WALL-CROSSING FORMULA FOR BPS STATES & SOME APPLICATIONS

CLAY WORKSHOP ON K3 & MODULAR FORMS
MARCH 20, 2008

BASED ON WORK DONE WITH
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1. INTRODUCTION

THE "SPACE OF BPS STATES" HAS BEEN A CENTRAL CONCEPT IN SUSY GAUGE THEORY & STRING THEORY FOR ALMOST 30 YEARS.

TODAY I'LL FOCUS ON RECENT PROGRESS IN UNDERSTANDING PHENOMENA ASSOCIATED TO MARGINAL STABILITY.

1. INTRODUCTION FOR MATHEMATICIANS

2. WALL-CROSSING FORMULAE:

PRIMITIVE, SEMI-PRIMITIVE, KONTSEVICH-SOIBELMAN

3. PHYSICAL DERIVATION

4. D6D2D0 SYSTEM: IDEAL SHEAVES & DT

5. D4D2D0 SYSTEM: MODULAR GEN. FUNCTIONS

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6. OSV

7. PROBLEMS AT WEAK COUPLING

8. CONCLUDE

A. DEFINING THE "SPACE OF BPS STATES"

FOR DEFINITENESS, WE FOCUS ON THEORIES WITH $d=4$, $\mathcal{N}=2$ SUSY IN (ASYMPTOTIC) MINKOWSKI SPACE M_4

HILBERT SPACE OF ONE-PARTICLE STATES, $\mathcal{H}$, IS A REP. OF THE $d=4$, $\mathcal{N}=2$ ALGEBRA.

$\hat{Z}$: CENTRAL CHARGE OPERATOR

$$\{\hat{Q}_{i\alpha}, \hat{Q}_{j\beta}\} = \delta_{ij} (C\Gamma^{\mu})_{\alpha\beta} \hat{P}_{\mu} + \epsilon_{ij} C_{\alpha\beta} \hat{Z}$$

DECOMPOSE $\mathcal{H} = \bigoplus_{z \in \mathbb{C}} \mathcal{H}_{\hat{Z} = z}$

LEMMA: $E \geq |z|$ ON $\mathcal{H}_z$

$$\boxed{\text{DEF'N: } \mathcal{H}_{BPS} \text{ IS THE SUBSPACE OF } \mathcal{H} \text{ WHERE } E = |z|.}$$

NOW - SPECIALIZE TO TYPE II STRING THEORY ON $M_4 \times X$.

- M_4 IS NONCOMPACT $\Rightarrow$ TO DEFINE THE HILBERT SPACE AS A REP. OF $N=2$ WE MUST SPECIFY BOUNDARY COND'S FOR THE MASSLESS FIELDS:

$$\lim_{\vec{x} \to \infty} \left(g_{\mu\nu}, \phi, B_{\mu\nu}, RR \right) := \Phi_\infty \in \tilde{\mathcal{M}}$$

$\mathcal{H}_{\Phi_\infty}$: 1-PARTICLE HILBERT SPACE DEPENDS ON Φ_∞

- GENERALIZED MAXWELL THEORY $\Rightarrow$ $\mathcal{H}_{\Phi_\infty}$ IS GRADED BY ELECTRIC/MAGNETIC CHARGE SECTORS:

$$\mathcal{H}_{\Phi_\infty} = \bigoplus_{\Gamma} \mathcal{H}^{\Gamma}_{\Phi_\infty}$$

$\Gamma \in$ (TWISTED) K-THEORY(X)

MOD TORSION: $\Gamma \in \Lambda$, A SYMPLECTIC LATTICE

NOW WE PUT THESE THINGS TOGETHER:

CONSIDER IIA STRINGS WITH

1. X = STATIC, COMPACT, CY 3-FOLD
2. FLAT B-FIELD: $B \in H^2(X, \mathbb{R})$
3. FLAT RR FIELDS

$\Longrightarrow$ $\mathcal{N}=2$, $d=4$ SUGRA

- EACH $\mathcal{H}^{\Gamma}_{\Phi_\infty}$ IS A REP OF $\mathcal{N}=2$
- CENTRAL CHARGE $\boxed{Z = Z(\Gamma; \Phi_\infty)}$

SO, WE STUDY THE BPS SPECTRUM

$$\mathcal{H}_{BPS} = \bigoplus_{\Gamma \in K^0(X)} \mathcal{H}^{\Gamma}_{\Phi_\infty, BPS}$$

FINITE DIMENSIONAL

B. DEPENDENCE ON MODULI

THE SPACES $\mathcal{H}^{\Gamma}_{\Phi_\infty, BPS}$ ARE LOCALLY CONSTANT BUT <u>NOT</u> GLOBALLY CONSTANT AS FUNCTIONS OF Φ_∞

MODULI SPACE $\widetilde{\mathcal{M}}$ IS A PRODUCT:

HYPERMULTIPLETS × VECTORMULTIPLETS

[CPLX STR., ϕ, RR FIELDS] [COMPLEXIFIED KÄHLER]

WE WORK AT A GENERIC HYPERMULTIPLET.

