Desperately Seeking Moonshine

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a project with Jeff Harvey

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Motivation

Search for a conceptual explanation of Mathieu Moonshine phenomena.

Eguchi, Ooguri, Tachikawa 2010

Proposal: It is related to the ``algebra of BPS states.''

Something like: M24 is a distinguished group of automorphisms of the algebra of spacetime BPS states in some string compactification using K3.

String-Math, 2014

Today's story begins in Edmonton, June 11, 2014. Sheldon Katz was giving a talk on his work with Albrecht Klemm and Rahul Pandharipande

He was describing how to count BPS states for type II strings on a K3 surface taking into account the so(4) = su(2) + su(2) quantum numbers of a particle in six dimensions.

Slide # 86 said ….

- Define the *refined K3 BPS invariants* $\mathrm{R}^{h\geq 0}_{j_L,j_R}$ by

$$\sum_{h=0}^{\infty}\sum_{j_L}\sum_{j_R}\mathrm{R}^{h}_{j_L,j_R}\,[j_L]_u[j_R]_y q^h =$$

$$\prod_{n=1}^{\infty}\frac{1}{(1-u^{-1}y^{-1}q^n)(1-u^{-1}yq^n)(1-q^n)^{20}(1-uy^{-1}q^n)(1-uyq}$$

where $[j]_x = x^{-j}+\ldots+x^{j}$

- For $h \leq 2$ the nonvanishing invariants are

$$\mathrm{R}^0_{0,0} = 1\,,$$

$$\mathrm{R}^1_{0,0} = 20\,,\quad \mathrm{R}^1_{\frac{1}{2},\frac{1}{2}} = 1\,,$$

$$\mathrm{R}^2_{0,0} = 231\,,\quad \mathrm{R}^2_{\frac{1}{2},\frac{1}{2}} = 21\,,\quad \mathrm{R}^2_{1,1} = 1$$

Heterotic/Type II Duality

Het/T4 = IIA/K3

DH states: Perturbative heterotic BPS states ⟺ D4-D2-D0 boundstates

Roughly: Cohomology groups of the moduli spaces of objects in $D^b(K3)$ with fixed K-theory invariant and stable wrt a stability condition determined by the complexified Kahler class.

Aspinwall-Morrison Theorem:

$$O_{\mathbb{Z}}(II^{20;4})\backslash O_{\mathbb{R}}(20;4)/\left(O_{\mathbb{R}}(20)\times O_{\mathbb{R}}(4)\right)$$

Heterotic Toroidal Compactifications

$$\mathbb{M}^{1,1+d} \times T^{8-d}$$

$$II^{24-d,8-d} \hookrightarrow \Gamma^{24-d;8-d} \subset \mathbb{R}^{24-d;8-d}$$

$$P = (P_L; P_R) \in \Gamma^{24-d;8-d}$$

Narain moduli space of CFT's:

$$O_{\mathbb{Z}}(II^{24-d;8-d}) \backslash O_{\mathbb{R}}(24-d;8-d) / \left(O_{\mathbb{R}}(24-d) \times O_{\mathbb{R}}(8-d)\right)$$

Crystal Symmetries Of Toroidal Compactifications

Construct some heterotic string compactifications with large interesting crystallographic group symmetries.

$$G \subset \mathrm{Aut}(\Gamma^{24-d;8-d})$$

$$G = G_L \times G_R$$

$$G_L \subset O_{\mathbb{R}}(24-d) \qquad G_R \subset O_{\mathbb{R}}(8-d)$$

Then G is a crystal symmetry of the CFT:

Example: Weyl group symmetries of enhanced YM gauge theories.

These are NOT the kinds of crystal symmetries we want

Conway Subgroup Symmetries

Start with a distinguished d=0 compactification:

$$\mathbb{M}^{1,1} \times T^8$$

$$\Gamma^{24,8} = (\Lambda; 0) \oplus (0; \Gamma_8)$$

Crystal symmetry:

$$\mathrm{Co}_0 \times W(E_8)$$

Note that Co_0 is not a Weyl group symmetry of any enhanced Yang-Mills gauge symmetry.

