral representations. In the present context there is no chiral algebra, and we should think of *every* element of $\mathcal{C}$ as a solution to $(T - \bar{T}) = 0$, and its generalizations.

The basis of "Ishibashi states" associated to definite representations of the chiral algebra is naturally associated to the basis ϕ_μ analogous to primary fields. In this basis the algebra is given by

$$\phi_\mu \phi_\nu = N_{\mu\nu}{}^{\lambda} \phi_\lambda \tag{4.7}$$

with positive integral $N_{\mu\nu}{}^{\lambda}$. Using these formulae we recover, essentially, Cardy's formula for Cardy boundary states in terms of character boundary states. Note that there is no need to use any relation to the modular group.

We close with one further brief remark. It is nice to see the standard relation that the closed string coupling is the square of the open string coupling in the present context. If we scale $\theta_{\mathcal{C}} \to \lambda^{-2} \theta_{\mathcal{C}}$ then $\chi_{\mathcal{C}} = \sum_\mu \phi_\mu \phi^\mu \to \lambda^{+2} \chi_{\mathcal{C}}$. We may therefore interpret λ^2 as the closed string couplling. On the other hand, the squareroot of θ_i in $B_{\mathcal{O}}$ shows that $B_{\mathcal{O}} \to \lambda B_{\mathcal{O}}$, and therefore λ is the open string coupling. Indeed, the partition function for a surface with g handles and h holes is $Z(\Sigma) = \theta_{\mathcal{C}}((\chi_{\mathcal{C}})^g (B_{\mathcal{O}})^h)$, and therefore scales as $Z(\Sigma) \to \lambda^{-\chi(\Sigma)} Z(\Sigma)$, as expected, where $\chi(\Sigma) = 2 - 2g - h$ is the Euler number of Σ..

5. Landau-Ginzburg theories

D-branes can be defined in general two-dimensional N=2 Landau-Ginzburg theories [31]. Such theories can be topologically twisted, producing topological Landau-Ginzburg theories. It is interesting to compare with the D-branes obtained from our results applied to the resulting closed topological theory. Here we confine ourselves to a few very elementary remarks. In the past few years, following an initial suggestion by Kontsevich, an elaborate theory of categories of topological Landau-Ginzburg branes has been developed. We refer to [32,33,34,35,36,37,38] for details. These categories are thought to capture more physical information about the D-branes. In the case when all the critical points of the superpotential are Morse there is a functor to the category of branes we construct.

Let us recall the definition of a topological LG theory. One begins with a superpotential $W(X_i)$ which is a holomorphic function of chiral superfields X_i. When W is a polynomial the Frobenius algebra is simply the Jacobian ideal

$$\mathcal{C} = \mathbb{C}[X_i]/(\partial_i W) \tag{5.1}$$

The Frobenius structure is defined by a residue formula. For example, in the one-variable case we define

$$\theta(\phi) := \mathrm{Res}_{X=\infty} \frac{\phi(X)}{W'(X)} \tag{5.2}$$

If the critical points of W are all Morse critical points then the algebra (5.1) is semisimple. Physically Morse critical points correspond to massive theories, while non-Morse critical points renormalize to nontrivial 2d CFT's in the infrared.

If all the critical points are Morse then the trace is easily written in terms of the critical points p_a as

$$\theta(\phi) = + \sum_{dW(p_a)=0} \frac{\phi(p_a)}{\det(\partial_i \partial_j W|_{p_a})} \tag{5.3}$$

In the semisimple one-variable case we can construct the basic idempotents as follows. Let

$$dW = \prod_{\alpha=1}^{n} (X - r_\alpha) \tag{5.4}$$

where we assume all the roots are distinct. Then it is easy to check that

$$\varepsilon_\beta := \prod_{\alpha:\alpha\neq\beta} \frac{(X - r_\alpha)}{(r_\beta - r_\alpha)} \tag{5.5}$$

are basic idempotents. (To prove this, write $(X - r_\alpha) = (X - r_\beta) + (r_\beta - r_\alpha)$).

Example: $W = \frac{1}{3}t^3 - qt$. For $n = 2$ we can explicitly write

$$\begin{aligned} \varepsilon_1 &= \frac{\sqrt{q} + t}{2\sqrt{q}} \\ \varepsilon_2 &= \frac{\sqrt{q} - t}{2\sqrt{q}} \end{aligned} \tag{5.6}$$

Note that $\theta_1 = 1/(2\sqrt{q})$ and $\theta_2 = -1/(2\sqrt{q})$.

Then from the general result above one finds $\mathcal{O} = \mathrm{End}(W_1) \oplus \mathrm{End}(W_2)$ and

$$\begin{aligned} \theta_{\mathcal{O}}(\Psi) &= \frac{1}{\sqrt{\theta_1}}\mathrm{Tr}(\Psi_1) + \frac{1}{\sqrt{\theta_2}}\mathrm{Tr}(\Psi_2) \\ \iota^*(\Psi) &= \sqrt{\theta_1}\mathrm{Tr}(\Psi_1)\varepsilon_1 + \sqrt{\theta_2}\mathrm{Tr}(\Psi_2)\varepsilon_2 \end{aligned} \tag{5.7}$$

Thus, the general boundary state is

$$B = w_1 \frac{\varepsilon_1}{\sqrt{\theta_1}} + w_2 \frac{\varepsilon_2}{\sqrt{\theta_2}} \tag{5.8}$$

where w_1, w_2 are integers. Note the interesting monodromy as $q \to e^{2\pi i}q$. Branes of type (w_1, w_2) map to branes of type $(w_2, -w_1)$.

Clearly, there will be similar phenomena for general Landau-Ginzburg theories. The space of superpotentials W has a codimension one "discriminant locus" where it has non-Morse critical points. Analytic continuation around this locus will permute the ε_i, but will only permute the $\sqrt{\theta_i}$ up to sign. One may understand in this elementary way some of the brane permutation/creation phenomena discussed in numerous places in the literature.

The "vector bundles on spacetime" that we have found can be taken quite literally in the context of the theory of strings moving in less than one dimension which was worked out in 1988-1991. (For a reviews [39][40][41].) Strings moving in a spacetime of n disjoint points can be modelled by matrix chains or by topological field theory. The latter point of view is described in, for example, [40][41]. In the latter point of view, one considers topological gravity coupled to topological matter. For n spacetime points the topological matter can be taken to be the $N = 2$ Landau-Ginzburg theories associated to W given by the unfolding of A_n singularities:

$$W = \frac{x^{n+1}}{n+1} + a_n x^n + \cdots + a_0 \tag{5.9}$$

For generic W we find vector bundles on n spacetime points. This is of course what we expect for the branes in such spacetimes!

It is worth mentioning that in these simplest of string theories (the "minimal string theories") considerable progress has been made in recent years in understanding the full spectrum of D-branes, going beyond the topological field theory truncation. See [42] for a review.

6. Going beyond semisimple Frobenius algebras

The examples of topological field theories coming from $N = 2$ conformal field theories — Landau-Ginzburg models and the quantum cohomology rings of Calabi-Yau manifolds — suggest that it is of interest to understand the possible solutions of the algebraic conditions in the case when $\mathcal{C}$ is not semisimple. In this section we shall make some partial progress with this problem, and shall also explain how it should perhaps be viewed in a wider context.

6.1. Examples related to the cohomology of manifolds

A natural example of a graded commutative Frobenius algebra is the cohomology with complex coefficients of a compact oriented manifold X. Thus, for $\mathcal{C}$ we can take the algebra $\mathcal{C} = H^*(X;\mathbb{C})$ with trace $\theta(\phi) = \int_X \phi$. What are the corresponding $\mathcal{O}$'s?

A natural guess, which turns out to be wrong, but for interesting reasons, is that we should take $\mathcal{O} = \mathcal{C} \otimes \mathrm{Mat}_N(\mathbb{C}) = \mathrm{Mat}_N(\mathcal{C})$ for some $N > 0$, together with

$$\theta_{\mathcal{O}}(\psi) = \int_X \mathrm{Tr}(\psi) \tag{6.1}$$

While $\mathcal{O}$ is indeed a Frobenius algebra, the only natural candidate for the map ι_* is $\iota_*(\phi) = \phi \otimes 1_N$. However, this fails to satisfy the Cardy condition: one computes $\iota^*(\psi) = \mathrm{Tr}(\psi)$ from the adjoint relation, and hence $\iota_*\iota^*(\psi) = \mathrm{Tr}\psi \otimes 1_N$. On the other hand, one also finds

$$\begin{aligned}\pi(\psi) &= \sum (-1)^{\deg\omega_i(\deg\psi+\deg\omega^i)}\omega_i \otimes e_{lm} \wedge \psi \wedge \omega^i \otimes e_{ml} \\ &= \chi(TX) \wedge \mathrm{Tr}\,\psi\end{aligned} \tag{6.2}$$

Here ω_i is a basis for $H^*(X;\mathbb{C})$, e_{ml} are matrix units, and $\chi(TX) \in H^{\mathrm{top}}(X;\mathbb{C})$ is the Euler class of TX. The map π annihilates forms of positive degree, and cannot agree with $\iota_*\iota^*$.

This example can be modified to give an open and closed theory by taking $\mathcal{O}$ to be associated with a *submanifold* of X. This is, after all, the standard picture of D-branes! Let us work in the algebraic category of $\mathbb{Z}$-graded vector spaces, and continue to take $\mathcal{C} = H^*(X;\mathbb{C})$, with X a compact connected oriented n-dimensional manifold, and the trace $\theta_{\mathcal{C}}(\phi) = \int_X \phi$ of degree $-n$ as above. Let us look for an open algebra of the form $\mathcal{O} = \mathrm{Mat}_N(\mathcal{O}_0)$, with $\mathcal{O}_0$ commutative. Then $\mathcal{O}_0$ is a Frobenius algebra, and we may as well assume that it is $H^*(Y;\mathbb{C})$ for some compact oriented manifold[14] Y of dimension m, and that $\iota : \mathcal{C} \to \mathcal{O}_0$ is f^* for some map $f : Y \to X$.

Thus $\mathcal{O} = H^*(Y;\mathbb{C}) \otimes Mat_N(\mathbb{C})$ with open string trace

$$\theta_{\mathcal{O}}(\Psi) = \theta_o \int_Y \mathrm{Tr}(\Psi) \tag{6.3}$$

of degree $-m$, where θ_o is a constant. This is a non-commutative Frobenius algebra.

By the adjoint relation ι^* is then determined to be

$$\iota^*(\Psi) = \theta_o f_*(\mathrm{Tr}(\Psi)) \tag{6.4}$$

where f_* is the adjoint of the ring homomorphism $f^* : H^*(X) \to H^*(Y)$ with respect to Poincaré duality. Thus ι^* has degree $n - m$. On the other hand , one sees at once that $\pi : \mathcal{O} \to \mathcal{O}$ has degree m, so if the Cardy condition is to hold we must have $n = 2m$. If that is true, then we can assume, by making a small generic perturbation of f, that f is an immersion of Y in X. We can now make the the adjoint map f_* more explicit:

$$f_*(\psi) = \pi^*(\psi) \wedge \Phi_{\mathcal{N}}, \tag{6.5}$$

where $\pi : \mathcal{N} \to Y$ is the projection of the normal bundle (identified with a tubular neighborhood of Y in X) and $\Phi_{\mathcal{N}}$ is the Thom class of the bundle, compactly supported in the tubular neighborhood, which represents the cohomology class of Y in X. One easily finds that

$$\iota_* \iota^*(\Psi) = \theta_o \chi(\mathcal{N}Y) \wedge \mathrm{Tr}(\Psi) \otimes 1. \tag{6.6}$$

where $\chi(\mathcal{N}Y)$ is the Euler class of the normal bundle of $Y \hookrightarrow X$, i.e. the homological self-intersection of Y in X.

---

[14] In fact we need to allow Y to have orbifold singularities to ensure this.

On the other hand,

$$\pi(\Psi) = \frac{1}{\theta_o}\mathrm{Tr}(\Psi)\chi(TY) \tag{6.7}$$

where $\chi(TY) \in H^{\mathrm{top}}(Y;\mathbb{C})$ is the Euler class of the tangent bundle TY, whose integral is the Euler number of Y.

Evidently the Cardy conditions are satisfied if we choose θ_o so that $\chi(TY) = \theta_o^2\chi(\mathcal{N}Y)$. This is always possible if $\chi(\mathcal{N}Y)$, which is the self-intersection number of Y in X, is non-zero, and also possible if Y is a Lagrangian submanifold of a symplectic manifold X, for then $\mathcal{N} \cong TY$. The boundary state is $B = \theta_o N \Phi_{\mathcal{N}}$.

One immediate consequence of this discussion is that if we start, say, with $\mathcal{O} = H^*(\mathbb{C}P^2)$ as our open algebra we can easily find two different closed algebras compatible with it, by regarding Y as a submanifold either of $X = \mathbb{C}P^4$ or of $X' = \mathbb{H}P^2$.

Unfortunately we do not know how to describe the *category* of boundary conditions for $\mathcal{C} = H^*(X)$. But it seems likely, in any case, that to get a significant result one would have to consider the theory on the cochain level. We next turn our attention to that case.

6.2. The Chas-Sullivan theory

There is an interesting example — due to Chas and Sullivan [43]— on the cochain level of a structure a little weaker than that of our open and closed theories which may illuminate the use of cochain theories. Let us start with a compact oriented manifold X, which we shall take to be connected and simply connected. We can define a category $\mathcal{B}$ whose objects are the oriented submanifolds of X, and whose vector space of morphisms from Y to Z is $\mathcal{O}_{YZ} = \mathrm{Ext}^*_{H^*(X)}(H^*(Y);H^*(Z))$ — the cohomology, as usual, has complex coefficients, and $H^*(Y)$ and $H^*(Z)$ are regarded as $H^*(X)$-modules by restriction. The composition of morphisms is given by the Yoneda composition of Ext groups. With this definition, however, it will not be true that $\mathcal{O}_{YZ}$ is dual to $\mathcal{O}_{ZY}$. (To see this it is enough to consider $\mathrm{Ext}^0 = \mathrm{Hom}$, when, say, $Y = X$ and Z is a point.)