RECENT PROGRESS HAS BEEN CONCERNED WITH THE DEPENDENCE ON VECTORMULTIPLETS, IN THIS TALK,

$$\boxed{t = B + iJ}$$

- THE JUMPING LOCUS IS REAL CODIMENSION ONE

DEFINE AN INDEX

$$\Omega(\Gamma;\Phi_\infty) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}^{\Gamma}_{\Phi_\infty,BPS}} (2J_3)^2(-1)^{2J_3}$$

- TECHNICAL POINT:

$$\mathcal{H}^{\Gamma}_{\Phi_\infty,BPS} = \underbrace{\mathcal{H}_{\frac{1}{2}HM}}_{\text{1/2 hyper}} \otimes \mathcal{H}(\Gamma,t_\infty)$$

$2(0)+(\frac{1}{2})$ as spin rep$^{\underline{n}}$

$$\Omega(\Gamma;t_\infty) = \mathrm{Tr}_{\mathcal{H}(\Gamma,t_\infty)}(-1)^F$$

HENCEFORTH FOCUS ON $\mathcal{H}(\Gamma;t_\infty)$

- CAN ALSO STUDY "HODGE POLYNOMIAL"

$$\mathrm{Tr}_{\mathcal{H}(\Gamma,t)}(-x)^{J_3+R}(-y)^{J_3-R}$$

- KEY POINT: Ω CHANGES ACROSS WALLS OF MARGINAL STABILITY

C. WHY DO WE CARE?

$\boxed{\text{PHYSICS MOTIVATION}}$

1. THE MAIN MOTIVATION FOR RECENT WORK IS THE PROGRAM, INITIATED BY STROMINGER–VAFA (1995) OF ACCOUNTING FOR BH ENTROPY VIA MICROSTATE COUNTING. THAT GOAL IS STILL NOT FULLY ACCOMPLISHED.

> WE DON'T KNOW BPS DEGENERACY FOR CERTAIN NATURAL CHARGE REGIMES, FOR EXAMPLE:
>
> $$\Gamma \longrightarrow \lambda\Gamma \qquad \lambda \longrightarrow \infty$$

2. OSV CONJECTURE:

RELATION BETWEEN

$$\Omega(\Gamma) \quad \& \quad \text{GW / DT / GV INVARIANTS}$$

$\Longrightarrow$ NONPTVE TOPOLOGICAL STRING?

MATH MOTIVATION

1. PHYSICAL STABILITY OF BPS STATES IS RELATED TO MATH. STABILITY IN THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES ON A C.Y.: KONTSEVICH, DOUGLAS, BRIDGELAND, THOMAS, PANDHARIPANDE ....

PHYSICS $\Rightarrow$ PREDICTIONS/CONSTRAINTS ON WHAT WE EXPECT SHOULD BE TRUE.

2. MANY INTERESTING CONNECTIONS TO AUTOMORPHIC FORMS AND ANALYTIC NUMBER THEORY; SOME RELATIONS TO ARITHMETIC CY'S.

3. THERE ARE SEVERAL OTHER MORE SPECULATIVE APPLICATIONS, E.G. BPS ALGEBRAS: GENERALIZING NAKAJIMA'S WORK AND SUGGESTED BY TYPE II/HET DUALITY SHOULD BE CLOSELY RELATED.

2. WALL-CROSSING FORMULAE: STATEMENT

$N=2$, $d=4$ Algebra $\Rightarrow$

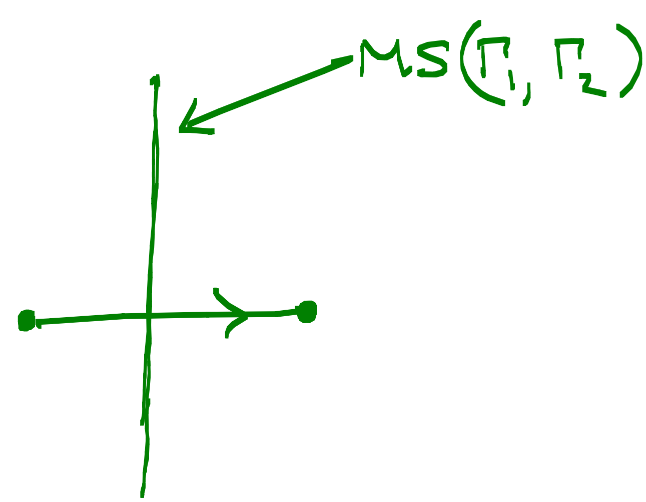
- MODULI OF VACUA $\widetilde{\mathcal{M}}$
- LATTICE OF ELECTRIC/MAGNETIC CHARGES Λ
- Central Charge $Z: \Lambda \times \widetilde{\mathcal{M}} \longrightarrow \mathbb{C}$