Now ``decompactify''

A Lattice Lemmino

$\mathfrak{F}_L \subset \Lambda$ & $\mathfrak{F}_R \subset \Gamma_8$ isometric of rank d

Then ***<u>there exists an even unimodular lattice with embedding</u>***

$$\Gamma^{24-d;8-d} \hookrightarrow \mathbb{R}^{24-d;8-d}$$

such that, if

$$G_L := \mathrm{Fix}(\mathfrak{F}_L) \subset \mathrm{Aut}(\Lambda) = \mathrm{Co}_0$$

$$G_R := \mathrm{Fix}(\mathfrak{F}_R) \subset \mathrm{Aut}(\Gamma_8) = W(E_8)$$

then $\Gamma^{24\text{-}d;8\text{-}d}$ has crystallographic symmetry

$$G_L \times G_R \subset O(24-d) \times O(8-d)$$

Easy Proof

Does not use the Nikulin embedding theorem.

Uses standard ideas of lattice theory.

See book of Miranda and Morrison for these ideas.

$$\mathcal{D}_+(\mathfrak{F}_L^\perp) \cong \mathcal{D}_-(\mathfrak{F}_L) \quad \cong \quad \mathcal{D}_-(\mathfrak{F}_R) \cong \mathcal{D}_+(\mathfrak{F}_R^\perp)$$

$$\Gamma \subset (\mathfrak{F}_L^\perp)^\vee \oplus (\mathfrak{F}_R^\perp)^\vee \subset \mathbb{R}^{24-d;8-d}$$

$$\Gamma = \{(x,y) | \bar{x} \cong \bar{y}\}$$

$$g : (x;y) \mapsto (g_L x; g_R y) \qquad \overline{g_L x} = \overline{x} \qquad \overline{g_R y} = \overline{y}$$

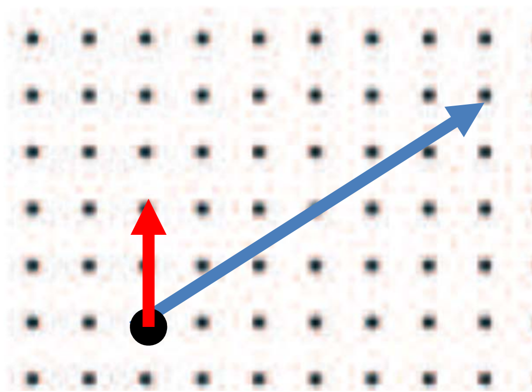
CSS Compactifications

This construction defines points of moduli space with
Conway **S**ubgroup **S**ymmetry:
call these CSS compactifications.

What crystal symmetries can you get?

In general, a sublattice preserves none of the crystal symmetries of the ambient lattice.

Consider, e.g., the lattice generated by (p,q) in the square lattice in the plane.



Fixed Sublattices Of The Leech Lattice

The culmination of a long line of work is the classification by Hohn and Mason of the 290 isomorphism classes of fixed-point sublattices of Λ

221	3	24	$[2^3 3]$ (#3)	8_3^{+3}	0	4	16	1	1	3	Mon_b*
222	2	9196830720	$U_6(2)$	$2_{II}^{-2}3^{+1}$	0	1	1	1	1	-	S*
223	2	898128000	McL	$3^{-1}5^{-1}$	1	1	1	1	1	-	S*
224	2	454164480	$2^{10}.M_{22}$	4_2^{+2}	0	1	1	1	1	-	Mon_a*
225	2	44352000	HS	$2_2^{-2}5^{+1}$	0	1	1	1	1	-	S*
226	2	20643840	$2^9.L_3(4).2$	$4_1^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a
227	2	10200960	M_{23}	23^{+1}	1	1	1	1	2	1	M_{23}*

Symmetries Of D4-D2-D0 Boundstates

99	4	245760	$[2^{12}].A_5$	$2_{II}^{-2}4_{II}^{-2}$	0	1	1	1	1	-	Mon_a*
100	4	30720	$[2^9].A_5$	$2_{II}^{-4}5^{-1}$	0	1	1	1	1	-	Mon_a*
101	4	29160	$3^4.A_6$	$3^{+2}9^{+1}$	1	1	1	1	1	-	S*
102	4	20160	$L_3(4)$	$2_{II}^{-2}3^{-1}7^{-1}$	2	1	1	1	2	1	M_{23}*
103	4	12288	$[2^{12}3]$	$2_{II}^{+2}4_3^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a
104	4	9216	$[2^{10}3^2]$	$2_{II}^{+4}3^{+2}$	0	1	2	1	1	-	Mon_a*

Therefore the space of D4D2D0 BPS states on K3 will naturally be a representation of the subgroups of Co_0 that preserve sublattices of rank 4.