We can do better by defining a cochain complex $\hat{\mathcal{O}}_{YZ}$ of "morphisms" by

$$\hat{\mathcal{O}}_{YZ} = \mathcal{B}_{\Omega(X)}(\Omega(Y);\Omega(Z)), \tag{6.8}$$

where $\Omega(X)$ denotes the usual de Rham complex of a manifold X, and $\mathcal{B}_A(B;C)$, for a differential graded algebra A and differential graded modules B and C, is the usual cobar resolution

$$\mathrm{Hom}(B;C) \to \mathrm{Hom}(A \otimes B;C) \to \mathrm{Hom}(A \otimes A \otimes B;C) \to \ldots, \tag{6.9}$$

(in which the differential is given by

$$
\begin{aligned}
df(a_1 \otimes \ldots \otimes a_k \otimes b) = a_1 f(a_2 \otimes \ldots \otimes a_k \otimes b) + \\
+ \sum (-1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_k \otimes b) + \\
+ (-1)^k f(a_1 \otimes \ldots \otimes a_{k-1} \otimes a_k b) \quad)
\end{aligned}
\tag{6.10}
$$

whose cohomology is $\mathrm{Ext}_A(B;C)$. This is different from $\mathcal{O}_{YZ} = \mathrm{Ext}^*_{H^*(X)}(H^*(Y);H^*(Z))$, but related to it by a spectral sequence whose E_2-term is $\mathcal{O}_{YZ}$ and which converges to $H^*(\hat{\mathcal{O}}_{YZ}) = \mathrm{Ext}_{\Omega(X)}(\Omega(Y);\Omega(Z))$. But more important is that $H^*(\hat{\mathcal{O}}_{YZ})$ is the homology of the space $\mathcal{P}_{YZ}$ of paths in X which begin in Y and end in Z. To be precise, $H^p(\hat{\mathcal{O}}_{YZ}) \cong H_{p+d_Z}(\mathcal{P}_{YZ})$, where d_Z is the dimension of Z. On the cochain complexes the Yoneda composition is associative up to cochain homotopy, and defines a structure of an A_∞-category $\hat{\mathcal{B}}$. The corresponding composition of homology groups

$$
H_i(\mathcal{P}_{YZ}) \times H_j(\mathcal{P}_{ZW}) \to H_{i+j-d_Z}(\mathcal{P}_{YW}) \tag{6.11}
$$

is the composition of the Gysin map associated to the inclusion of the codimension d_Z submanifold $\mathcal{M}$ of pairs of composable paths in the product $\mathcal{P}_{YZ} \times \mathcal{P}_{ZW}$ with the concatenation map $\mathcal{M} \to \mathcal{P}_{YW}$.

Let us try to fit a "closed string" cochain algebra $\mathcal{C}$ to this A_∞ category. The algebra of endomorphisms of the identity functor of $\mathcal{B}$, denoted $\mathcal{E}$ in §3, is easily seen to be just the cohomology algebra $H^*(X)$. We have mentioned in Section 2 that this is the Hochschild cohomology $HH^0(\mathcal{B})$.

The definition of Hochschild cohomology for a linear category $\mathcal{B}$ was given at the end of §3. In fact the definition of the Hochschild complex makes sense for an A_∞ category such as $\hat{\mathcal{B}}$, and it is one candidate for the closed algebra $\mathcal{C}$.

In the present situation $\mathcal{C}$ is equivalent to the usual Hochschild complex of the differential graded algebra $\Omega(X)$, whose cohomology is the homology of the free loop space $\mathcal{L}X$ with its degrees shifted downwards[15] by the dimension d_X of X, so that the cohomology $H^i(\mathcal{C})$ is potentially non-zero for $-d_X \leq i < \infty$. This algebra was introduced by Chas and Sullivan in precisely the present context — they were trying to reproduce the structures of string theory in the setting of classical algebraic topology. There is a map $H^i(X) \to H^{-i}(\mathcal{C})$ which embeds the ordinary cohomology ring of X in the Chas-Sullivan

[15] Thus the identity element of the algebra, in $H^0(\mathcal{C})$, is the fundamental class of X, regarded as an element of $H_n(\mathcal{L}X)$ by thinking of the points of X as point loops in $\mathcal{L}X$.

ring, and there is also a ring homorphism $H^i(\mathcal{C}) \to H_i(\mathcal{L}_0 X)$ to the Pontrjagin ring of the based loop space $\mathcal{L}_0 X$, based at any chosen point in X.

The other candidate for $\mathcal{C}$ mentioned in Section 2 was the cyclic cohomology of the algebra $\Omega(X)$, which is well-known [44] to be the *equivariant* homology of the free loop space $\mathcal{L}X$ with respect to its natural circle-action. This may be an improvement on the non-equivariant homology.

The structure we have arrived at is, however, *not* a cochain-level open and closed theory, as we have no trace maps inducing inner products on $H^*(\hat{\mathcal{O}}_{YZ})$. When one tries to define operators corresponding to cobordisms it turns out to be possible only when each connected component of the cobordism has non-empty outgoing boundary. (A theory defined on this smaller category is often called a *non-compact* theory.) The nearest theory in our sense to the Chas-Sullivan one is the so-called "A-model" defined for a symplectic manifold X. There the A_∞ category is the *Fukaya category*, whose objects are the Lagrangian submanifolds of X equipped with bundles with connection, and the cochain complex of morphisms from Y to Z is the Floer complex which calculates the "semi-infinite" cohomology of the path space $\mathcal{P}_{YZ}$. In good cases the cohomology of this Floer complex has a vector space basis indexed by the points of intersection of Y and Z, and the cohomology of the corresponding closed complex is just the ordinary cohomology of X. From our perspective the essential feature of the Floer theory is that it satisfies Poincaré duality for the infinite dimensional manifold $\mathcal{L}X$.

6.3. Remarks on the B-model

Let X be a complex variety of complex dimension d with a trivialization of its canonical bundle. That is, we assume there is a nowhere-vanishing holomorphic d-form Ω. The *B-model* [45] is a $\mathbb{Z}$-graded topological field theory arising from the N=2 supersymmetric σ-model of X. The natural boundary conditions for the theory are provided by holomorphic vector bundles on X.

The category of holomorphic vector bundles is not a Frobenius category. There is, however, a very natural $\mathbb{Z}$-graded Frobenius category associated to X: the category $\mathcal{V}_X$ whose objects are the vector bundles on X, but whose space of morphisms from E to F is

$$\mathcal{O}_{EF} = \mathrm{Ext}^*_X(E; F) = H^{0,*}(X; E^* \otimes F). \tag{6.12}$$

The trace $\theta_E : \mathcal{O}_{EE} \to \mathbb{C}$, of degree $-d$, is defined by

$$\theta_E(\Psi) = \int_X \mathrm{Tr}(\Psi) \wedge \Omega. \tag{6.13}$$

This is nondegenerate by Serre duality, but the category is still not semisimple — in fact the non-vanishing of groups Ext^i for $i > 0$ precisely expresses the non-semisimplicity of the category. (For a non-zero element of $\mathrm{Ext}^1_X(E;F)$ corresponds to an exact sequence $0 \to F \to G \to E \to 0$ which does not split, i.e. to a vector bundle G with a subbundle F with no complementary bundle.)

What are the endomorphisms of the identity functor of $\mathcal{V}_X$? Multiplication by any element of $H^{0,*}(X)$ clearly defines such an endomorphism. A holomorphic vector field ξ on X also defines an endomorphism of degree 1, for any bundle E has an "Atiyah class"[16] $a_E \in \mathrm{Ext}^1(E; E \otimes T_X^*)$ — its curvature — which we can contract with ξ to give $e_\xi = \iota_\xi a_E \in \mathrm{Ext}^1(E;E)$. More generally, a class

$$\eta \in H^{0,q}(X; \bigwedge^p T_X) = \mathrm{Ext}^q(\bigwedge^p T_X^*; \mathbb{C})$$

can be contracted with $(a_E)^p \in \mathrm{Ext}^p(E; E \otimes (T_X^*)^{\otimes p})$ to give

$$e_\eta = \iota_\eta (a_E)^p \in \mathrm{Ext}^{p+q}(E;E).$$

Now Witten has shown in [45] that $H^{0,*}(X; \bigwedge^* T_X)$ is indeed the closed string algebra of the B-model. To understand this in our context we must once again pass to the cochain-level theory of which the Ext groups are the cohomology. A good way to do this is to replace a holomorphic vector bundle E by its $\overline{\partial}$-complex $\hat{E} = \Omega^{0,*}(X;E)$, which is a differential graded module for the differential graded algebra $A = \Omega^{0,*}(X)$. Then we define $\hat{\mathcal{O}}_{EF}$ as the cochain complex $\mathrm{Hom}_A(\hat{E};\hat{F})$, whose cohomology groups are $\mathrm{Ext}^*_X(E;F)$. If we are going to do this, it is natural to allow a larger class of objects, namely all finitely generated projective differential graded A-modules. Any coherent sheaf E on X defines such a module: one first resolves E by a complex E^* of vector bundles, and then takes the total complex of the double complex $\hat{E}^*$. The resulting enlarged category is essentially the *bounded derived category* of the category of coherent sheaves on X. In this setting, we find without difficulty that the endomorphisms of the identity morphism are given, just as in the topological example above, by the Hochschild complex

$$\hat{\mathcal{C}} = \{A \to A \otimes A \to A \otimes A \otimes A \to \dots\},$$

whose cohomology is $H^*(X; \bigwedge^* T_X)$. There is still, however, work to do to understand the trace maps on $\hat{\mathcal{C}}$, and the adjoint maps ι_E and ι^E. We feel that this has not yet been properly elucidated in the literature. For some progress on this question see [46][47].

[16] Corresponding to the extension of bundles $E \otimes T_X^* \to J^1 E \to E$, where $J^1 E$ is the bundle of 1-jets of holomorphic sections of E.

7. Equivariant 2-dimensional topological open and closed theory

An important construction in string theory is the "orbifold" construction. Abstractly, this can be carried out whenever the closed string background has a group G of automorphisms. There are two steps in defining an orbifold theory. First, one must extend the theory by introducing "external" gauge fields, which are G-bundles (with connection) on the world-sheets. Next, one must construct a new theory by summing over all possible G-bundles (and connections).

We begin by describing carefully the first step in forming the orbifold theory. The second step — summing over the G-bundles — is then very easy in the case of a finite group G.

7.1. Equivariant closed theories

Let us begin with some general remarks. In d-dimensional topological field theory one begins with a category $\mathcal{S}$ whose objects are oriented $(d-1)$-manifolds and whose morphisms are oriented cobordisms. Physicists say that a theory admits a group G as a *global symmetry group* if G acts on the vector space associated to each $(d-1)$-manifold, and the linear operator associated to each cobordism is a G-equivariant map. When we have such a "global" symmetry group G we can ask whether the symmetry can be "gauged", i.e. whether elements of G can be applied "independently" — in some sense — at each point of space-time. Mathematically the process of "gauging" has a very elegant description: it amounts to extending the field theory functor from the category $\mathcal{S}$ to the category $\mathcal{S}_G$ whose objects are $(d-1)$-manifolds equipped with a principal G bundle, and whose morphisms are cobordisms with a G-bundle.[17] We regard $\mathcal{S}$ as a subcategory of $\mathcal{S}_G$ by equipping each $(d-1)$-manifold S with the trivial G-bundle $S \times G$. In $\mathcal{S}_G$ the group of automorphisms of the trivial bundle $S \times G$ contains G, and so in a gauged theory G acts on the state space $\mathcal{H}(S)$: this should be the original "global" action of G. But the gauged theory has a state space $\mathcal{H}(S, P)$ for each G-bundle P on S: if P is non-trivial one calls $\mathcal{H}(S, P)$ a "twisted sector" of the theory. In the case $d = 2$, when $S = S^1$ we have the bundle $P_g \to S^1$ obtained by attaching the ends of $[0, 2\pi] \times G$ via multiplication by g. Any bundle is isomorphic to one of these, and P_g is isomorphic to $P_{g'}$ if and only if g' is conjugate to g. But note that the state space depends on the bundle and *not* just its

[17] We are assuming here that the group G is *discrete*: if G is a Lie group we should define $\mathcal{S}_G$ as the category of manifolds equipped with a principal G-bundle with a connection.

isomorphism class, so we have a twisted sector state space $\mathcal{C}_g = \mathcal{H}(S, P_g)$ labelled by a group element g rather than by a conjugacy class.

We shall call a theory defined on the category $\mathcal{S}_G$ a G-*equivariant* TFT. It is important to distinguish the equivariant theory from the corresponding "gauged theory," described below. In physics, the equivariant theory is obtained by coupling to nondynamical background gauge fields, while the gauged theory is obtained by "summing" over those gauge fields in the path integral.

An alternative and equivalent viewpoint which is especially useful in the two-dimensional case is that $\mathcal{S}_G$ is the category whose objects are oriented $(d-1)$-manifolds S equipped with a map $p : S \to BG$, where BG is the classifying space of G. In this viewpoint we have a bundle over the space $\mathrm{Map}(S, BG)$ whose fiber at p is $\mathcal{H}_p$. To say that $\mathcal{H}_p$ depends only on the G-bundle p^*BG on S pulled back from the universal G-bundle EG on BG by p is the same as to say that the bundle on $\mathrm{Map}(S, BG)$ is equipped with a flat connection allowing us to identify the fibres at points in the same connected component by parallel transport; for the set of bundle isomorphisms $p_0^*EG \to p_1^*EG$ is the same as the set of homotopy classes of paths from p_0 to p_1. When $S = S^1$ the connected components of the space of maps correspond to the conjugacy classes in G: each bundle P_g corresponds to a specific point p_g in the mapping space, and a group-element h defines a specific path from p_g to $p_{hgh^{-1}}$.

The second viewpoint makes clear that G-equivariant topological field theories are examples of "homotopy topological field theories" in the sense of Turaev [48]. We shall use his two main results: first, an attractive generalization of the theorem that a two-dimensional TFT "is" a commutative Frobenius algebra, and, secondly, a classification of the ways of gauging a given global G-symmetry of a semisimple TFT. We shall now briefly review his work .

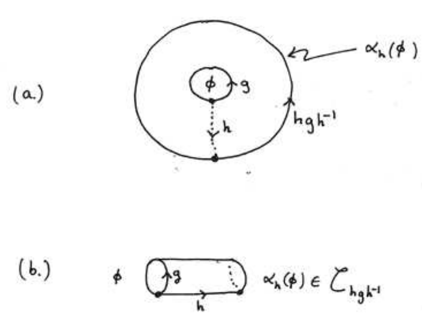

Fig. 16: (a) The action of α_h on a state $\phi \in \mathcal{C}_g$. This can also be represented by the cylinder as in (b).

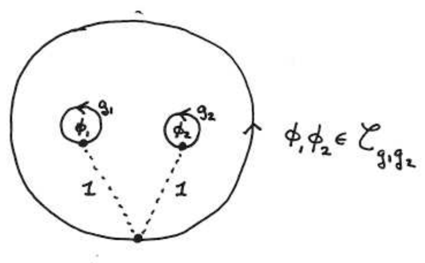

Fig. 15: Definition of the product in the G-equivariant closed theory. The heavy dot is the basepoint on S^1. To specify the morphism unambiguously we must indicate consistent holonomies along a set of curves whose complement consists of simply connected pieces. This means that the product is not commutative. We need to fix a convention for holonomies of a composition of curves, i.e. whether we are using left or right path-ordering. We will take $h(\gamma_1 \circ \gamma_2) = h(\gamma_1) \cdot h(\gamma_2)$.

A G-equivariant TFT gives us for each element $g \in G$ a vector space $\mathcal{C}_g$, associated to the circle equipped with the bundle P_g whose holonomy is g. The usual pair-of-pants cobordism, equipped with the evident G-bundle which restricts to P_{g_1} and P_{g_2} on the two incoming circles, and to $P_{g_1 g_2}$ on the outgoing circle, induces a product

$$\mathcal{C}_{g_1} \otimes \mathcal{C}_{g_2} \to \mathcal{C}_{g_1 g_2} \tag{7.1}$$

making $\mathcal{C} := \oplus_{g\in G}\mathcal{C}_g$ into a G-graded algebra.