WALLS WHERE $\mathcal{H}_{BPS}$ <u>MIGHT</u> JUMP

$$MS(\Gamma_1, \Gamma_2) := \{ t \mid Z(\Gamma_1, t) = \lambda Z(\Gamma_2, t),\ \lambda \in \mathbb{R}_+ \}$$

CECOTTI, INTRILIGATOR, VAFA ; SEIBERG & WITTEN:

A BOUNDSTATE OF PARTICLES WITH CHARGES Γ_1, Γ_2 CAN DECAY

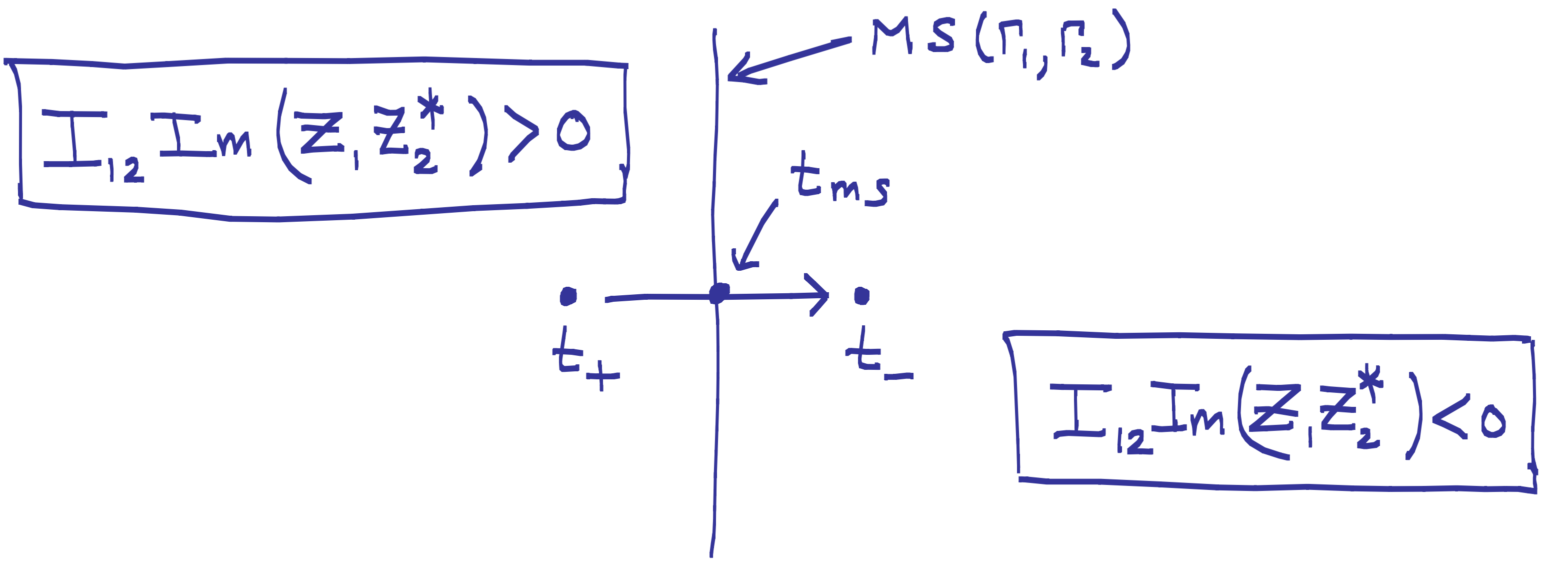


WE WANT TO SAY HOW MANY STATES DECAY.

PRIMITIVE WALL-CROSSING FORMULA:

Λ HAS SYMPLECTIC FORM $\langle \cdot , \cdot \rangle$

LET $I_{12} := \langle \Gamma_1, \Gamma_2 \rangle$



Γ_1, Γ_2 PRIMITIVE, t_{ms} GENERIC $\Longrightarrow$

$$\boxed{\mathcal{H}_+ - \mathcal{H}_- = (J_{12}) \otimes \mathcal{H}(\Gamma_1 ; t_{ms}) \otimes \mathcal{H}(\Gamma_2 ; t_{ms})}$$

$$J_{12} = \frac{1}{2}\left(|I_{12}| - 1 \right)$$

$$\Delta \Omega = (-1)^{I_{12}-1} |I_{12}| \, \Omega(\Gamma_1, t_{ms}) \Omega(\Gamma_2, t_{ms})$$