These discrete groups will be automorphisms of the algebra of BPS states at the CSS points.

Symmetries Of Derived Category

Theorem [Gaberdiel-Hohenegger-Volpato]: If $G \subset O_{\mathbb{Z}}(20;4)$ fixes a positive 4-plane in $\mathbb{R}^{20,4}$ then G is a subgroup of Co_0 fixing a sublattice with $4 \leq$ rank.

Remark: GHV generalize the arguments in Kondo's paper proving Mukai's theorem that the symplectic automorphisms of K3 are subgroups of M23 with at least 5 orbits on Ω

Interpreted by Huybrechts in terms of the bounded derived category of K3 surfaces

$$G \cong \mathrm{Aut}_{H^{2,0}\oplus H^{0,2}}(D^b(K3)) \cap \mathrm{Aut}_{B+\mathrm{i}J}(D^b(K3))$$

But Is There Moonshine In KKP Invariants?

99	4	245760	$[2^{12}].A_5$	$2_{II}^{-2}4_{II}^{-2}$	0	1	1	1	1	-	Mon_a*
100	4	30720	$[2^9].A_5$	$2_{II}^{-4}5^{-1}$	0	1	1	1	1	-	Mon_a*
101	4	29160	$3^4.A_6$	$3^{+2}9^{+1}$	1	1	1	1	1	-	S*
102	4	20160	$L_3(4)$	$2_{II}^{-2}3^{-1}7^{-1}$	2	1	1	1	2	1	M_{23}*
103	4	12288	$[2^{12}3]$	$2_{II}^{+2}4_3^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a
104	4	9216	$[2^{10}3^2]$	$2_{II}^{+4}3^{+2}$	0	1	2	1	1	-	Mon_a*

$$2^{12} : A_5 \cong 2^8 : M_{20}$$

So the invariants of KKP will show ``Moonshine'' with respect to this symmetry……

But this is a little silly: All these groups are subgroups of O(20). If we do not look at more structure, that includes the momenta/characteristic classes we might as well consider the degeneracies as O(20) representations.

Silly Moonshine

$$\prod_{n=1}^{\infty}\left(\frac{1}{(1-q^n)^{20}(1-yzq^n)(1-yz^{-1}q^n)(1-y^{-1}zq^n)(1-y^{-1}z^{-1}q^n)}\right)$$

is just the SO(4) character of a Fock space of 24 bosons.

$$\iota : O(20) \times O(4) \hookrightarrow O(24)$$

$$\iota^*(V) = \mathbf{20} \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathbf{4} \qquad \mathcal{F}_q(\mathbf{20} \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathbf{4})$$

$$\mathcal{F}_q(V) := \mathrm{Sym}_q^*(V) \otimes \mathrm{Sym}_{q^2}^*(V) \otimes \cdots$$

All the above crystal groups are subgroups of O(20) so the ``Moonshine'' wrt those groups is a triviality.

IS THERE MORE GOING ON ??

Baby Case: T7 & d=1

272	2	80	$[2^4 5]$ (#34)	$4_2^{+2}5^{+2}$	0	1	8	1	1	-	Mon_a
273	1	$\lvert Co_2\rvert$	Co_2	4_1^{+1}	0	1	1	1	1	-	S*
274	1	$\lvert Co_3\rvert$	Co_3	$2_3^{-1}3^{-1}$	0	1	1	1	1	-	S*

Decompose partition function of BPS states wrt reps of transverse rotation group O(1)

$$\mathcal{F}_q(V_{23}\otimes \mathbf{T}\oplus \mathbf{1}\otimes \mathbf{S}) = \mathbf{1}\otimes \mathbf{T} \oplus q\,[V_{23}\otimes \mathbf{T}\oplus \mathbf{1}\otimes \mathbf{S}]$$

$$\oplus\, q^2\,[\mathbf{300}\otimes \mathbf{T}\oplus \mathbf{24}\otimes \mathbf{S}] \oplus q^3\,[\mathbf{2876}\otimes \mathbf{T}\oplus \mathbf{324}\otimes \mathbf{S}]$$