As in the usual case there is a trace $\theta : \mathcal{C}_1 \to \mathbb{C}$ defined by the disk diagram with one ingoing circle. Note that the holonomy around the boundary of the disk must be 1. Making the standard assumption that the cylinder corresponds to the unit operator we obtain a nondegenerate pairing $\mathcal{C}_g \otimes \mathcal{C}_{g^{-1}} \to \mathbb{C}$.

A new element in the equivariant theory is that G acts as an automorphism group on $\mathcal{C}$. That is, there is a a homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{C})$ such that

$$\alpha_h : \mathcal{C}_g \to \mathcal{C}_{hgh^{-1}}. \tag{7.2}$$

Diagramatically, α_h is defined by the surface in fig. 16.

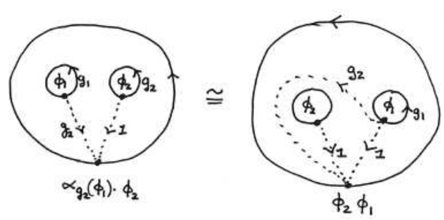

Fig. 18: Demonstrating twisted centrality.

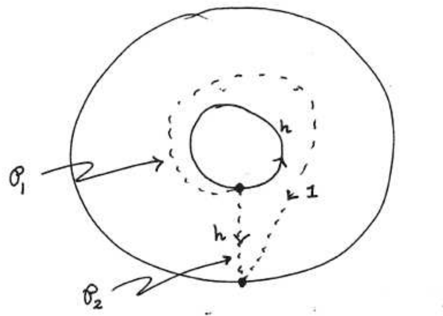

Fig. 17: If the holonomy along path $\mathcal{P}_2$ is h then the holonomy along path $\mathcal{P}_1$ is 1. However, a Dehn twist around the inner circle maps $\mathcal{P}_1$ into $\mathcal{P}_2$. Therefore, $\alpha_h(\phi) = \alpha_1(\phi) = \phi$, if $\phi \in \mathcal{C}_h$.

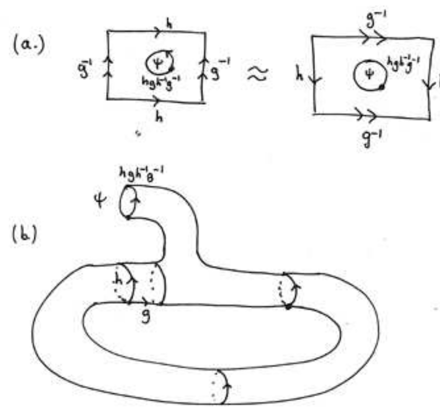

Fig. 19: Deforming the LHS of (a) into a spacetime evolution diagram yields (b), whose value is $\mathrm{Tr}_{\mathcal{C}_h}(L_\psi \alpha_g)$. Similarly deforming the RHS of (a) gives a diagram whose value is $\mathrm{Tr}_{\mathcal{C}_{g^{-1}}}(R_\psi \alpha_h)$.

Now let us note some properties of α. First, if $\phi \in \mathcal{C}_h$ then $\alpha_h(\phi) = \phi$. The reason for this is explained in fig. 17.

Next, while $\mathcal{C}$ is not commutative, it is "twisted-commutative" in the following sense. If $\phi_1 \in \mathcal{C}_{g_1}$ and $\phi_2 \in \mathcal{C}_{g_2}$ then

$$\alpha_{g_2}(\phi_1)\phi_2 = \phi_2\phi_1. \tag{7.3}$$

The necessity of this condition is illustrated in fig. 18.

The last property we need is a little more complicated. The trace of the identity map of $\mathcal{C}_g$ is the partition function of the theory on a torus with the bundle with holonomy $(g, 1)$. Cutting the torus the other way, we see that this is the trace of α_g on $\mathcal{C}_1$. Similarly, by considering the torus with a bundle with holonomy (g, h), where g and h are two commuting elements of G, we see that the trace of α_g on $\mathcal{C}_h$ is the trace of α_h on $\mathcal{C}_{g^{-1}}$. But we need a strengthening of this property. Even when g and h do not commute we can form a bundle with holonomy (g, h) on a torus with one hole, around which the holonomy will be $c = hgh^{-1}g^{-1}$. We can cut this torus along either of its generating circles to get a cobordism operator from $\mathcal{C}_c \otimes \mathcal{C}_h$ to $\mathcal{C}_h$ or from $\mathcal{C}_{g^{-1}} \otimes \mathcal{C}_c$ to $\mathcal{C}_{g^{-1}}$. If $\psi \in \mathcal{C}_{hgh^{-1}g^{-1}}$ let us introduce two linear transformations L_ψ, R_ψ associated to left- and right-multiplication by ψ. On the one hand, $L_\psi \alpha_g : \phi \mapsto \psi\alpha_g(\phi)$ is a map $\mathcal{C}_h \to \mathcal{C}_h$. On the other hand $R_\psi \alpha_h : \phi \mapsto \alpha_h(\phi)\psi$ is a map $\mathcal{C}_{g^{-1}} \to \mathcal{C}_{g^{-1}}$. The last sewing condition states that these two endomorphisms must have equal traces:

$$\mathrm{Tr}_{\mathcal{C}_h}\Big(L_\psi \alpha_g\Big) = \mathrm{Tr}_{\mathcal{C}_{g^{-1}}}\Big(R_\psi \alpha_h\Big). \tag{7.4}$$

The reason for this can be deduced by pondering the diagram in fig. 19.

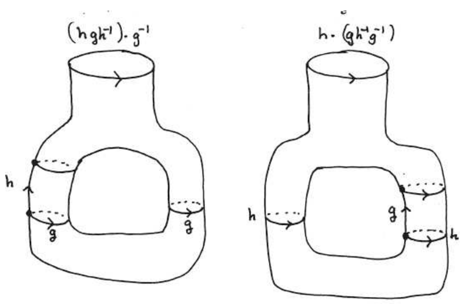

Fig. 20: A simpler axiom than Turaev's torus axiom.

The equation ((7.4)) was taken by Turaev as one of his axioms. It can, however, be reexpressed in a way that we shall find more convenient. Let $\Delta_g \in \mathcal{C}_g \otimes \mathcal{C}_{g^{-1}}$ be the "duality" element corresponding to the identity cobordism of (S^1, P_g) with both ends regarded as outgoing. We have $\Delta_g = \sum \xi_i \otimes \xi^i$, where ξ_i and ξ^i run through dual bases of $\mathcal{C}_g$ and $\mathcal{C}_{g^{-1}}$. Let us also write

$$\Delta_h = \sum \eta_i \otimes \eta^i \in \mathcal{C}_h \otimes \mathcal{C}_{h^{-1}}.$$

Then ((7.4)) is easily seen to be equivalent to

$$\sum \alpha_h(\xi_i)\xi^i = \sum \eta_i \alpha_g(\eta^i), \tag{7.5}$$

in which both sides are elements of $\mathcal{C}_{hgh^{-1}g^{-1}}$. This equation is illustrated by the isomorphic cobordisms of fig. 20.

In summary, the sewing theorem for G-equivariant 2d topological field theories is given by the following theorem:

Theorem 4 ([48]) To give a 2d G-equivariant topological field theory is to give a G-graded algebra $\mathcal{C} = \oplus_g \mathcal{C}_g$ together with a group homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{C})$ such that

1. There is a G-invariant trace $\theta : \mathcal{C}_1 \to \mathbb{C}$ which induces a nondegenerate pairing $\mathcal{C}_g \otimes \mathcal{C}_{g^{-1}} \to \mathbb{C}$.
2. The restriction of α_h to $\mathcal{C}_h$ is the identity.
3. For all $\phi \in \mathcal{C}_g, \phi' \in \mathcal{C}_h$, $\alpha_h(\phi)\phi' = \phi'\phi$.
4. For all $g, h \in G$ we have

$$\sum \alpha_h(\xi_i)\xi^i = \sum \eta_i \alpha_g(\eta^i) \in \mathcal{C}_{hgh^{-1}g^{-1}}, \tag{7.6}$$

where $\Delta_g = \sum \xi_i \otimes \xi^i \in \mathcal{C}_g \otimes \mathcal{C}_{g^{-1}}$ and $\Delta_h = \sum \eta_i \otimes \eta^i \in \mathcal{C}_h \otimes \mathcal{C}_{h^{-1}}$ as above.

Remarks:

1. We will give a proof of the sewing theorem in the appendix.
2. Warning: Turaev calls the above a *crossed G Frobenius algebra*, but it is not a crossed-product algebra in the sense of C^* algebras (see below). We will refer to an algebra satisfying the conditions of the theorem as a *Turaev algebra*.
3. Axioms 1 and 3 have counterparts in the non-equivariant theory, but axioms 2 and 4 are new elements.

7.2. The orbifold theory

Before going any further, let us describe how we obtain the orbifold theory from the Turaev algebra.

Let us return to the general discussion at the beginning of §7.1, where we outlined the definition of an equivariant theory. Roughly speaking, the gauged theory is obtained from the equivariant theory by summing over the gauge fields. More precisely, the state-space which a gauged theory associates to a $(d-1)$-manifold S consists of "wave-functions" ψ which associate to each G-bundle P on S an element ψ_P of the state-space $\mathcal{H}_{S,P}$ of the equivariant theory. The map ψ must be "natural" in the sense that when $\theta : P \to P'$ is a bundle isomorphism the induced isomorphism $\mathcal{H}_{S,P} \to \mathcal{H}_{S,P'}$ takes ψ_P to $\psi_{P'}$. This is often referred to as the "Gauss law." In the two-dimensional case, the Gauss law amounts to saying that the state-space $\mathcal{C}_{\text{orb}}$ for the circle is the G-invariant part of the Turaev algebra $\mathcal{C} = \oplus \mathcal{C}_g$. In other words,

$$\mathcal{C}_{\text{orb}} = \oplus \{\mathcal{C}_g\}^{Z_g}, \tag{7.7}$$

where now g runs through a set of representatives for the conjugacy classes in G, and we take the invariant part of $\mathcal{C}_g$ under the centralizer Z_g of g in G. The algebra $\mathcal{C}_{\text{orb}}$ is not a graded algebra if G is non-abelian. One must check that the product in $\mathcal{C}_{\text{orb}}$ is simply the restriction of the product in $\mathcal{C}$. The trace $\mathcal{C}_{\text{orb}} \to \mathbb{C}$ is the restriction of the trace $\mathcal{C} \to \mathbb{C}$ which is the given trace on $\mathcal{C}_1$ and is zero on $\mathcal{C}_g$ when $g \neq 1$. Then $\mathcal{C}_{\text{orb}}$ is a commutative Frobenius algebra which encodes the orbifold theory.

7.3. Solutions of the closed string G-equivariant sewing conditions

Having found the sewing conditions in the G-equivariant case we can ask what examples there are of the structure. The Frobenius algebra $\mathcal{C}_1$ with its G-action corresponds to a topological field theory with a global G-symmetry. In the case when $\mathcal{C}_1$ is a semisimple Frobenius algebra — and therefore the algebra of functions on a finite G-set X — Turaev finds a nice answer: ways of gauging the symmetry, i.e. of extending $\mathcal{C}_1$ to a Turaev algebra, correspond to *equivariant B-fields* on X, i.e. to equivariant 2-cocycles of X with values in $\mathbb{C}^\times$, and two such B-fields define isomorphic Turaev algebras if and only if they represent the same class in $H^2_G(X;\mathbb{C}^\times) \cong H^3_G(X;\mathbb{Z})$. We now review this result and take the opportunity to introduce a more geometric picture of Turaev's algebra $\mathcal{C}$ (in the semisimple case). We shall first recall some very general constructions.

7.3.1. *General constructions*

Whenever a group G acts on a set X we can form a category $X//G$, whose objects are the points x of X, and whose morphisms $x_0 \to x_1$ are

$$\text{Hom}(x_0, x_1) := \{g \in G : gx_0 = x_1\}. \tag{7.8}$$

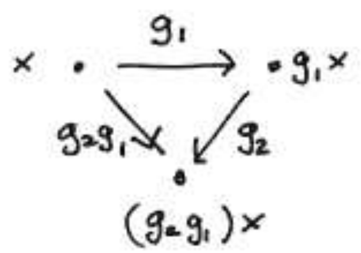

Fig. 21: An oriented two-simplex Δ_{x,g_1,g_2} in the space $|X//G|$.

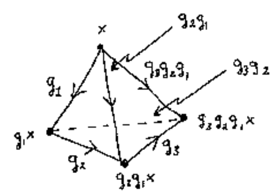

Fig. 22: An oriented 3-simplex in $|X//G|$.

Next, for any category $\mathcal{C}$, one can form the *space of the category*, denoted $|\mathcal{C}|$. This is an oriented simplicial complex whose p-simplexes are in 1-1 correspondence with the composable p-tuples of morphisms in the category. To be specific, the vertices are the objects of the category. The edges are the morphisms. Triples of morphisms (f_1, f_2, f_3) with $f_3 = f_2 \circ f_1$ correspond to 2-simplices, and so forth. In the present case, when we form the simplicial complex $|X//G|$ the 2-simplices are the triples (g_1, g_2, x) illustrated in fig. 21. Three-simplices are shown in fig. 22, etc.

The space $|X//G|$ is a model for $(X \times EG)/G$. Hence the (cellular) cohomology of this space $H^*(|X//G|;\mathbb{C}^\times)$ is the equivariant cohomology $H^*_G(X;\mathbb{C}^\times)$.

Another object which we can associate to any category C is its *algebra* $A(C)$ over the field $\mathbb{C}$. This has a vector space basis $\{\varepsilon_f\}$ indexed by the morphisms of C, and the product is given by $\varepsilon_{f_1}\varepsilon_{f_2} = \varepsilon_{f_1 \circ f_2}$ when f_1 and f_2 are composable, and $\varepsilon_{f_1}\varepsilon_{f_2} = 0$ otherwise. For the category $X//G$ the algebra $A(X//G)$ is the usual *crossed-product* algebra $A(X) \ltimes G$ in the sense of operator algebra theory, where $A(X)$ is the algebra of complex-valued functions on the set X. [18]

The construction of the category-algebra $A(C)$ can be generalized. A *B-field* on a category C is a rule which associates a complex line L_f to each morphism f of C, and associative isomorphisms

$$L_{f_1} \otimes L_{f_2} \to L_{f_1 \circ f_2}$$

to each pair (f_1, f_2) of composable morphisms . In concrete terms, to give such a product is to give a 2-cocycle on the space $|C|$. Indeed, choosing basis elements $\ell_f \in L_f$, we must have

$$\ell_{f_1} \cdot \ell_{f_2} = b(f_1, f_2, f_3)\ell_{f_3} \tag{7.10}$$

where $b(f_1, f_2, f_3) \in \mathbb{C}^\times$ defines a 2-cochain on $|C|$. (We choose values in $\mathbb{C}^\times$ rather than $\mathbb{C}$ so the product is not degenerate.) Associativity of (7.10) holds iff b is a 2-cocycle. A change of basis of the L_f modifies b by a coboundary. Hence the isomorphism classes of B-fields on C are in 1-1 correspondence with cohomology classes $[b] \in H^2(|C|;\mathbb{C}^\times)$. When we have a B-field b on C we can form a *twisted* category-algebra $A_b(C)$, which as a vector space is $\oplus L_f$, and where the multiplication is defined by means of the associative maps $L_{f_1} \otimes L_{f_2} \to L_{f_1 \circ f_2}$.