SEMI-PRIMITIVE WALL-CROSSING FORMULA

IN ADDITION TO $\Gamma_1+\Gamma_2$ BOUNDSTATES WE CAN ALSO FORM $N_1\Gamma_1+N_2\Gamma_2$ BOUNDSTATES

$$MS(\Gamma_1,\Gamma_2) = MS(N_1\Gamma_1, N_2\Gamma_2) \quad N_1,N_2\in\mathbb{Z}_+$$

CONSIDER $N_1=1$, $N_2\geq 1$:

$$\bigoplus_{N_2} u^{N_2}\,\Delta\mathcal{H}\Big|_{\Gamma\to\Gamma_1+N_2\Gamma_2}$$

CLAIM: THIS IS A $\mathbb{Z}_2$-GRADED FOCK SPACE

$$\mathcal{H}(\Gamma_1;t_{ms})\bigotimes_{k=1}^{\infty}\mathcal{F}\Big(u^k\underbrace{\left(J_{\Gamma_1,k\Gamma_2}\right)\otimes\mathcal{H}(k\Gamma_2;t_{ms})}_{\text{GRADED SPACE OF OSCILLATORS}}\Big)$$

IN PARTICULAR:

$$\boxed{\begin{aligned}&\Omega_1+\sum_{N>0}u^N\Delta\Omega(\Gamma_1+N\Gamma_2) = \\ &=\Omega(\Gamma_1)\prod_{k>0}\left(1-(-1)^{\langle\Gamma_1,k\Gamma_2\rangle}u^k\right)^{|\langle\Gamma_1,k\Gamma_2\rangle|\,\Omega(k\Gamma_2)}\end{aligned}}$$

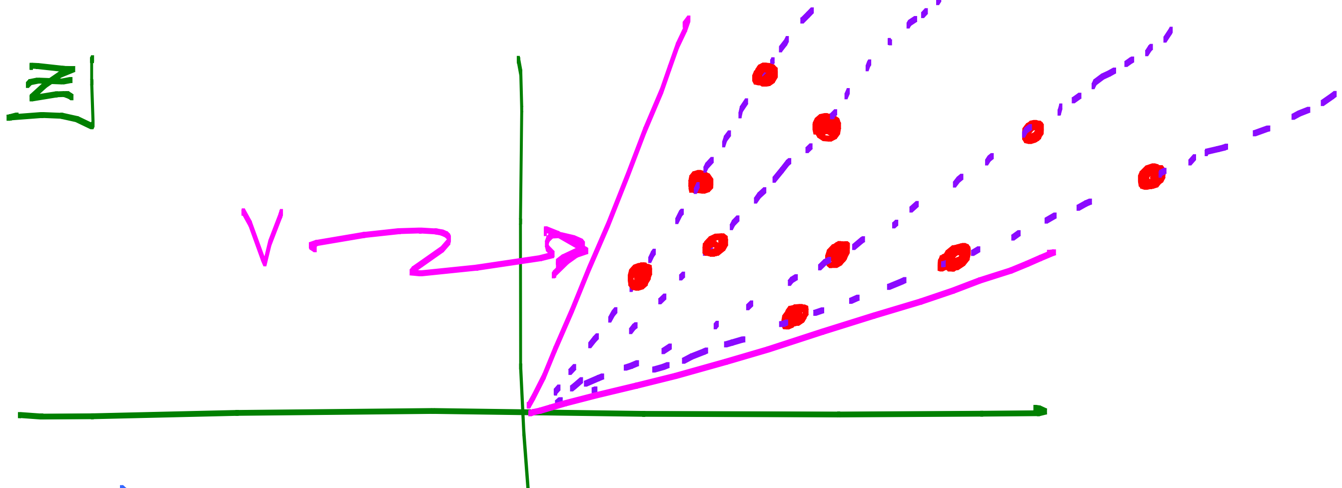
THE KONTSEVICH–SOIBELMAN FORMULA

FOR THE LATTICE Λ OF CHARGES INTRODUCE A LIE ALGEBRA $\mathbb{Z}[\Lambda]$ WITH ONE GENERATOR FOR EACH $\gamma \in \Lambda$:

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2}$$

@ FIXED t, $Z: \Lambda \longrightarrow \mathbb{C}$,

CHOOSE ANY CONVEX ANGULAR SECTOR V



$$\overrightarrow{\prod_{\gamma \in Z^{-1}(V)\cap\Lambda}} \left(\exp \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2}\right)^{\Omega^-(\gamma)} \quad \text{INCREASING SLOPE}$$

$$= \overleftarrow{\prod_{\gamma \in Z^{-1}(V)\cap\Lambda}} \left(\exp \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2}\right)^{\Omega^+(\gamma)} \quad \text{DECREASING SLOPE}$$

3. PHYSICAL DERIVATION OF WCF

A. SUPERGRAVITY TOOLS

D-BRANES ARE OBJECTS IN A CATEGORY

IN TYPE IIA/CY, THE SUBCATEGORY OF SUSY BRANES IS PROBABLY THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES.