$\oplus \cdots$ These numbers dutifully decompose nicely as representions of Co_2:

That's trivial because $Co_2 \subset O(23)$

$$\mathbf{300} \cong S^2 V_{23} \oplus V_{23} \oplus \mathbf{1}$$

But is there a Co_0 x O(1) symmetry? Co_0 is *NOT* a subgroup of O(23). Co_0 x O(1) symmetry CANNOT come from a linear action on V_{24}.

The SumDimension Game

$$\mathrm{Irrep}(\mathrm{Co}_0) = \{\mathbf{1}, \mathbf{24}, \mathbf{276}, \mathbf{299}, \mathbf{1771}, \mathbf{2024}, \mathbf{2576}, \mathbf{4576},$$

$$\mathbf{1} \otimes \mathbf{T} \;\oplus\; q\,[V \otimes \mathbf{T} \oplus \mathbf{1} \otimes \mathbf{S}] \oplus q^2\,[\mathbf{300} \otimes \mathbf{T} \oplus \mathbf{24} \otimes \mathbf{S}] \;\oplus$$
$$q^3\,[\mathbf{2876} \otimes \mathbf{T} \oplus \mathbf{324} \otimes \mathbf{S}] \;\oplus \cdots$$

$$\mathbf{300} = \mathbf{299} + \mathbf{1} \qquad \mathbf{300} = \mathbf{276} + \mathbf{24}$$

$$\mathbf{2876} = \mathbf{2576} + \mathbf{299} + \mathbf{1} \qquad \mathbf{2876} = \mathbf{2576} + \mathbf{276} + \mathbf{24}$$

$$\mathbf{324} = \mathbf{299} + \mathbf{24} + \mathbf{1} \qquad \mathbf{324} = \mathbf{276} + \mathbf{24} + \mathbf{24}$$

ETC.

Defining Moonshine

Any such decomposition defines the massive states of $\mathcal{F}_q(V)$ as a representation of Co_0 x O(1).

Problem: There are infinitely many such decompositions! What physical principle distinguishes which, if any, are meaningful?

Definition: You have committed ***Moonshine*** (for d=1) if you exhibit the massive sector of $\mathcal{F}_q(V)$ as a representation of Co_0 x O(1)such that the graded character of any element g:

$$\mathrm{Tr}_{\mathcal{F}_q(V)} g q^{L_0 - 1}$$

is a modular form for $\Gamma_0(m)$ where m = order of g .

Virtual Representations

Most candidate $Co_0 \times O(1)$ representations will fail to be modular.

But if we allow *virtual* representations:

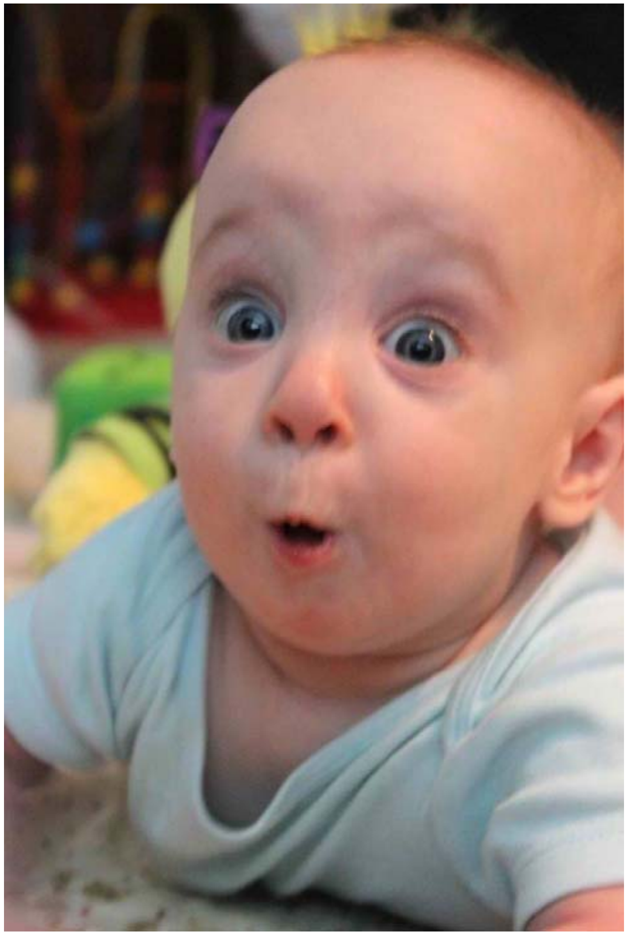
$$V_{23} \to V_{24} - \mathbf{1}$$

$$\mathcal{F}_q(V_{23} \otimes \mathbf{T} \oplus \mathbf{1} \otimes \mathbf{S}) \to \mathcal{F}_q(V_{24}) \otimes \frac{\mathcal{F}_q(\mathbf{1} \otimes \mathbf{S})}{\mathcal{F}_q(\mathbf{1} \otimes \mathbf{T})}$$

The characters are guaranteed to be modular!

But massive representations have no reason to be true representations.

In fact, the negative representations cancel and ALL the massive levels are in fact true representations!!



But! The same argument also shows they are also true representations of O(24) x O(1).

Lessons

Modularity of characters is crucial.

Virtual Fock spaces are modular.

There can be nontrivial cancellation of the negative representations.

A ``mysterious'' discrete symmetry can sometimes simply be a subgroup of a continuous symmetry.

What About d=4 ?

$$\mathrm{Co}_0 \times O(1) \to M_{24} \times O(4)$$

$$\mathcal{F}_q(V_{20} \otimes \mathbf{1} \oplus \mathbf{1} \otimes V_4) \to \mathcal{F}_q(V_{23}) \otimes \frac{\mathcal{F}_q(\mathbf{1} \otimes V_4)}{\mathcal{F}_q(\mathbf{1} \otimes \mathbf{1})^3}$$

Magical positivity fails:

$$\dim \mathrm{R}^2_{0,0} = 231$$

$$\mathrm{R}^2_{0,0} = V_{252} - V_{23} + 2V_1$$

But we are desperately seeking Moonshine…

<u>So we ask</u>: Could it still be that, magically, <u>some</u> positive combination of representations from the SumDimension game is nevertheless modular?

Characters Of An Involution

$$\begin{aligned} Z_{2A} &= 8 + 1/q + 36q + 144q^2 + 282q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 426q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 218q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 362q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 266q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 410q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 202q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 346q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 378q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 522q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 314q^3 \\ &= 8 + 1/q + 36q + 144q^2 + 458q^3 \end{aligned}$$

Should be modular form for $\Gamma_0(2)$.

Weight?

(assumed half-integral)

Multiplier system?

What Is Your Weight?

$$\tau_0 = \frac{1}{2}(1+\mathrm{i}) \qquad (ST^2S)\tau_0 = \tau_0 - 1$$

One can deduce the multiplier system from the weight, and derive the weight numerically from:

$$w = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \log\left|\frac{Z(\tau_0 - \mathrm{i}\epsilon)}{Z(\tau_0+\mathrm{i}\epsilon)}\right|$$

$$q_0 = e^{2\pi \mathrm{i}\tau_0} = -e^{-\pi} = -0.04...$$

Convergence is good so can compute the weight numerically.
For Z_{2A} it converges to -8.4...... Not pretty. Not half-integral !!

No positive combination of reps is modular. No M24 Moonshine.

Application To Heterotic-Type II Duality

Existence of CSS points have some interesting math predictions.