Applying the above construction to the category $X//G$, an associative product on the lines $L_{g,x}$ is the same thing as a 2-cocycle in $H^2_G(X;\mathbb{C}^\times)$. In terms of the basis elements $\ell_{g,x}$ for the lines $L_{g,x}$ we shall write the multiplication

$$\begin{aligned} \ell_{g_2,x_2}\ell_{g_1,x_1} &= b_{x_1}(g_2, g_1)\ell_{g_2 g_1, x_1} & \text{if} \quad & x_2 = g_1 x_1 \\ &= 0 & & \text{otherwise} \end{aligned} \tag{7.11}$$

[18] For any commutative algebra A with G-action, $A \ltimes G$ is spanned by elements $a \otimes g$ with $a \in A$ and $g \in G$, and the product is given by

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = a_1 g_1(a_2) \otimes g_1 g_2. \tag{7.9}$$

The isomorphism $A(X//G) \to A(X) \ltimes G$ takes $\varepsilon_{g,x}$ to $\chi_{gx} \otimes g$, where χ_x is the characteristic function supported at x.

Here $b_{x_1}(g_2, g_1) = b(\Delta_{x,g_1,g_2})$ is the value of the cocycle on the oriented 2-simplex of fig. 21.

Notice that if G_x is the isotropy group of some point $x \in X$ then restricting (7.11) to the elements $\ell_{g,x}$ with $g \in G_x$ shows that b_x defines an element of the group cohomology $H^2(G_x;\mathbb{C}^\times)$, corresponding to the central extension of G_x by $\mathbb{C}^\times$ whose elements are pairs (g, λ) with $g \in G_x$ and $\lambda \in L_x - \{0\}$. This central extension of the isotropy group G_x does not in general extend to any central extension of the whole group G. It does so, however, in the particular case when the B-field b is pulled back from a 2-cocycle of G by the map $X \to$ (point), i.e. when $b_x(g_2, g_1)$ is independent of x. In general the cocycle $b : G \times G \times X \to \mathbb{C}^\times$ can be regarded as a cocycle of the group G with values in the abelian group $A(X)^\times = \mathrm{Map}(X;\mathbb{C}^\times)$ with its natural G-action. Thus it defines a (non-central) extension

$$1 \to A(X)^\times \to \tilde{G} \to G \to 1.$$

One technical point to notice is that for any B-field we have $L_f = \mathbb{C}$ canonically when f is an identity morphism. Thus $L_{g,x} = \mathbb{C}$ when $g = 1$. We shall always choose $\ell_{g,x} = 1$ when $g = 1$, thereby normalizing the cocycle so that $b_x(g_1, g_2) = 1$ if either g_1 or g_2 is 1.

The algebra $A_b(X//G) = \oplus_{g\in G,x\in X} L_{g,x}$ with the multiplication rule defined by (7.11) can be identified with the twisted crossed-product algebra $A(X) \ltimes_b G$ via

$$\ell_{g,x} \mapsto \chi_{gx} \otimes g,$$

where χ_x is the characteristic function supported at x. The twisted cross-product is defined by

$$(f_1 \otimes g_1)(f_2 \otimes g_2) = \alpha_{g_1 g_2}(b(g_1, g_2)) f_1\, \alpha_{g_1}(f_2) \otimes g_1 g_2, \tag{7.12}$$

where $b(g_1, g_2)$ denotes the function $x \mapsto b_x(g_1, g_2)$ in $A(X)^\times$, and the group G acts on $A(X)$ in the natural way

$$\alpha_g(f)(x) = f(g^{-1}x),$$

so that $g \cdot \chi_x = \chi_{gx}$.

If we wish to apply these considerations to the spin case described in sections 2.6 and 3.4 then we must consider the lines L_f to be $\mathbb{Z}/2$ graded. In this case the theory will admit a further twisting by $H^1(|C|;\mathbb{Z}/2)$. However, we will not discuss this generalization further.

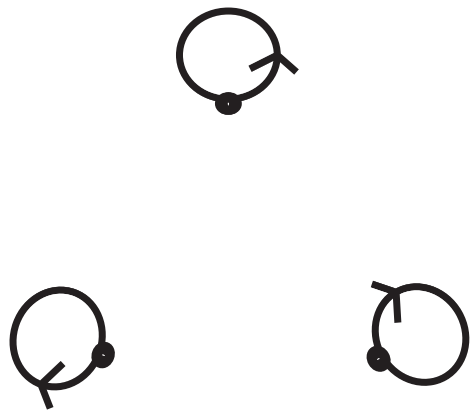

Fig. 23: The algebra of little loops for $X = S_3/S_2$, where S_n is the permutation group on n letters.

7.3.2. *The Turaev algebra associated to a G-space*

The algebra $A_b(X//G)$ does *not* satisfy the sewing conditions and is not a Turaev algebra. In particular (7.3) is usually not satisfied for a crossed-product algebra. However, the subcategory defined by the morphisms with the same initial and terminal object does lead to a Turaev algebra for any B-field b on $X//G$. We call this the "algebra of little loops". Thus we define $\mathcal{C} = \oplus_g \mathcal{C}_g \subset A_b(X//G)$ by

$$\mathcal{C}_g := \oplus_{x:gx=x} L_{g,x} \tag{7.13}$$

and define the trace by

$$\theta(\ell_{g,x}) = \delta_{g,1}\theta(\varepsilon_x) \tag{7.14}$$

where on the right θ is the given G-invariant trace on $\mathcal{C}_1$, and the ε_x are the usual idempotents in the semisimple Frobenius algebra $\mathcal{C}_1 = A(X)$, i.e. $\varepsilon_x = 1 \in L_{1,x} = \mathbb{C}$. The algebra of little loops can be visualized as in fig. 23.

An equivalent way to describe $\mathcal{C}$ is as the commutant of $\mathcal{C}_1 = A(X)$ in $A_b(X//G) = A(X) \ltimes_b G$. As $A(X)$ is in the centre of $\mathcal{C}$, it is natural to think of $\mathcal{C}$ as the sections of a bundle of algebras on X; the fibre of this bundle at $x \in X$ is the twisted group algebra $\mathbb{C}_{b_x}[G_x]$, where G_x is the isotropy group of x. Furthermore, the bundle of algebras has a natural G-action, covering the G-action on X. To see this, notice that the extension $\tilde{G} = \{f, g) : f \in A(X)^\times, g \in G\}$ of the group G by the multiplicative group $A(X)^\times$ defined by the B-field sits inside the multiplicative group of $A(X) \ltimes_b G$, normalizing the subalgebra $A(X)$. As $A(X)$ is in the centre of $\mathcal{C}$, this means that G acts by conjugation on the algebra $\mathcal{C}$. Notice, however, that only $\tilde{G}$ and not G acts on the larger algebra $A_b(X//G)$.

In terms of explicit formulae, the action of G on the algebra $\mathcal{C}$ is given by

$$\alpha_{g_1}(\ell_{g_2,x}) = \ell_{g_1,x}\ell_{g_2,x}\ell_{g_1,x}^{-1} = z_x(g_2,g_1)\ell_{g_1g_2g_1^{-1},g_1x}, \tag{7.15}$$

where

$$z_x(g_2,g_1) = \frac{b_x(g_1,g_2)b_x(g_1g_2,g_1^{-1})}{b_x(g_1,g_1^{-1})}. \tag{7.16}$$

In this way we obtain a Turaev algebra, which we shall denote by $\mathcal{C} = T(X,b,\theta)$. The only non-trivial point is to verify the "torus" axiom (7.4) . But in fact it is easy to see that both sides of the equation are equal to

$$\sum_x \ell_{h,x}\ell_{g,x}\ell_{h,x}^{-1}\ell_{g,x}^{-1},$$

where x runs through the set $\{x \in X : hx = gx = x\}$.

Turaev has shown that the above construction is the most general one possible in the semisimple case.

Theorem 5 ([48], Theorem 3.6) Let $\mathcal{C}$ be a Turaev algebra. If $\mathcal{C}_1$ is semisimple then $\mathcal{C}$ is the twisted algebra $T(X,b,\theta)$ of little loops on $X = \mathrm{Spec}(\mathcal{C}_1)$ for some cocycle $b \in Z^2_G(X;\mathbb{C}^\times)$.

Proof: If $\mathcal{C}_1$ is semisimple we may decompose it in terms of the basic idempotents ε_x. Then $\mathcal{C}_g$ is a module over $\mathcal{C}_1$, and hence it should be identified with the cross sections of the vector bundle over the finite set X whose fibre at x is $\mathcal{C}_{g,x} = \varepsilon_x\mathcal{C}_g$. (This is a trivial case of what is called the Serre-Swan theorem.) Now we consider the torus axiom (7.4) in the case $h = 1$. We have $\Delta_1 = \sum \theta(\varepsilon_x)^{-1}\varepsilon_x \otimes \varepsilon_x$, and hence

$$\sum \theta(\varepsilon_x)^{-1}\alpha_g(\varepsilon_x)\varepsilon_x = \sum \theta(\varepsilon_x)^{-1}\varepsilon_x,$$

where the second sum is over x such that $gx = x$. On the other hand we readily calculate that if $\{a_{x,i}\}$ is a basis of $\mathcal{C}_{g,x}$ and $\{a^*_{x,i}\}$ is the dual basis of $\mathcal{C}_{g^{-1},x}$ then $a_{g,i}a^*_{g,i} = \theta(\varepsilon_x)^{-1}\varepsilon_x$, so that the other side of the torus axiom is

$$\sum \dim(\mathcal{C}_{g,x})\varepsilon_x.$$

Thus the axiom tells us that $\mathcal{C}_{g,x}$ is a one-dimensional space $L_{g,x}$ when $gx = x$, and is zero otherwise. The multiplication in $\mathcal{C}$ makes these lines into a G-equivariant B-field on the category of small loops in $X//G$. Finally, it is not hard to show that the category of B-fields on $X//G$ is equivalent to the category of G-equivariant B-fields on the category of small loops; but we shall omit the details. ♠.

Let us now consider the orbifold theory coming from the gauged theory defined by the Turaev algebra $\mathcal{C} = T(X, b, \theta)$. We saw in Section 7.2 that it is defined by the commutative Frobenius algebra $\mathcal{C}_{\mathrm{orb}}$ which is the G-invariant subalgebra of $\mathcal{C}$. In the case of the Turaev algebra of a G-space X we have

Theorem 6 The orbifold algebra $\mathcal{C}_{\mathrm{orb}}$ is the centre of the crossed-product algebra $A(X) \ltimes_b G$. It is the algebra of functions on a finite set $(X/G)_{\mathrm{string}}$ which is a "thickening" of the orbit space X/G with one point for each pair ξ, ρ consisting of an orbit ξ and an irreducible projective representation ρ of the isotropy group G_x of a point $x \in \xi$, with the projective cocycle b_x defined by the B-field.

Proof The Turaev algebra $\mathcal{C}$ consists of the elements of $A(X) \ltimes_b G$ which commute with $A(X)$. But an element of $A(X) \ltimes_b G$ belongs to its centre if an only if it commutes with $A(X)$ and also commutes with the elements of G, i.e. is G-invariant.

Now we saw that $\mathcal{C}$ is the product over the points $x \in X$ of the twisted group-algebras $\mathbb{C}_{b_x}[G_x]$. The invariant part is therefore the product over the orbits ξ of the G_x-invariant part of $\mathbb{C}_{b_x}[G_x]$, i.e. of the centre of $\mathbb{C}_{b_x}[G_x]$, which consists of one copy of $\mathbb{C}$ for each irreducible representation ρ with the cocycle b_x. ♠

The Turaev algebra $\mathcal{C} = T(X, b, \theta)$ sits between $\mathcal{C}_{\mathrm{orb}}$ and $A(X) \ltimes_b G$. We shall see in Section 7.6 that $A(X) \ltimes_b G$ is semisimple, and hence Morita equivalent[19] to its centre $\mathcal{C}_{\mathrm{orb}}$. But the Turaev algebra retains more information than the orbifold theory: it encodes X and its G-action. The difference is plainest when G — of order n — acts freely on X; then $A(X) \ltimes G$ is the product of a copy of the algebra of $n \times n$ matrices for each G-orbit in X, and provides us with no way of distinguishing the individual points of X. We shall see in

[19] This means that the category of representations of $A(X) \ltimes_b G$ is equivalent to the category of representations of $\mathcal{C}_{\mathrm{orb}}$, uniquely up to tensoring with a "line bundle" — a representation L of $\mathcal{C}_{\mathrm{orb}}$ such that $L \otimes_{\mathcal{C}_{\mathrm{orb}}} L' \cong \mathcal{C}_{\mathrm{orb}}$ for some L'.

section 7.5 that the category of boundary conditions for the gauged theory $\mathcal{C}$ is a natural enrichment of the category for the orbifold theory, at least in the semisimple case.

It might come as a surprise that the cross-product algebra of spacetime $A(X) \ltimes G$ is not the appropriate Frobenius algebra for G-equivariant topological field theory, in view of the occurrence of the crossed-product algebra as a central concept in the theory of D-branes on orbifolds developed in [49][50][51]. In fact, this fits in very nicely with the philosophy of this paper. The Turaev algebra remembers the points of X, and so allows only the "little loops" above. In this way the sewing conditions - which are meant to formalize worldsheet locality - also encode a crude form of *spacetime* locality.

We shall conclude this section by making contact with the usual path integral expression for the orbifold partition function on a torus. To do this we compute $\dim \mathcal{C}_{\rm orb}$ by computing the projection onto G-invariant states in $\mathcal{C}$. Note that $\alpha_g(\ell_{h,x})$ is only proportional to $\ell_{h,x}$ when $[g,h]=1$ and $gx=x$, and then

$$\alpha_g(\ell_{h,x}) = \frac{b_x(g,h)}{b_x(h,g)}\ell_{h,x} \tag{7.17}$$

where we have combined (7.16) with the cocycle identity. Thus we find

$$\dim \mathcal{C}_{\rm orb} = \frac{1}{|G|}\sum_{gh=hg}\sum_{x=gx=hx}\frac{b_x(g,h)}{b_x(h,g)} \tag{7.18}$$

Fig. 24: The wavy line is a constrained boundary. If there is holonomy g along the dotted path $\mathcal{P}$ then this morphism gives the G-action on $\mathcal{O}$.

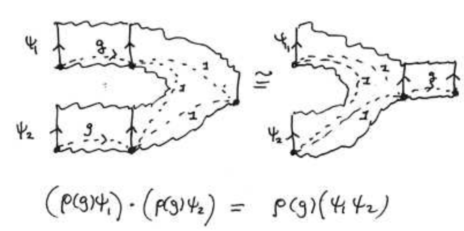

Fig. 26: Showing that G acts on $\mathcal{O}$ as a group of automorphisms.

Fig. 25: The definition of the multiplication in $\mathcal{O}$. The holonomy on all dotted paths is 1. Note the order of multiplication.