BUT WE WANT TO DESCRIBE THE (PHYSICALLY) STABLE OBJECTS.

AT WEAK STRING COUPLING, AND $J \to \infty$ $\exists$ A BEAUTIFUL DESCRIPTION OF STABLE BPS STATES USING SUGRA.

IN THE SEMICLASSICAL LIMIT

$\psi \in \mathcal{H}_{BPS} \rightsquigarrow$ BPS SOLUTION OF SUGRA EQUATIONS

* SUPERGRAVITY ALLOWS ONE TO IDENTIFY MANY "STABLE OBJECTS" THANKS TO THE ATTRACTOR MECHANISM.

ATTRACTOR MECHANISM: (F.K.S. ; STROMINGER)

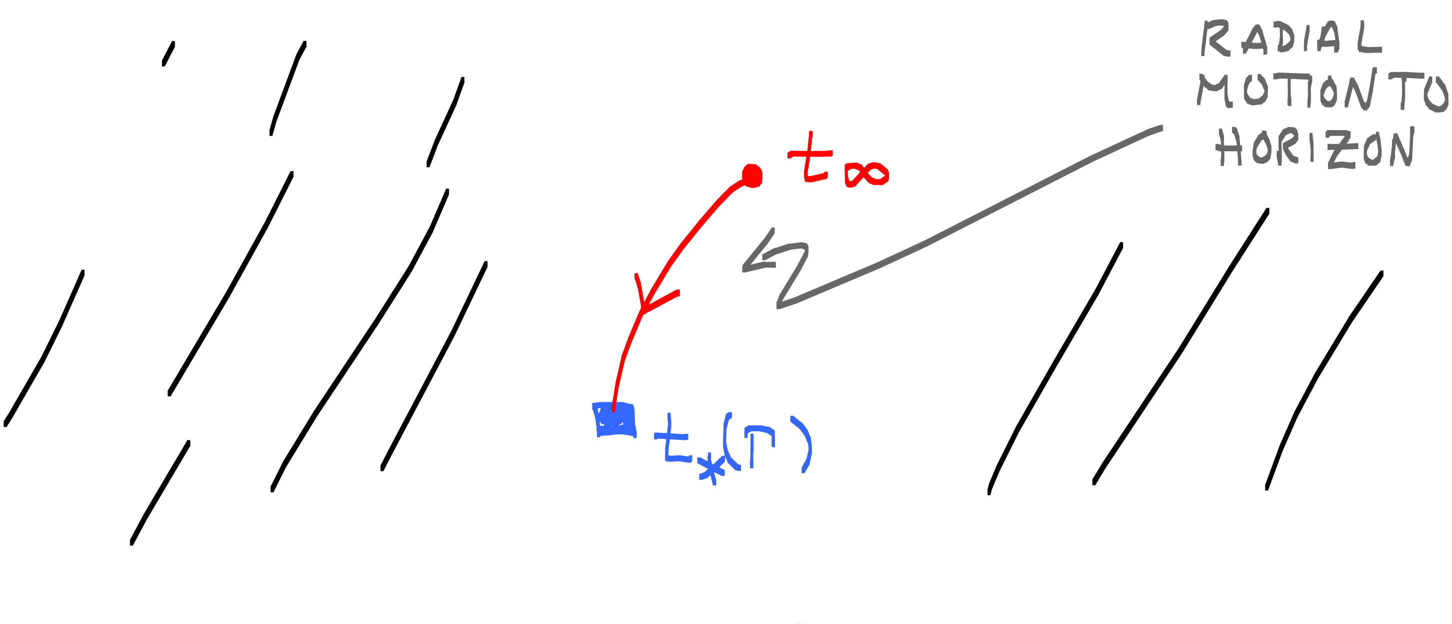
Γ, t_∞ & SPHERICAL SYMMETRY

$\Longrightarrow$ $\exists$ AT MOST ONE BPS SOLUTION OF SUGRA.

IF IT EXISTS....

SCALAR FIELDS $t = t(r)$, AND EVOLUTION FROM $r=\infty$ to $r=0$ APPROACHES AN ATTRACTIVE FIXED POINT $t_*(\Gamma)$:

$\widetilde{\mathcal{M}}_{VM}$



ATTRACTOR FLOW = GRADIENT FLOW FOR

$$\log |Z(\Gamma;t)|^2 \Rightarrow$$

BASIC TRICHOTOMY

1. $t_*(\Gamma) \in \mathrm{Interior}(\widetilde{\mathcal{M}})$

and $Z(\Gamma; t_*(\Gamma)) \neq 0$

"REGULAR ATTRACTOR POINT"

2. $\exists$ NONEMPTY SUBVARIETY $\subset \widetilde{\mathcal{M}}$

$$Z(\Gamma;t) = 0$$

3. $t_*(\Gamma) \in \partial \widetilde{\mathcal{M}}$

(1.) $\exists$ SPHERICALLY SYMMETRIC BPS BLACK HOLES IN $\mathcal{H}_{BPS}(\Gamma;t)$ FOR ALL t

(2.) $\mathcal{H}_{BPS}(\Gamma;t) = \phi$ IN AN OPEN REGION OF THE ZERO LOCUS.