$\mathfrak{X}$ K3 and ellipticaly fibered CY3

$$\mathrm{Het}/T^2 \times K3' \quad \Longleftrightarrow \quad \mathrm{IIA}/\mathfrak{X}$$

Perturbative heterotic string states $\Longleftrightarrow$ Vertical D4-D2-D0 boundstates

Orbifolds At CSS Points

Orbifold at the CSS points for d=2 (T^6 compactification)

For simplicity: $\mathbb{Z}_2$ orbifold

$$X \to RX + \delta$$

$$R = (g_L; g_R) \in G_L \times G_R$$

g_R an involution in W(E8) with eigenvalues $-1^4, +1^4$

We can realize all but 6 of the 51 rank 2 HM classes:

(q_{11}, q_{22}, q_{12})	HM Number	σ_1 ?	σ_2 ?
$(4,4,0)$	224	Y	
$(4,4,1)$	223	N	N
$(4,4,2)$	222	Y	
$(4,6,0)$	229	Y	
$(4,6,1)$	227	N	N
$(4,6,2)$	225	Y	
$(4,8,0)$	226, 236	Y	
$(4,8,2)$	232	N	N
$(4,10,2)$	241	Y	
$(4,12,0)$	234	Y	
$(4,12,2)$	233	Y	
$(4,16,0)$	250	Y	
$(4,16,2)$	237	N	N
$(4,20,0)$	257	Y	
$(4,24,0)$	244	Y	
$(6,6,0)$	228, 242	Y	
$(6,6,2)$	235	Y	
$(6,6,3)$	230	Y	
$(6,8,2)$	251	Y	
$(6,10,2)$	243	Y	
$(6,12,0)$	253	Y	
$(6,12,3)$	240	N	N
$(6,18,0)$	263	Y	
$(8,8,0)$	238, 258	Y	
$(8,8,2)$	246	N	N
$(8,8,4)$	231, 252	Y	
$(8,12,0)$	248	Y	
$(8,12,4)$	239, 249	Y	
$(8,16,0)$	269	Y	
$(8,20,4)$	266	Y	
$(10,10,0)$	262	Y	
$(10,10,2)$	259	Y	
$(10,12,6)$	256	Y	
$(12,12,0)$	247, 254, 261, 271	Y	
$(12,12,2)$	255	Y	
$(12,12,6)$	245, 260	Y	
$(12,18,6)$	265	Y	
$(12,24,0)$	270	Y	
$(20,20,0)$	272	Y	
$(20,20,10)$	264	Y	
$(24,24,12)$	267, 268	Y	

Explicit Example – 1/2

Gram matrix of $\mathfrak{F}_R = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Can embed in Leech and E8 lattices

There exists g_R in W(E8) fixing $\mathfrak{F}_R$ with ev's $+1^4$, -1^4. Mod out by this on the right.

Need to choose involution g_L.

Structure of Golay code implies the only possibilities:

$$g_L \sim \mathrm{Diag}\{-1^8, +1^{16}\} \quad g_L \sim \mathrm{Diag}\{-1^{12}, +1^{12}\} \quad g_L \sim \mathrm{Diag}\{-1^{16}, +1^8\}$$

Flipping 8,12,16 coordinates x^i according to octad/dodecad/ octad complement $\mathcal{C}$-sets of the Golay code

Explicit Example – 2/2

Massless sector:

Abelian gauge symmetry.

	$VM = h^{1,1}(\mathfrak{X})$	$HM = h^{2,1}(\mathfrak{X}) + 1$	$\chi = 2(1 + VM - HM)$
A_1	15	$256 + 8$	-496
A_2	15	$64 + 8$	-112
B	7	528	-1040
C	11	12	0

Personal Ad: Two Hodge numbers seek friendly compatible CY3…

(7,527)? Huh? $\chi(\mathfrak{X}_B) = -1040$

TaxMan violates Wati's bound: $h^{1,2}(\mathfrak{X}) \leq 491$

Generalized Huybrechts Theorem

$$\mathrm{Aut}_\sigma(D^b(\mathfrak{X})_{\mathrm{vertical}})$$

Should be the centralizer of an element of a subgroup of Co_0 fixing a rank two sublattice of Λ

Some General Questions

Clarify TaxMan example: Should every heterotic model on K3 x T2 have a type II dual?

Decreasing d $\Longrightarrow$ larger CSS groups: raises a general question about D-brane categories: What to do on $\mathfrak{X} \times S^1$?

There ARE A & B Models!

In the type II interpretation CSS only arise for special values of the flat RR fields. How do flat RR fields affect the BPS D-brane states?

$\langle X |$  $\rangle = e^{ip\cdot x}$

$p \in \Gamma^{20,4}$

HAPPY BIRTHDAY DAVE!!