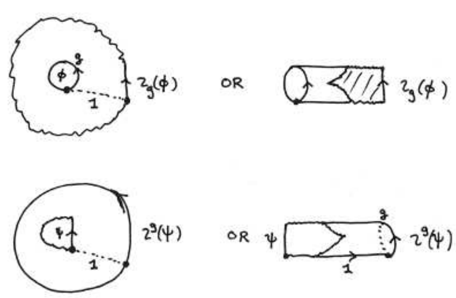

Fig. 27: The open/closed transitions ι_g and ι^g.

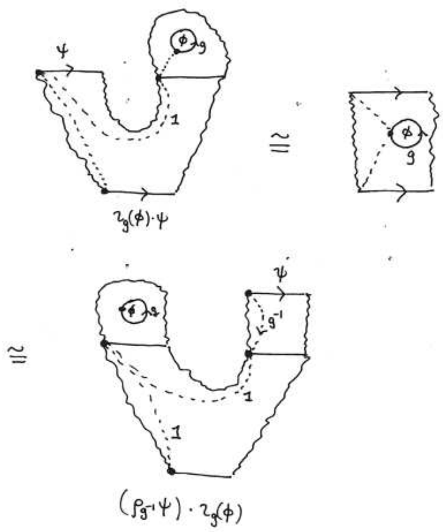

Fig. 28: The G-twisted centrality axiom.

7.4. Sewing conditions for equivariant open and closed theory

Let us now pass on to consider G-equivariant open and closed theories. We enlarge

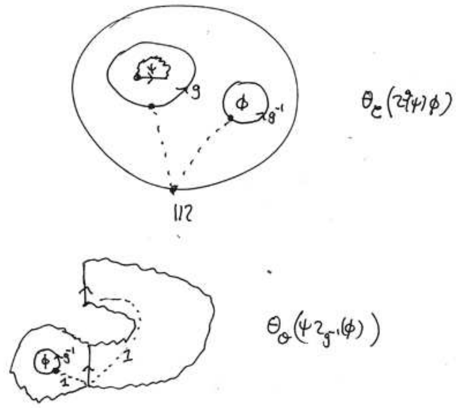

Fig. 29: The G-twisted adjoint relation.

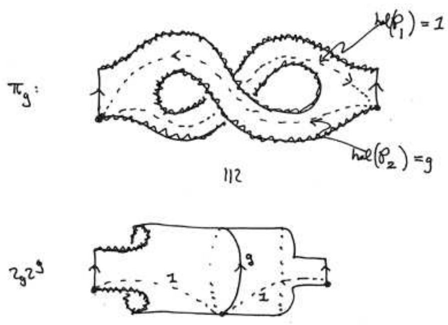

Fig. 30: The G-twisted Cardy condition. In the double-twist diagram the holonomy around $\mathcal{P}_1$ is 1 and the holonomy around $\mathcal{P}_2$ is g.

the category $\mathcal{S}_G$ so that the objects are oriented 1-manifolds w ith boundary, with labelled ends, equipped with principal G-bundles. The morphisms are the same cobordisms as in the non-equivariant case, but equipped with G-bundles. Up to isomorphism there is only one G-bundle on the interval: it is trivial, and admits G as an automorphism group. So an equivariant theory gives us for each pair a, b of labels a vector space $\mathcal{O}_{ab}$ with a G-action. The action of $g \in G$ on $\mathcal{O}_{ab}$ can be regarded as coming from the "square" cobordism with the bundle whose holonomy is g along each of its "constrained" edges. There is also a composition law $\mathcal{O}_{ab} \times \mathcal{O}_{bc} \to \mathcal{O}_{ac}$, which is G-equivariant. These are illustrated in fig. 24 and fig. 25.

In the open/closed case the conditions analogous to equations (2.2) to (2.12) are the following.

Focussing first on a single label a, we have a not necessarily commutative Frobenius algebra $(\mathcal{O}_{aa} = \mathcal{O}, \theta_{\mathcal{O}})$ together with a G-action $\rho : G \to \mathrm{Aut}(\mathcal{O})$:

$$\rho_g(\psi_1\psi_2) = \big(\rho_g\psi_1\big)\big(\rho_g\psi_2\big) \tag{7.19}$$

which preserves the trace $\theta_{\mathcal{O}}(\rho_g\psi) = \theta_{\mathcal{O}}(\psi)$. See fig. 26.

There are also G-twisted open/closed transition maps

$$\begin{aligned} \iota_{g,a} = \iota_g &: \mathcal{C}_g \to \mathcal{O}_{aa} = \mathcal{O} \\ \iota^{g,a} = \iota^g &: \mathcal{O}_{aa} = \mathcal{O} \to \mathcal{C}_g \end{aligned} \tag{7.20}$$

which are equivariant:

$$\begin{array}{ccc} \mathcal{C}_{g_1} & \stackrel{\alpha_{g_2}}{\to} & \mathcal{C}_{g_2 g_1 g_2{}^{-1}} \\ \downarrow \iota_{g_1} & & \downarrow \iota_{g_2 g_1 g_2{}^{-1}} \\ \mathcal{O} & \stackrel{\rho_{g_2}}{\to} & \mathcal{O} \end{array} \tag{7.21}$$

$$\begin{array}{ccc} \mathcal{C}_{g_2{}^{-1} g_1 g_2} & \stackrel{\alpha_{g_2}}{\to} & \mathcal{C}_{g_1} \\ \iota^{g_2{}^{-1} g_1 g_2} \uparrow & & \uparrow \iota^{g_1} \\ \mathcal{O} & \stackrel{\rho_{g_2}}{\to} & \mathcal{O} \end{array} \tag{7.22}$$

These maps are illustrated in fig. 27. The open/closed maps must satisfy the G-twisted versions of conditions 3a-3e of section 2.2. In particular, the map $\iota : \mathcal{C} \to \mathcal{O}$ obtained by putting the ι_g together is a ring homomorphism, i.e.

$$\iota_{g_1}(\phi_1)\iota_{g_2}(\phi_2) = \iota_{g_2 g_1}(\phi_2\phi_1) \qquad \forall \phi_1 \in \mathcal{C}_{g_1}, \phi_2 \in \mathcal{C}_{g_2}. \tag{7.23}$$

Since the identity is in $\mathcal{C}_1$ the condition (2.8) is unchanged. The G-twisted centrality condition is

$$\iota_g(\phi)(\rho_g\psi)) = \psi\iota_g(\phi) \qquad \forall \phi \in \mathcal{C}_g, \psi \in \mathcal{O}. \tag{7.24}$$

and is illustrated in fig. 28.

The G-twisted adjoint condition is

$$\theta_{\mathcal{O}}\big(\psi\iota_{g^{-1}}(\phi)\big) = \theta_{\mathcal{C}}\big(\iota^g(\psi)\phi)\big) \qquad \forall \phi \in \mathcal{C}_{g^{-1}}. \tag{7.25}$$

and is shown in fig. 29.

Finally, the G-twisted Cardy conditions restrict not only each algebra $\mathcal{O}_{aa}$, but also the spaces of morphisms $\mathcal{O}_{ab}$ between labels b and a. For each $g \in G$ we must have

$$\pi_{g,b}{}^{a} = \iota_{g,b}\iota^{g,a} \tag{7.26}$$

Here ${\pi_{g,b}}^a$ is defined by

$$ {\pi_{g,b}}^a(\psi) = \sum_\mu \psi^\mu \psi(\rho_g \psi_\mu), \tag{7.27} $$

where we sum over a basis ψ_μ for $\mathcal{O}_{ab}$, and take ψ^μ to be the dual basis of $\mathcal{O}_{ba}$. See fig. 30.

We may now formulate

Theorem 7 The above conditions form a complete set of sewing conditions for G-equivariant open/closed 2d TFT.

This will be proved in the appendix. Note that the above axioms are slightly redundant since (7.21) and (7.25) together imply (7.22).

7.5. Solution of the sewing conditions for semisimple $\mathcal{C}$

We now show that, when $\mathcal{C}$ is semisimple, the solutions of the above sewing conditions are provided by G-equivariant bundles on $X = \mathrm{Spec}(\mathcal{C}_1)$ twisted by the B-field defined by $\mathcal{C}$.

Let us first say a word about these bundles. To give a finite dimensional representation of the cross-product algebra $A(X) \ltimes G$ is to give a representation of $A(X)$ — i.e. a vector bundle E on X — together with an intertwining action of G. Thus representations of $A(X) \ltimes G$ are precisely G-vector-bundles on X. For a finite group G there are many equivalent ways of defining the notion of a *twisted* G-vector-bundle on X, twisted by a B-field b representing an element of $H^2_G(X;\mathbb{C}^\times)$: the simplest for our purposes is to say that a twisted bundle is just a representation of $A(X) \ltimes_b G$. (Unfortunately this description does not work when G is not finite, and so it is not the one used in [52] . We shall explain the relationship with the description of [52] at the end of this subsection.)

The problem is easily reduced to consideration of a single G-orbit, so we may assume $X = G/H$ for some subgroup H of G. Accordingly, the closed string Frobenius data is specified by a 2-cocycle b and a single constant $\theta_c \in \mathbb{C}^\times$ defining the trace: $\theta(\ell_{g,x}) = \delta_{g,1}\theta_c$. As usual, the isomorphism class only depends on $[b] \in H^2(H;\mathbb{C}^\times)$.

Theorem 8 Let $\mathcal{C} = T(X, b, \theta_c)$ be a Turaev algebra with $\mathcal{C}_1$ semisimple and $X = G/H$. For a single label a the most general solution $\mathcal{O} = \mathcal{O}_{aa}$ of the sewing constraints is determined by a choice of square-root $\theta_o = \sqrt{\theta_c}$ and a projective representation V of H with the cocycle b_o which is the restriction of b.

The algebra $\mathcal{O}$ is the algebra of sections of the G-equivariant bundle of algebras over X:

$$\mathcal{O} := \Gamma\big(G \times_H (\mathrm{End}(V))\big) = \mathrm{Ind}_H^G\big(\mathrm{End}(V)\big), \tag{7.28}$$

and the trace is determined by θ_o:

$$\theta_{\mathcal{O}}(\Psi) = \theta_o \sum_{x \in G/H} \mathrm{Tr}_V(\Psi(x)). \tag{7.29}$$

Proof

Let us suppose that we have a Turaev algebra $\mathcal{C}$ with $\mathcal{C}_1$ semisimple, together with $\mathcal{O}, \theta_{\mathcal{O}}, \iota_g, \iota^g$ satisfying the sewing conditions. Let X be the G-space $\mathrm{Spec}(\mathcal{C}_1)$. Then, from our results in the non-equivariant case, we know that $\mathcal{O} = \mathrm{End}_{\mathcal{C}_1}(\Gamma(\mathrm{End}(E)))$ for some vector bundle $E \to X$, unique up to tensoring wtih a line bundle $L \to X$. Thus $\mathcal{O} = \oplus \mathcal{O}_x$, where $\mathcal{O}_x = \mathrm{End}(E_x)$. We also know that the trace on $\mathcal{O}$ must be given by (3.7) . The same square-root θ_o of θ_x must be taken for each $x \in X$ to make $\theta : \mathcal{O} \to \mathbb{C}$ invariant under G. Now G acts compatibly on $\mathcal{C}_1$ and $\mathcal{O}$ by algebra isomorphisms, so $g \in G$ maps $\mathcal{O}_{x_0}$ to $\mathcal{O}_x$ by an algebra isomorphism. This proves (7.28) , where $V = E_{x_0}$. Finally, the Turaev algebra $\mathcal{C}$ is the product $\oplus \mathcal{C}_x$, where $\mathcal{C}_x$ is the twisted group-ring of G_x with the twisting b_x. The algebra homomorphism $\mathcal{C} \to \mathcal{O}$ makes $\mathcal{C}_x$ act on E_x, and so V is a projective representation of $H = G_{x_0}$ with the cocycle b_{x_0}.

This proves that $\mathcal{O}$ is of the form stated. One must still check that the definition (7.28) does provide a solution of the sewing conditions, but that presents no problems.

Remark Although in the hypothesis of the theorem we were given a cocycle b representing an element of $H^2_G(X;\mathbb{C}^\times)$, the conclusion uses only its restriction b_{x_0}. This should not surprise us, as cohomologous cocycles b define isomorphic Turaev algebras, and $H^2_G(X;\mathbb{C}^\times)$ is canonically isomorphic to the group cohomology $H^2_H(\mathrm{point};\mathbb{C}^\times)$ when $X = G/H$.

We can now deduce a complete description of the category of boundary conditions, using exactly the same arguments by which we obtained Theorem 3 from Theorem 2.

Theorem 9 If $\mathcal{C}$ is a Turaev algebra with $\mathcal{C}_1$ semisimple, corresponding to a space-time X with a B-field b, then the category of boundary conditions for $\mathcal{C}$ is equivalent to the category of b-twisted G-vector-bundles on X, uniquely up to tensoring with a G-line-bundle on X. Its Frobenius structure is determined by a choice of the dilaton field θ.

The meaning of this theorem needs to be explained. The linear category of equivariant boundary conditions for a given Turaev algebra is an example of what is called an "enriched" category: for each pair of objects a, b the vector space $\mathcal{O}_{ab}$ has an action of the group G. Now the category Vect_G of finite dmensional vector spaces with G-action is a symmetric tensor category, with the neutral object $\mathbb{C}$. To say that we have a category *enriched* in a tensor category such as Vect_G means that we have

(i) a set of objects,

(ii) for each pair a, b of objects an object $\mathcal{O}_{ab}$ of Vect_G, and

(iii) for each triple a, b, c of objects an associative "composition" morphism

$$\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$$

of G-vector spaces.

The axioms are almost identical to the axioms for a category, but the space of morphisms has extra structure. In such a situation the category is said to be an enrichment of the ordinary linear category in which the morphisms from b to a are $F(\mathcal{O}_{ab})$, where $F : \mathrm{Vect}_G \to \mathrm{Vect}$ is the functor defined by $F(V) = \mathrm{Hom}_G(\mathbb{C}; V) = V^G$. There is, however, another ordinary category associated to the enriched category by simply forgetting the G-action, so that the morphisms from b to a are simply $\mathcal{O}_{ab}$ as a vector space.

An example of a category enriched in Vect_G is the category of finite dimensional representations of $\tilde{G}$, where $\tilde{G}$ is a central extension (with a fixed cocycle) of G by the circle, where the central circle acts by scalar multiplication. Indeed, given two such representations $V_1^* \otimes V_2$ is a representation of G.

Theorem 9 should really be expanded as follows. The category of b-twisted G-vector-bundles on X has a natural enrichment in Vect_G, in which the G-vector space of morphisms consists of the homomorphisms of b-twisted vector bundles which are not necessarily equivariant with respect to the G-action. This enrichment is equivalent to the category of equivariant boundary conditions. The underlying ordinary category is the category of boundary conditions for the orbifold theory.