$\mathcal{H}_{BPS}$ MIGHT BE NONEMPTY FURTHER AWAY

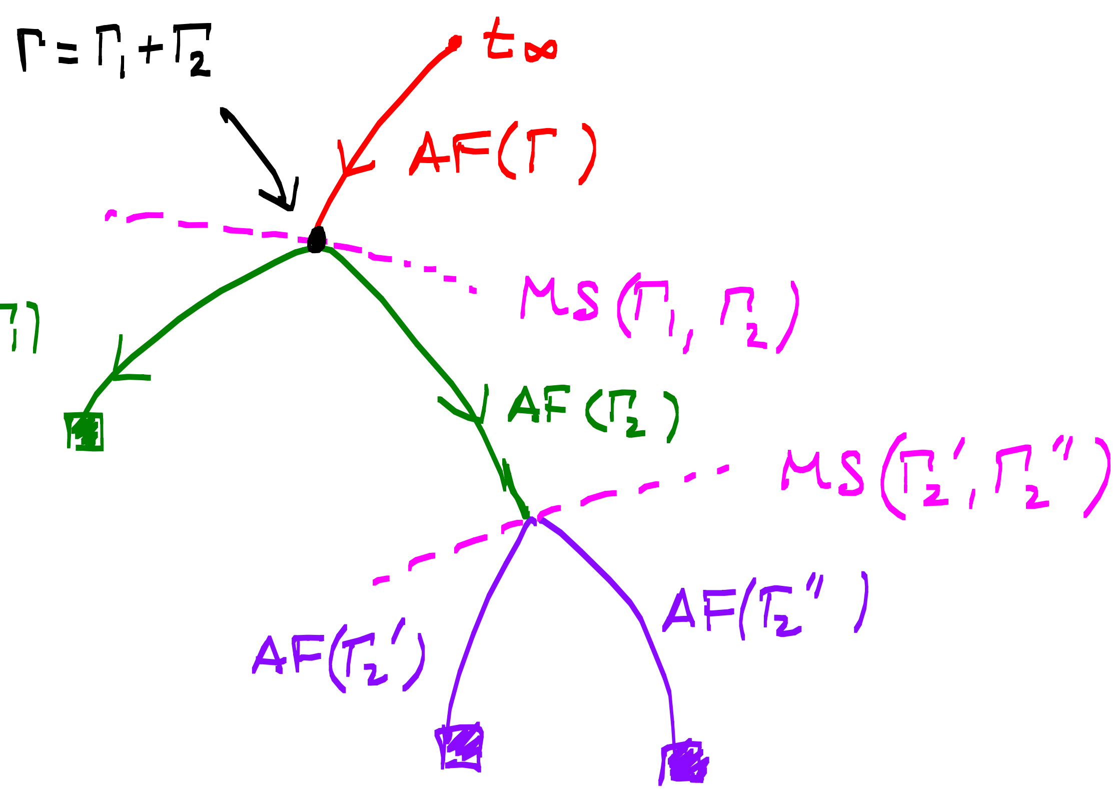
(3.) CANNOT USE SUGRA TO ESTABLISH EXISTENCE: MUST USE MICROSCOPIC ARGUMENTS.

SPLIT ATTRACTOR FLOWS

IF $Z(\Gamma;t)=0$ HAS SOLUTIONS IN THE INTERIOR OF MODULI SPACE THEN USE:

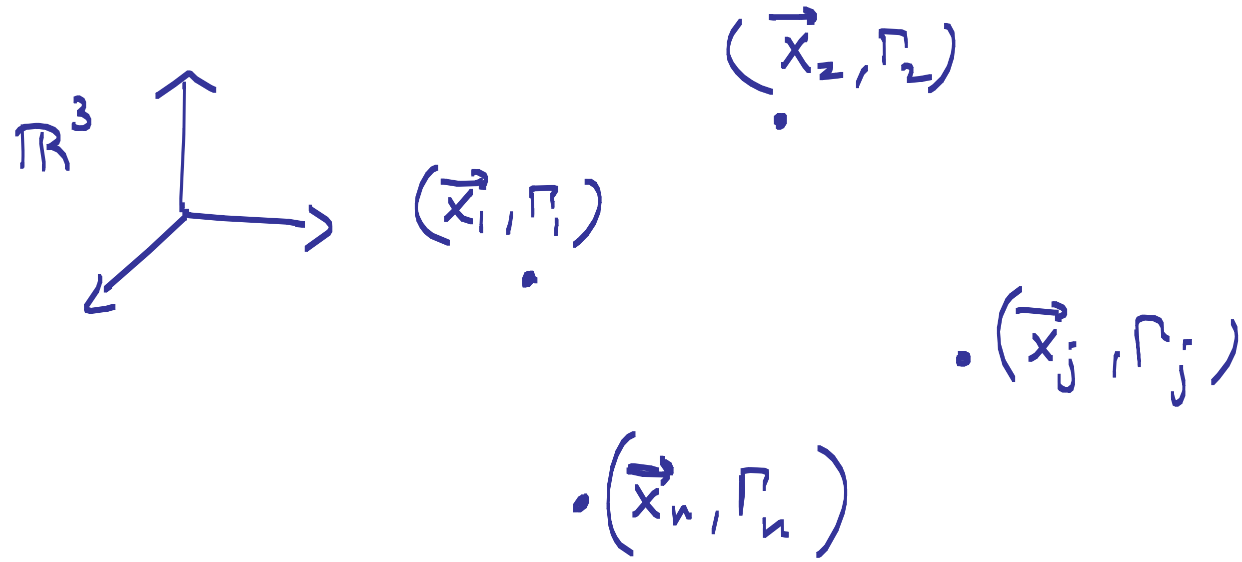
DENEF'S RULE: $\mathcal{H}(\Gamma;t)\neq 0 \Longleftrightarrow$ $\exists$ A SPLIT ATTRACTOR FLOW (S.A.F.)

S.A.F.: A PIECEWISE ATTRACTOR FLOW, JOINED ALONG WALLS OF M.S., CONSERVING CHARGE AT THE VERTICES, TERMINATING ON R.A.P.'S:



- IF SUCH ATTRACTOR FLOW TREES EXIST WE CAN CONSTRUCT A CORRESPONDING SOLUTION OF SUGRA.

- SPACETIME PICTURE:



- NEAR EACH POINT $\vec{x}_i$ THE SOLUTION LOOKS LIKE THE SINGLE-CENTERED SOLUTION: "BLACK-HOLE MOLECULES"

"INTEGRABILITY CONDITION"

$$\boxed{\forall i: \quad \sum_{j \neq i} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|} = 2\, \mathrm{Im}\left(\frac{Z_i \bar{Z}}{|Z|} \right)\Bigg|_{t\infty}}$$

B. DERIVATION OF PRIMITIVE WCF:

CONSIDER BOUNDSTATE OF TWO PRIMITIVE CHARGES:

$$\Gamma_1 \bullet \xrightarrow{\vec{R}} \bullet\, \Gamma_2$$

$$R = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\operatorname{Im}(Z_1 \bar{Z}_2)_\infty}$$

- NOTE: $\langle \Gamma_1, \Gamma_2 \rangle \operatorname{Im}(Z_1 \bar{Z}_2)_\infty > 0$

- NOTE THAT BY CHANGING t_∞ WE CAN MAKE $\operatorname{Im}(Z_1 \bar{Z}_2)\big|_{t_\infty} \to 0$ WHILE $|Z_1 + Z_2|_{t_\infty} \neq 0$

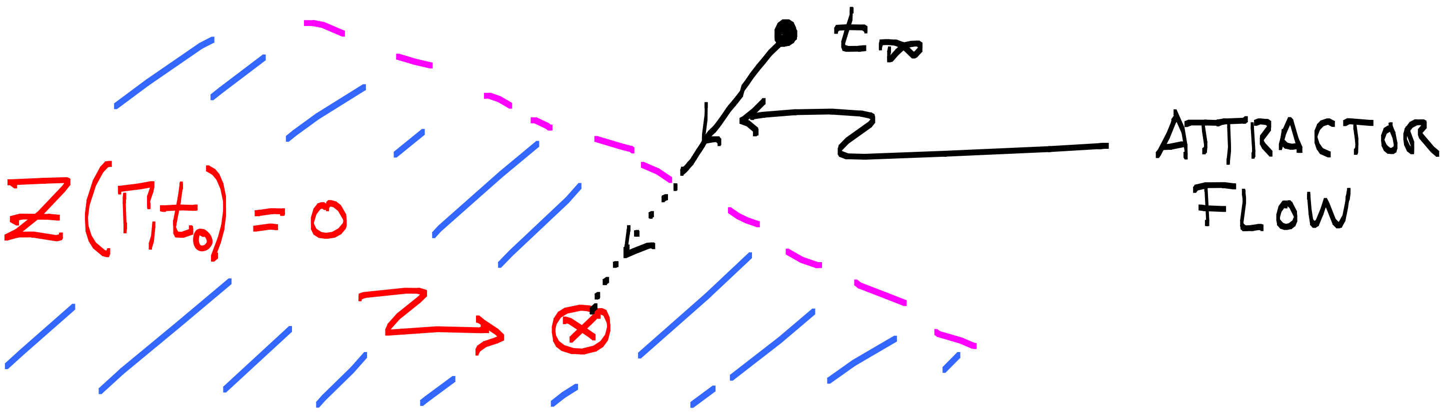
ILLUSTRATES THE KEY POINT OF MARGINAL STABILITY:

$$MS(\Gamma_1,\Gamma_2) := \left\{ t \in \mathcal{M}_{VM} \;\middle|\; \frac{Z_1}{Z_2} \in \mathbb{R}_+ \right\}$$