Theorem 9 has a converse, which is the G-equivariant extension of the discussion of §3.3

Theorem 10 *If $\mathcal{B}$ is a linear category enriched in* Vect_G*, with G-equivariant traces making it a Frobenius category, and the linear category obtained from $\mathcal{B}$ by forgetting the*

G-action is semisimple with finitely many irreducible objects, then $\mathcal{B}$ is equivalent to the category of equivariant boundary conditions for a canonical equivariant topological field theory. The Turaev algebra defining the theory is $\oplus_g \mathcal{C}_g$, where an element of $\mathcal{C}_g$ is a family $\phi_a \in \mathcal{O}_{aa}$, indexed by the objects a of $\mathcal{B}$, satisfying

$$\phi_a \circ f = (g.f) \circ \phi_b \tag{7.30}$$

for each $f \in \mathcal{O}_{ab}$.

To prove this, one must show that (7.30) really does define a Turaev algebra. The details are straightforward and we will omit them.

7.6. Equivariant boundary states

To conclude our discussion, let us consider the equivariant analogues of the "boundary states" discussed in §4. Our notion of the category of boundary conditions for a G-gauged theory is intrinsically G-invariant, and we have already pointed out that it gives us exactly the same category as we would obtain from the orbifold theory in which we have summed over the gauge fields. To reformulate this in terms of boundary states we begin with the definition.

In the gauged theory associated to a Turaev algebra $\mathcal{C} = T(X, b, \theta)$ the observables at any point of the world-sheet are precisely the elements of $\mathcal{C}$. The boundary state $B_a \in \mathcal{C}$ associated to a boundary condition a is characterized by the property that the correlation function of observables $\phi_1, \ldots, \phi_k$ evaluated at points of a surface Σ with boundary S^1 with the boundary condition a (and arbitrary holonomy around the boundary) is equal to that of the same observables on the closed surface obtained by capping-off the boundary, with the additional insertion of B_a at the centre of the cap. It suffices (because of the factorization properties of a field theory) to check the case when Σ is a disc. The correlation function on the disc is obtained by propagating $\phi_1 \ldots \phi_k \in \mathcal{C}_g$ to $\mathcal{H}(\emptyset) = \mathbb{C}$ by the annulus whose non-incoming boundary circle is constrained by the condition a, along which the holonomy is necessarily g. Our rules tell us that the result is

$$\theta_{\mathcal{O}_{aa}}(\iota_{g,a}(\phi_1 \ldots \phi_k)).$$

Equating this to $\theta_{\mathcal{C}_1}(\phi_1 \ldots \phi_k B_a)$, we see that

$$B_a = \sum_g \iota_{g,a}(1).$$

The map $a \mapsto B_a$ evidently has its image in the G-invariant part — i.e. the centre — of the Turaev algebra. It extends to a homomorphism

$$K_{G,h}(X) \to T(X,b,\theta)^G,$$

and we have

Theorem 11 The G-invariant boundary states generate a lattice in $T(X,b,\theta)^G$ related to the twisted equivariant K-theory via:

$$K_{G,h}(X) \otimes \mathbb{C} = T(X,b,\theta)^G. \tag{7.31}$$

Remark

1. Equation (7.31) is related to an old observation of [53]. If $X = G$ with G acting on itself by conjugation then $T(X,0)^G$ is the Verlinde algebra occuring in the conformal field theory of orbifolds for chiral algebras with one representation [53]. The different orbits are the conjugacy classes of G. Focusing on one conjugacy class $[g]$ we can compare with the above results. One basis of states is provided by a choice of a character of the centralizer of g. These are just the G-invariant boundary states found above.
2. The translation of the above results to the language of branes at orbifolds is the following. The boundary states corresponding to the different b-irreps V_i are the "fractional branes" of [49]. The use of projective representations was proposed in [49], and further explored in [54]. A different proof of the fact that the cocycle for the open sector and that of the closed sector b are cohomologous can be found in [55].

To conclude this section, let us return to explain the relation between the definition of twisted equivariant K-theory by $A(X) \ltimes_b G$-modules and the definition given in [52] .

In [52] the elements of the twisted equivariant theory are described as follows. First, the twisting class $b \in H^3_G(X;\mathbb{Z})$ is represented by a bundle P of projective Hilbert spaces on X equipped with a G-action covering the G-action on X. Then elements of $K_{G,P}(X)$ are represented by families $\{T_x\}_{x\in X}$ of fibrewise Fredholm operators in the bundle P. Let us show how to associate such a pair $(P,\{T_x\})$ to a finitely generated $A(X) \ltimes_b G$-module. Such a module is the same thing as a finitely generated $A(X)$-module equipped with a compatible action of the extended group $\tilde{G}$ associated to b which we have already described. Equivalently, it is a finite dimensional vector bundle E on X with an action of $\tilde{G}$ on the total space which covers the action of G on X. Let us choose a fixed infinite dimensional Hilbert space $\mathcal{H}$. Then $\hat{E} = E \otimes \mathcal{H}$ is a Hilbert bundle on X, and the associated bundle $P = \mathbb{P}(\hat{E})$ of projective spaces has a natural action of G, and it represents the class of b in $H^3_G(X; \mathbb{Z})$. (Cf. the proof of Proposition 6.3 in [52] .) If $T : \mathcal{H} \to \mathcal{H}$ is a fixed surjective Fredholm operator with a one-dimensional kernel, then $(\text{identity})_E \otimes T : P \to P$ represents an element of $K_{G,P}(X)$ according to the definition of [52] .

If the cocycle b is a coboundary — or even if $b_x(g_1, g_2)$ is independent of $x \in X$ — it is plain that the two rival definitions of equivariant K coincide. A Mayer-Vietoris argument can then be used to show that they coincide for all b.

The essential point here is that when X and G are finite the twisting class b is of finite order, and that makes it possible to represent the K-classes by families of Fredholm operators of constant rank, and hence by finite dimensional vector bundles.

Appendix A. Morse theory proof of the sewing theorems

In this appendix we shall use Morse theory to give uniform proofs of four theorems. The first is the very well-known result that a two-dimensional topological field theory is precisely encoded in a commutative Frobenius algebra. The second is the corresponding statement for open and closed theories: this is Theorem 1 of Section 2 above. The third and fourth are the equivariant analogues of the first two, i.e. Theorems 4 and 7 of Section 7.

A.1. The classical theorem

We wish to prove that when we have a commutative Frobenius algebra $\mathcal{C}$ we can assign to an oriented cobordism Σ from S_0 to S_1 a linear map

$$U_\Sigma : \mathcal{C}^{\otimes p} \to \mathcal{C}^{\otimes q},$$

where the oriented 1-manifolds S_0 and S_1 have p and q connected components respectively.

We can always choose a smooth function $f : \Sigma \to [0, 1] \subset \mathbb{R}$ such that $f^{-1}(0) = S_0$ and $f^{-1}(1) = S_1$, and which has only "Morse" singularities, i.e. the gradient df vanishes at only finitely many points $x_1, \ldots, x_n \in \Sigma$, and

(i) the Hessian $d^2 f(x_i)$ is a non-degenerate quadratic form for each i, and

(ii) the critical values $c_1 = f(x_1), \ldots, c_n = f(x_n)$ are distinct, and not equal to 0 or 1.

Each critical point has an *index*, equal to 0, 1, or 2, which is the number of negative eigenvalues of the Hessian $d^2 f(x_i)$.

The choice of the function f gives us a decomposition of the cobordism into "elementary" cobordisms. If

$$0 = t_0 < c_1 < t_1 < c_2 < t_2 < \ldots < c_n < t_n = 1,$$

and $S_t = f^{-1}(t)$, then each S_{t_i} is a collection of, say, m_i disjoint circles, with $m_i = m_{i-1} \pm 1$, and $\Sigma_i = f^{-1}([t_{i-1}, t_i])$ is a cobordism from $S_{t_{i-1}}$ to S_{t_i} which is trivial (i.e. a union of cylinders) except for one connected component of one of the four forms of fig. 2.

For a given Frobenius algebra $\mathcal{C}$ we know how to define an operator

$$U_{\Sigma_i} : \mathcal{C}^{\otimes m_{i-1}} \to \mathcal{C}^{\otimes m_i}$$

in each case. (In the third case the map we assign is

$$\phi \mapsto \sum \phi\phi_i \otimes \phi^i,$$

where $\{\phi_i\}$ and $\{\phi^i\}$ are dual bases of $\mathcal{C}$ such that $\theta_{\mathcal{C}}(\phi^i \phi_j) = \delta_{ij}$.) We should notice two points. First, we need $\mathcal{C}$ to be commutative, for otherwise we would need to have an *order* on the two incoming circles of a pair of pants, and no such order is given. Secondly, the assignments we make have the property that reversing the direction of time in a cobordism replaces the operator by its adjoint with respect to the Frobenius inner product on the state-spaces. This property will be a firm principle in all our constructions, and it reduces the number of cases we have to check in the tedious arguments below.

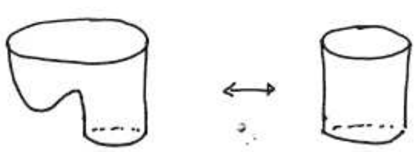

Fig. 31:

The important task now is to show that the composite operator $U_{\Sigma_n} \circ \cdots \circ U_{\Sigma_1}$ is independent of the chosen Morse function f.

Two Morse functions f_0 and f_1 can always be connected by a smooth path $\{f_s\}_{0 \le s \le 1}$ in which f_s is a Morse function except for a finite set of parameter values s at which one of the following two things happens:

(i) f_s has one degenerate critical point where in local coordinates (u, v) it has the form $f_s(u, v) = \pm u^2 + v^3$, or

(ii) two distinct critical points x_i, x_j of f_s have the same critical value $f_s(x_i) = f_s(x_j) = c$.

In the first case, two critical points of adjacent indices are created or annihilated as the parameter passes through the non-Morse value s, and the cobordism changes by fig. 31.
or vice-versa, or by the time-reversal of these pictures. The well-definedness of U_Σ under this kind of change is ensured by the identity $1.a = a$ in the algebra $\mathcal{C}$.

Case (ii) is more problematical. Because operators of the form $U \otimes 1$ and $1 \otimes U'$ commute, we easily see that there is nothing to prove unless the two critical points x_i and x_j are connected in the "bad" critical contour S_c, in which case they must both have index 1.

Let us consider the resulting two-step cobordism which is factorized in different ways before and after the critical parameter value s. It will have just one non-trivial connected component, which, because an elementary cobordism changes the number of circles by 1, must be a cobordism from p circles to q circles, where $(p, q) = (1, 1), (2, 2), (1, 3)$ or $(3, 1)$. We need to check only one of (1,3) and (3,1), as they differ only by time-reversal. Because the Euler number of a cobordism is the number of critical points of its Morse function (counted with the sign $(-1)^{\text{index}}$), the non-trivial component has Euler number -2, so is a 2-holed torus when $(p, q) = (1, 1)$ and a 4-holed sphere in the other cases.

Fig. 33:

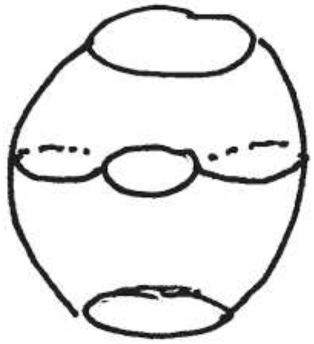

Fig. 32:

Fig. 34:

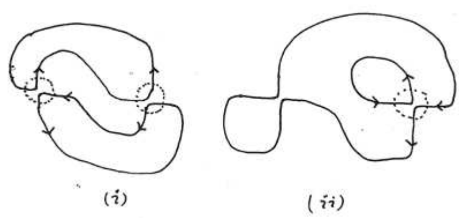

Fig. 35:

In the case (1,1), depicted in fig. 32, a circle splits into two which then recombine. There is nothing to check, because, though a torus with two holes can be cut into two pairs of pants by many different isotopy classes of cuts, there is only one possible composite cobordism, and we have only one possible composite map $\mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$.

In the case (3,1), two circles of the three combine, then the resulting circle combines with the third. The picture is fig. 33. Clearly this case is covered by the associative law in $\mathcal{C}$.

In the case (2,2) we are again factorizing a 4-holed sphere into two elementary cobordisms. This can be done in many ways, as we see from the pictures fig. 34. The best way of making sure we are not overlooking any possibility is to think of the contour just below the doubly-critical level, which, if it consists of two circles, must have one of the two forms (i) or (ii) in fig. 35. (Consider the possible ways of connecting the "terminals" inside the dotted circles.) But, whatever happens, the only algebraic maps the cobordism can lead to are

$$\mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$$

and

$$\mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C},$$

given by

$$\phi \otimes \phi' \mapsto \phi\phi' \mapsto \sum \phi\phi'\phi_i \otimes \phi^i$$

and

$$\phi \otimes \phi' \mapsto \sum \phi\phi_i \otimes \phi^i \otimes \phi' \mapsto \sum \phi\phi_i \otimes \phi^i\phi'$$

respectively, where $\{\phi_i\}$ and $\{\phi^i\}$ are dual bases of $\mathcal{C}$ such that $\theta_{\mathcal{C}}(\phi^i\phi_j) = \delta_{ij}$. These two maps are equal because of the identity

$$\sum \phi'\phi_i \otimes \phi^i = \sum \phi_i \otimes \phi^i\phi', \tag{A.1}$$

which holds in any Frobenius algebra because the inner product of each side with $\phi^j \otimes \phi_k$ is $\theta_{\mathcal{C}}(\phi^j\phi'\phi_k)$.

That completes the proof of the theorem: notice that we have used all the axioms of a commutative Frobenius algebra.

A.2. Open and closed theories

As in the preceding argument we consider a cobordism Σ from S_0 to S_1, but now S_0 and S_1 are collections of circles and intervals, and the boundary $\partial\Sigma$ has a constrained part $\partial_{\mathrm{constr}}\Sigma$, which we shall abbreviate to $\partial'\Sigma$, which is a cobordism from ∂S_0 to ∂S_1. We choose $f : \Sigma \to [0,1]$ as before, but now there are two kinds of critical points of f: interior points of Σ at which the gradient df vanishes, and points of $\partial'\Sigma$ at which the gradient of the restriction of f to the boundary vanishes. For an internal critical point, "nondegenerate" has its usual meaning. A critical point x on the boundary is called nondegenerate if it is a nondegenerate critical point of the restriction of f to $\partial'\Sigma$, and in addition the derivative of f normal to the boundary does not vanish at x.

As before, f is a *Morse function* if all its critical points are non-degenerate, and all the critical values are distinct and $\neq 0, 1$. We can always choose such a function.

There are now four kind of boundary critical points, which we can denote $0\pm, 1\pm$, recording the index and the sign of the normal derivative. Six things can happen as we pass through one of them. At those of type $0+$ or $1-$, an open string is created or annihilated. At type $0-$ either two open strings join end-to-end, or else an open string becomes a closed string. Type $1+$ is the time-reverse of $0-$. If we have a Frobenius catgegory $\mathcal{B}$, we know what to do in each of the six cases.

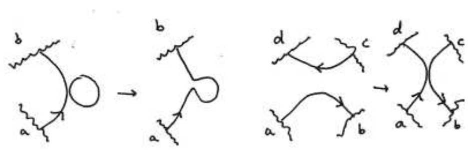

Fig. 36:

An internal critical point has index 0,1, or 2, as before. Only if the index is 1 can the corresponding cobordism involve an open string. Up to time reversal, there are three index 1 processes: two closed strings can become one, an open string can "absorb" a closed string, and two open strings can "reorganize themselves" to form two new open strings as in fig. 36.

For a given Frobenius category $\mathcal{B}$, we assign to (open)+(closed)→(open) the map

$$\mathcal{O}_{ab} \otimes \mathcal{C} \to \mathcal{O}_{ab}$$

given by $\phi \otimes \psi \mapsto \phi\psi$. (Here, as we usually do, we are regarding $\mathcal{O}_{ab}$ as a $\mathcal{C}$-module, writing

$$\phi\psi = \iota_a(\phi)\psi = \iota_b(\phi)\psi.)$$

To (open)+(open)→(open)+(open) we assign the map

$$\mathcal{O}_{ab} \otimes \mathcal{O}_{cd} \to \mathcal{O}_{ad} \otimes \mathcal{O}_{cb}$$

given by

$$\psi \otimes \psi' \mapsto \sum \psi\psi_i \otimes \psi'\psi^i,$$

where ψ_i and ψ^i are dual bases of $\mathcal{O}_{bd}$ and $\mathcal{O}_{db}$.

We must now consider what happens when we change the Morse function. As before, two Morse functions can be connected by a path $\{f_s\}$ in which each f_s is a Morse function except for finitely many values of s at which either one critical point is degenerate or else two critical values coincide. We begin with the degenerate case. There are now three kinds of degeneracy which we must allow, for besides internal degeneracies which are just as in the closed string case we can have two kinds of degeneracy on the boundary: either $f|\partial'\Sigma$ has a cubic inflexion, or else the normal derivative vanishes at a boundary critical point.

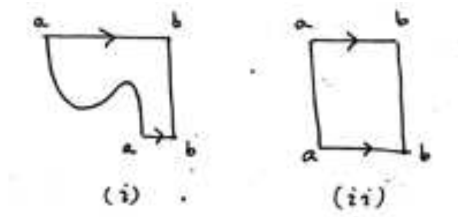

Fig. 37:

When s passes through a boundary inflexion, two nondegenerate boundary critical points of opposite index but with normal derivatives of the same sign are created or annihilated. This means that the cobordism changes between figures (i) and (ii) of fig. 37 (or the time-reversal). These changes are covered by the axiom that the category $\mathcal{B}$ has identity morphisms.

When the normal derivative vanishes at a boundary critical point what happens is that an internal critical point has moved "across the boundary of Σ, i.e. it moves into coincidence with a boundary critical point and changes the sign of the normal derivative there. There are four cases:

$$(0-) \ + \ (\text{index } 0) \ \to \ (0+),$$

$$(0+) \ + \ (\text{index } 1) \ \to \ (0-),$$

and the time-reversals of these. In the first case, the composite cobordism in which a small closed string is created and then breaks open is replaced by the elementary cobordism in which an open string is created. This corresponds to the axiom that $\mathcal{C} \to \mathcal{O}_{aa}$ takes $1_{\mathcal{C}}$ to 1_a. In the second case, in the composite cobordism, an open string is created, and then it either "absorbs" an existing closed string or else "rearranges" itself with an existing open string; these composites are to be equivalent, respectively, to the elementary breaking of a closed or open string. Putting $\psi = 1_a$ in the formulae above we see that this is allowed by the Frobenius category axioms.

When we have an internal degenerate critical point, what happens, up to time-reversal, is that a closed string is created and then joins an existing open or closed string; this should be the same as the trivial cobordism. Again, the unit axioms cover this.

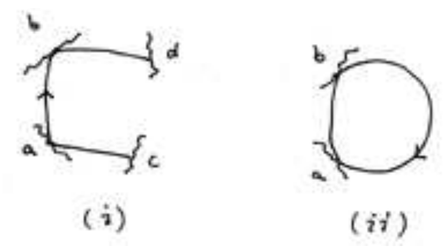

Fig. 38:

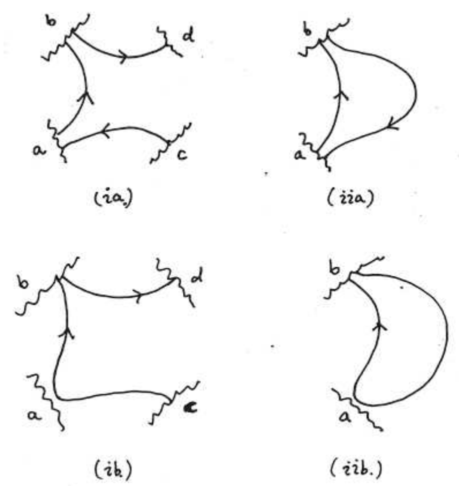

Fig. 39:

Finally, we have to consider what happens when two critical values cross. They can be two boundary critical points, two internal ones, or one of each.

If two boundary critical points are linked by a critical contour, it has the form fig. 38. These give us four cases to check, where the contour below the critical level is fig. 39. Case $(i)_a$ is accounted for by the associativity of composition in the category $\mathcal{B}$; case $(i)_b$ by the open-string analogue of the identity (A.1); case $(ii)_a$ by the trace axiom $\iota^a(\psi_1\psi_2) = \iota^b(\psi_2\psi_1)$, which follows by combining (cyclic),(center),and (adjoint); and case $(ii)_b$ by the Cardy identity.

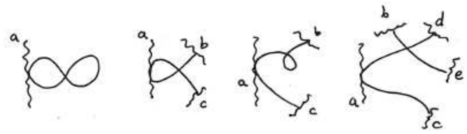

Fig. 40:

When we have one boundary and one internal critical point at the same level we may as well assume the boundary point is of type $0-$ and the internal critical point is of index 1, and that they are joined in the critical contour,which must have one of the four forms fig. 40.

At the boundary point either an open string becomes closed, or else two open strings join. We shall consider each possibility in turn. In the first case, if the boundary point is encountered first, then at the interior point three things can happen: the closed string can split into two closed strings, or it can combine with another closed or open string. Thus the possibilities are

$$o \;\to\; c \;\to\; c + c$$

$$o + c \;\to\; c + c \;\to\; c$$

$$o + o \;\to\; c + o \;\to\; o.$$

When the internal point is encountered first there is only one possibility in each case, and the three sequences are replaced respectively by

$$o \;\to\; o + c \;\to\; c + c$$

$$o + c \;\to\; o \;\to\; c$$

$$o \;+ o \;\to\; o + o \;\to\; o.$$

We have to check three identities. The first two reduce to the fact that $\iota^a : \mathcal{O}_{aa} \to \mathcal{C}$ is a map of modules over $\mathcal{C}$. The third is the Cardy condition.

Now let us consider the case where two open strings join at the boundary critical point. If we meet the boundary point first, there are again three things that can happen at the internal critical point: the open string can emit a closed string, or else it can interact with another closed or open string. The possibilities are

$$o + o \to o \to o + c$$

$$o + o + c \to o \ + c \to o$$

$$o + o + o \to o + o \to o + o.$$

In the second and third of these cases there is only one thing that can happen when the order of the critical points is reversed: they become

$$o + o + c \to o + o \to o$$

$$o + o + o \to o + o + o \to o + o.$$

The identities relating the corresponding algebraic maps $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \otimes \mathcal{C} \to \mathcal{O}_{ac}$ and $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \otimes \mathcal{O}_{de} \to \mathcal{O}_{ae} \otimes \mathcal{O}_{dc}$ are immediate.

The first sequence, however, can become either

$$o + o \to o + o + c \to o + c$$

or

$$o + o \to o + o \to o + c.$$

The first of these presents nothing of interest algebraically, but to deal with the second we need to check that

$$\sum \psi\psi'\phi_i \otimes \phi^i = \sum \psi\psi^k \otimes \iota^b(\psi'\psi_k)$$

for $\psi \in \mathcal{O}_{ab}$, $\psi' \in \mathcal{O}_{bc}$, and dual bases ϕ^i, ϕ_i of $\mathcal{C}$ and ψ^k, ψ_k of $\mathcal{O}_{bc}, \mathcal{O}_{cb}$. This relation holds because the inner product of the left-hand side with $\psi_m \otimes \phi_j$ is $\theta_b(\psi\psi'\phi_j\psi_m)$, while the inner product of the right-hand side with $\psi_m \otimes \phi_j$ is

$$\begin{aligned}\sum_k \theta_b(\psi\psi^k\psi_m)\theta(\iota^b(\psi'\psi_k)\phi_j) &= \sum_k \theta_b(\psi_m\psi\psi^k)\theta_b(\psi_k\phi_j\psi') \\ &= \theta_b(\psi_m\psi\phi_j\psi') = \theta_b(\psi\psi'\phi_j\psi_m).\end{aligned} \tag{A.2}$$

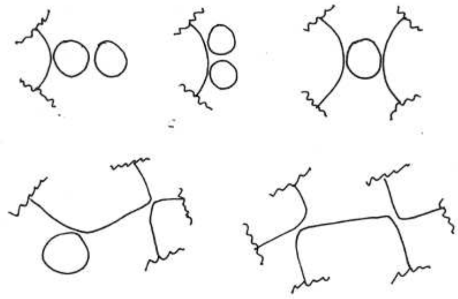

Fig. 41:

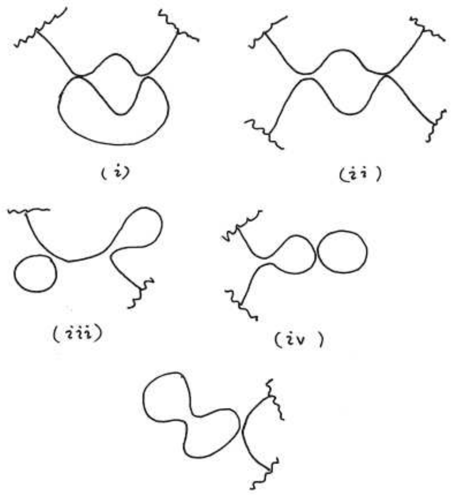

Fig. 42:

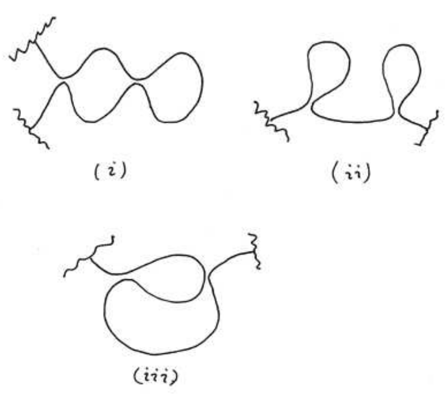

Fig. 43:

Finally, we must consider what happens when there are two internal critical points on the same level. Here we have the possibilities which we have already discussed in the closed case, but must also allow any or all of the strings involved to be open. We can analyse the situation according to the number of connected components of the part of the contour immediately below the doubly critical level which pass close to the critical points. There must be one, two, or three such components. If there are three they can form five configurations (apart from the case when all three are closed), as depicted in fig. 41. The well-definedness of the composite map in all these cases follows immediately from the associative law of composition in the Frobenius category.

If there are two components below the critical level then they can again form five configurations (for either the two components meet twice, or else they meet once, and one of them has a self-interaction), depicted in fig. 42. But we have only three cases to check, as the second is the time-reversal of one from fig. 41, and the last two are time-reversals of each other. Case 15(i) corresponds to the fact that the composition

$$\mathcal{O}_{ab} \otimes \mathcal{C} \to \mathcal{O}_{ab} \to \mathcal{O}_{ab} \otimes \mathcal{C}$$

can be effected by cutting the composite cobordism in different ways, but there is nothing to check, as there is only one possible algebraic map.

In fig. 42 case(iii), one order of the critical points gives us the same composition

$$\mathcal{O}_{ab} \otimes \mathcal{C} \to \mathcal{O}_{ab} \to \mathcal{O}_{ab} \otimes \mathcal{C}$$

as before, while the other order gives

$$\mathcal{O}_{ab} \otimes \mathcal{C} \to \mathcal{O}_{ab} \otimes \mathcal{C} \otimes \mathcal{C} \to \mathcal{O}_{ab} \otimes \mathcal{C};$$

but it is very easy to check that both maps take $\psi \otimes \phi$ to $\sum \psi\phi\phi_i \otimes \phi^i$ in the notaion we have already used.

In fig. 42 case(iii) we must again compare compositions

$$\mathcal{O}_{ab} \otimes \mathcal{C} \to \mathcal{O}_{ab} \otimes \mathcal{C} \otimes \mathcal{C} \to \mathcal{O}_{ab} \otimes \mathcal{C}$$

and

$$\mathcal{O}_{ab} \otimes \mathcal{C} \to \mathcal{O}_{ab} \to \mathcal{O}_{ab} \otimes \mathcal{C}.$$

This time we must check that

$$\sum \psi\phi_i \otimes \phi^i\phi = \sum \psi\phi\phi_i \otimes \phi^i.$$

This is the same formula which we met at the end of our discussion of closed string theories.

Finally, suppose that the contour below the critical level has only one connected component. There are three possible configurations, corresponding to the three ways of pairing four points on an interval. They are fig. 43. The first two of these are time-reversals of cases we have already treated. The last one leads — in either order — to a factorization

$$\mathcal{O}_{ab} \to \mathcal{O}_{ab} \otimes \mathcal{C} \to \mathcal{O}_{ab}.$$

There is only one possibility for this, so there is nothing to check.

That completes the proof of the theorem about open and closed theories.

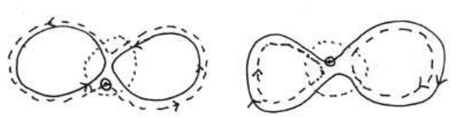

Fig. 44:

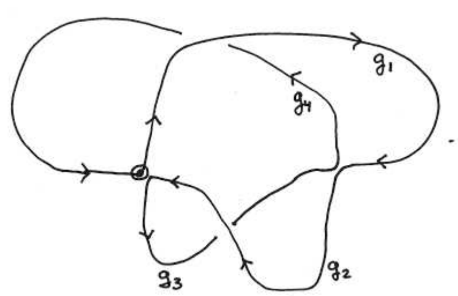

Fig. 45:

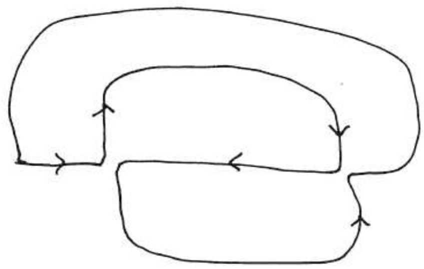

Fig. 46:

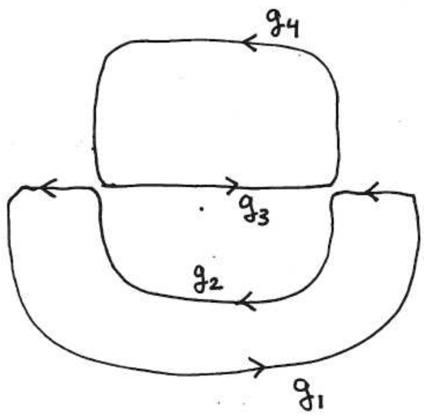

Fig. 47:

A.3. Equivariant closed theories

We must now redo the discussion in the first part of this appendix, but for surfaces and circles equipped with a principal G-bundle, where G is a given finite group.

The first observation is that any circle with a bundle is isomorphic to a standard bundle S_g with holonomy $g \in G$ on the standard circle S^1. Furthermore the set of morphisms from S_g to $S_{g'}$ is $\{h \in G : \ hgh^{-1} = g'\}$. In other words, the category of bundles on S^1 is equivalent to the category $G//G$ formed by the group G acting on itself by conjugation. An equivariant theory therefore gives us a vector space $\mathcal{C}_g$ for each g, and together the $\mathcal{C}_g$ form a G-vector-bundle on G. Conversely, given the G-vector-bundle $\{\mathcal{C}_g\}$ and a circle S with a bundle P on it, the theory gives us the vector space $\mathcal{H}(S,P)$ whose elements are rules which associate $\psi_{x,t} \in \mathcal{C}_{g_{x,t}}$ to each $x \in S$ and trivialization $t : P_x \to G$, where $g_{x,t}$ is the holonomy of P with base-point (x,t), and we require that

$$\psi_{x',t'} = g\psi_{x,t}$$

if g is the holonomy of P along the positive path from (x,t) to (x',t'). For this to be well-defined we need the condition that $g_{x,t}$ acts trivially on $\mathcal{C}_{g_{x,t}}$, whose necessity we have already explained in Section 7.

Next we consider the trivial cobordism from S_g to $S_{g'}$. The possible extensions of the bundles on the ends over the cylinder correspond to the possible holonomies from the incoming base-point to the outgoing base-point, i.e. to the set of morphisms $\{h \in G : hgh^{-1} = g'\}$ in $G//G$. Clearly these cylinders induce the isomorphisms $\mathcal{C}_g \to \mathcal{C}_{g'}$ which we already know. But two such cobordisms are to regarded as equivalent if there is a diffeomorphism from the cylinder (with its bundle) to itself which is the identity on the ends. The mapping-class group of the cylinder is generated by the Dehn twist around it, so the morphism corresponding to h is equivalent to that for $hg = g'h$. This means that g must act trivially on $\mathcal{C}_g$, as we already know.

Now we come to the possible bundles on the four elementary cobordisms of diagram 1. The bundle on a cap must of course be trivial. The pair-of-pants cobordisms that are relevant to us arise as the regions between nearby level curves separated by a critical level. We can draw them as in fig. 44, where the solid contour is below the critical level, and the dashed one is above it. We can trivialize the G-bundle in the neighbourhood of the critical point (i.e. within the dotted circles), and then the bundle on the cobordism is determined by giving the holonomies g_1, g_2 along the ribbons, as indicated. The operator we associate to case (i) is the multiplication map

$$m_{g_1,g_2} : \mathcal{C}_{g_1} \otimes \mathcal{C}_{g_2} \to \mathcal{C}_{g_1 g_2}$$

of ((7.1)). In writing it this way we are choosing an ordering of the ribbons, i.e. a base-point on the outgoing loop. The two orderings are related by the conjugation

$$\alpha_{g_2} : \mathcal{C}_{g_1 g_2} \to \mathcal{C}_{g_2 g_1},$$

so the consistency condition for us to have a well-defined assignment is that

$$m_{g_2,g_1}(\psi_2 \otimes \psi_1) = \alpha_{g_2}(m_{g_1,g_2}(\psi_1 \otimes \psi_2)).$$

We see that this holds in any Turaev algebra by combining ((7.3)) with the facts that G acts on the algebra by algebra-automorphisms, and that α_{g_2} acts trivially on $\mathcal{C}_{g_2}$. As the mapping-class group of the pair of pants is generated by the three Dehn twists parallel to its boundary circles, there are no new conditions needed to make the assignment of the operator to the pair of pants well-defined.

The homomorphism

$$c_{g_1,g_2} : \mathcal{C}_{g_1 g_2} \to \mathcal{C}_{g_1} \otimes \mathcal{C}_{g_2}$$

corresponding to the coordism 17(ii) is fixed by the requirement of adjunction, bearing in mind that the dual space to $\mathcal{C}_g$ is $\mathcal{C}_{g^{-1}}$. It is given by

$$c_{g_1,g_2}(\phi) = \sum \phi\phi^i \otimes \phi_i,$$

where $\{\phi_i\}$ is a basis for $\mathcal{C}_{g_2}$, and $\{\phi^i\}$ is the dual basis of $\mathcal{C}_{g_2^{-1}}$.

Any cobordism with a bundle can be factorized by Morse theory just as before; bundles are inherited by the elementary cobordisms. The difficult part of the discussion is considering what happens when we change the Morse function. But in fact the only step which presents anything significant is the consideration of the interchange of two critical points of index 1 on the same level, i.e. the cobordisms of fig. 32, fig. 33, fig. 34.

Let us consider the case fig. 32, where a string divides and then rejoins — i.e. a torus with two holes, one incoming and one outgoing. We draw the picture in the form fig. 45. (We do not draw it in the apparently more perspicuous form fig. 46, as then the neighbourhoods of the two critical points would have opposite orientation in the plane.)

The cobordism corresponds to a map $\mathcal{C}_{4321} \to \mathcal{C}_{2341}$, where, as in the following, we have abbreviated $\mathcal{C}_{g_4 g_3 g_2 g_1}$ to $\mathcal{C}_{4321}$. If the left-hand critical point is encountered first, the map we obtain is

$$\mathcal{C}_{4321} \to \mathcal{C}_{43} \otimes \mathcal{C}_{21} \cong \mathcal{C}_{34} \otimes \mathcal{C}_{12} \to \mathcal{C}_{3412} \cong \mathcal{C}_{2341},$$

$$\phi \mapsto \sum \phi\phi^i \otimes \phi_i \mapsto \sum \alpha_3(\phi\phi^i) \otimes \alpha_1(\phi_i) \mapsto \sum \alpha_3(\phi\phi^i)\alpha_1(\phi_i) \mapsto \sum \alpha_2(\alpha_3(\phi\phi^i)\alpha_1(\phi_i)),$$

where ϕ_i runs through a basis for $\mathcal{C}_{21}$, and we write α_3 for α_{g_3}, and so on. (The maps indicated by $\cong$ in the previous line correspond to moving the choice of base-point on the various strings.)

With the other order, we get

$$\mathcal{C}_{4321} \cong \mathcal{C}_{3214} \to \mathcal{C}_{32} \otimes \mathcal{C}_{14} \cong \mathcal{C}_{23} \otimes \mathcal{C}_{41} \to \mathcal{C}_{2341}$$

$$\phi \mapsto \alpha_4^{-1}(\phi) \mapsto \sum \alpha_4^{-1}(\phi)\psi^i \otimes \psi_i \mapsto \sum \alpha_2(\alpha_4^{-1}(\phi)\psi^i) \otimes \alpha_4(\psi_i) \mapsto \sum \alpha_2(\alpha_4^{-1}(\phi)\psi^i)\alpha_4(\psi_i),$$

where ψ_i runs through a basis of $\mathcal{C}_{14}$.

Thus we must prove that

$$\sum \alpha_{23}(\phi\phi^i)\phi_i = \sum \alpha_{24^{-1}}(\phi)\alpha_2(\psi^i)\alpha_4(\psi_i).$$

We can deduce this from the axiom (newax) of §7, with $h = g_2g_4^{-1}g_1^{-1}g_2^{-1}$ and $g = g_1^{-1}g_2^{-1}$, as follows. We rewrite the right-hand side of the equation as

$$\sum \alpha_{24^{-1}}(\phi)\xi^i\alpha_g(\xi_i),$$

where ξ^i is the basis $\alpha_2(\psi^i)$ of $\mathcal{C}_h$, so that $\xi_i = \alpha_2(\psi_i)$ and $\alpha_g(\xi_i) = \alpha_1^{-1}(\psi_i) = \alpha_4(\psi_i)$. By the axiom this equals

$$\sum \alpha_{24^{-1}}(\phi)\alpha_h(\eta^i)\eta_i = \sum \alpha_{24^{-1}}\alpha_h(\phi^i)\phi_i.$$

Finally,

$$\alpha_{24^{-1}}(\phi)\alpha_h(\phi^i) = \alpha_{24^{-1}}(\phi\phi^i) = \alpha_{23}(\phi\phi^i),$$

because $\phi\phi^i \in \mathcal{C}_{43}$, and so $\alpha_{24^{-1}}(\phi\phi^i) = \alpha_{24^{-1}}\alpha_{43}(\phi\phi^i) = \alpha_{23}(\phi\phi^i)$. Thus we have dealt with the case of fig. 32.

If fact this case is decidedly the most complicated of the set. We shall do one more, namely case (i) of fig. 35, in which two strings join and then split. We draw the diagram as in fig. 47, corresponding to the two compositions

$$\mathcal{C}_{43} \otimes \mathcal{C}_{21} \to \mathcal{C}_{4321} \cong \mathcal{C}_{1432} \to \mathcal{C}_{14} \otimes \mathcal{C}_{32} \cong \mathcal{C}_{41} \otimes \mathcal{C}_{23}$$

$$\mathcal{C}_{43} \otimes \mathcal{C}_{21} \cong \mathcal{C}_{34} \otimes \mathcal{C}_{12} \to \mathcal{C}_{3412} \cong \mathcal{C}_{4123} \to \mathcal{C}_{41} \otimes \mathcal{C}_{23}.$$

The first sequence gives us

$$\psi \otimes \psi' \mapsto \psi\psi' \mapsto \alpha_1(\psi\psi') \mapsto \sum \alpha_1(\psi\psi')\phi^i \otimes \phi_i \mapsto \sum \alpha_4(\alpha_1(\psi\psi')\phi^i) \otimes \alpha_2(\phi_i),$$

where ϕ_i is a basis for $\mathcal{C}_{32}$. The second sequence gives

$$\psi \otimes \psi' \mapsto \alpha_3(\psi) \otimes \alpha_1(\psi') \mapsto \alpha_3(\psi)\alpha_1(\psi') \mapsto \psi\alpha_{3^{-1}1}(\psi') \mapsto \sum \psi\alpha_{3^{-1}1}(\psi')\psi^i \otimes \psi_i,$$

where ψ_i is a basis for $\mathcal{C}_{23}$. But we can assume that $\psi_i = \alpha_2(\phi_i)$, and hence that $\psi^i = \alpha_2(\phi^i)$. So, noticing that $\alpha_1(\psi\psi')\phi^i \in \mathcal{C}_{14}$, and hence that

$$\alpha_4(\alpha_1(\psi\psi')\phi^i) = \alpha_{-1}(\alpha_1(\psi\psi')\phi^i),$$

what we need to prove is just that

$$\psi'\alpha_{-1}(\phi^i) = \alpha_{3^{-1}}(\alpha_1(\psi')\phi^i).$$

This is true because $\alpha_1(\psi')\phi^i \in \mathcal{C}_{13^{-1}}$, and so is fixed by $\alpha_{13^{-1}}$.

We shall leave the remaining verifications to the reader.

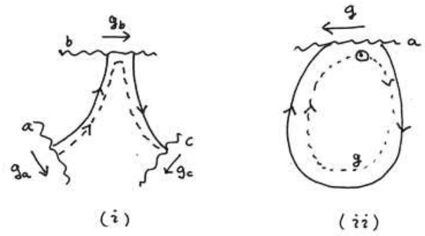

Fig. 48:

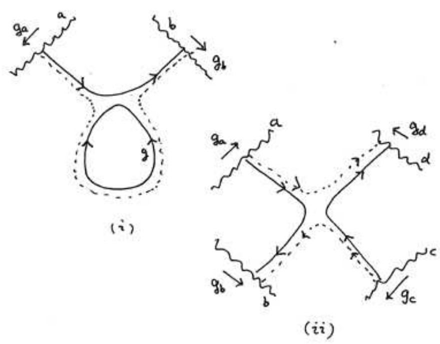

Fig. 49:

A.4. Equivariant open and closed theories

We now have to redo the open and closed case taking account of G-bundles on the cobordisms.

We assign the vector space $\mathcal{O}_{ab}$ to an open string from b to a equipped with a trivialization of the bundle on it. Changing the trivialization by an element $g \in G$ corresponds to the action ρ_g of g on $\mathcal{O}_{ab}$, which also corresponds to the map induced by a rectangular cobordism with holonomy g along its constrained edges.

We must consider the maps to be associated to the elementary cobordisms corresponding to the critical points of a Morse function. Up to time-reversal, two interesting things can happen at a boundary critical point: either two open strings join end-to-end or an open string becomes closed. We have the pictures of fig. 48. As before, the solid line is the contour below the critical point, and the dashed line that above it. In 20(i), g_a, g_b, g_c are the holonomies between nearby points on the respective D-branes, expressed in terms of the chosen trivializations on the strings. (They satisfy $g_c g_b = g_a$.) The map $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$ that we associate to this situation is

$$\psi \otimes \psi' \mapsto \rho_{g_a}(\psi)\rho_{g_b}(\psi').$$

The dual operation $\mathcal{O}_{ac} \to \mathcal{O}_{ab} \otimes \mathcal{O}_{bc}$ is

$$\psi \mapsto \sum \rho_{g_1}(\psi \xi^i) \otimes \rho_{g_2}(\xi_i),$$

where ξ_i and ξ^i are dual bases of $\mathcal{O}_{bc}$ and $\mathcal{O}_{cb}$.

In case (ii) of fig. 48, the open string becomes a closed string whose holonomy is g with respect to the indicated base-point and the trivialization coming from the beginning of the open string. The corresponding map is ι^g, with adjoint ι_g.

There are also the two kinds of operation coming from internal critical points which involve open strings. They are illustrated in fig. 49.
The map $\mathcal{C}_g \otimes \mathcal{O}_{ab} \to \mathcal{O}_{ab}$ corresponding to 21(i) is $\phi \otimes \psi \mapsto \rho_{g_a}(\iota_g(\phi)\psi))$, while the map $\mathcal{O}_{ab} \otimes \mathcal{O}_{cd} \to \mathcal{O}_{ad} \otimes \mathcal{O}_{cb}$ corresponding to 21(ii) is

$$\psi \otimes \psi' \mapsto \sum \rho_{g_a}(\psi)\psi_i \otimes \rho_{g_c}(\psi')\rho_{g_b g_a^{-1}}(\psi^i),$$

where $\{\psi_i\}$ is a basis of $\mathcal{O}_{bd}$.

We now have all the same verifications to make as in the non-equivariant case. They are very tedious, but are in 1-1 correspondence with what we have already done, and present nothing new. As an example of the modifications needed, let us point out that the very frequently used formula A.1 which holds in any Frobenius category when $\phi' \in \mathcal{O}_{ab}$ and ϕ^i and ϕ_i are dual bases for $\mathcal{O}_{ab}$ and $\mathcal{O}_{ba}$, generalizes — with the same proof — when there is a G-action on the category to

$$\sum \phi'\phi_i \otimes \alpha_g(\phi^i) = \sum \phi_i \otimes \alpha_g(\phi^i \phi')$$

for any $g \in G$.

We shall say no more about the proof.

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