CHANGE BC's @ $r=\infty \Longrightarrow$

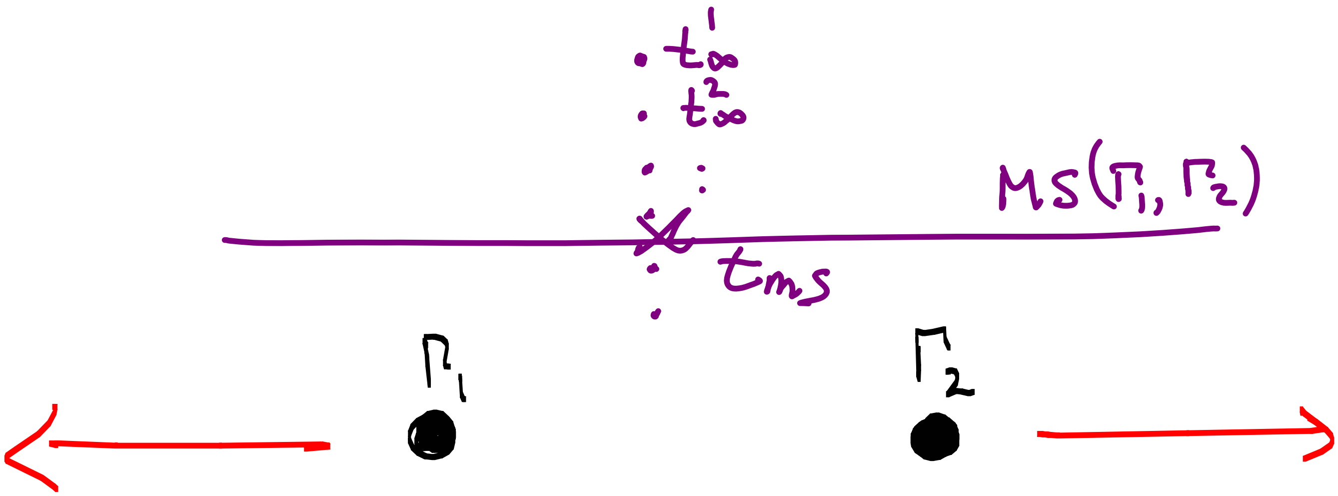
$R_{12} \longrightarrow \infty$

IF $Z(\Gamma;t)$ HAS A ZERO THEN THERE IS NO BOUNDSTATE OF TYPE $\Gamma_1+\Gamma_2$ IN THE BLUE REGION.



MACROSCOPIC ARGUMENT FOR WCF:

$$R_{12} = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\mathrm{Im}(Z_1 \bar{Z}_2)_\infty}$$



ELECTROMAGNETIC FIELD OF TWO DYONS HAS SPIN:

$$J_{12} = \frac{1}{2}\left(|\langle \Gamma_1, \Gamma_2 \rangle| - 1 \right)$$

quantum correction

LOCALITY $\Rightarrow$ FOR Γ_1, Γ_2 PRIMITIVE:

STATES LOST FROM $\mathcal{H}(\Gamma; t_\infty)$ ARE

$$(J_{12}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms})$$

MICROSCOPIC ARGUMENT FOR WCF:

WHEN $\vartheta = \arg Z_2/Z_1 \to 0$, MODEL LIGHT D.O.F. BY A QUIVER GAUGE THRY:

$$\Gamma_1 \quad \overset{d=1}{\bullet} \; \underset{n_-}{\overset{n_+}{\rightleftarrows}} \; \overset{d=1}{\bullet} \quad \Gamma_2$$

TRANSLATION TO SUPERGRAVITY:

STABILITY DATA: $(\vartheta, -\vartheta)$

$$n_+ - n_- = I_{12}$$

GENERICALLY $n_+ = 0$ or $n_- = 0$.

SUPPOSE $n_- = 0$:

$$\vartheta > 0 \qquad \mathcal{M} = \mathbb{C}P^{n_+ - 1}$$

$$\vartheta < 0 \qquad \mathcal{M} = \emptyset$$

$$\Delta \mathcal{H} = H^*\left(\mathbb{C}P^{n_+ - 1}\right)$$

spin(3) $\approx$ Lefshetz

QUIVER QUANTUM MECHANICS

0+1 SUSY QED WITH

1 VM $(A_0, \vec{x}, \lambda)$

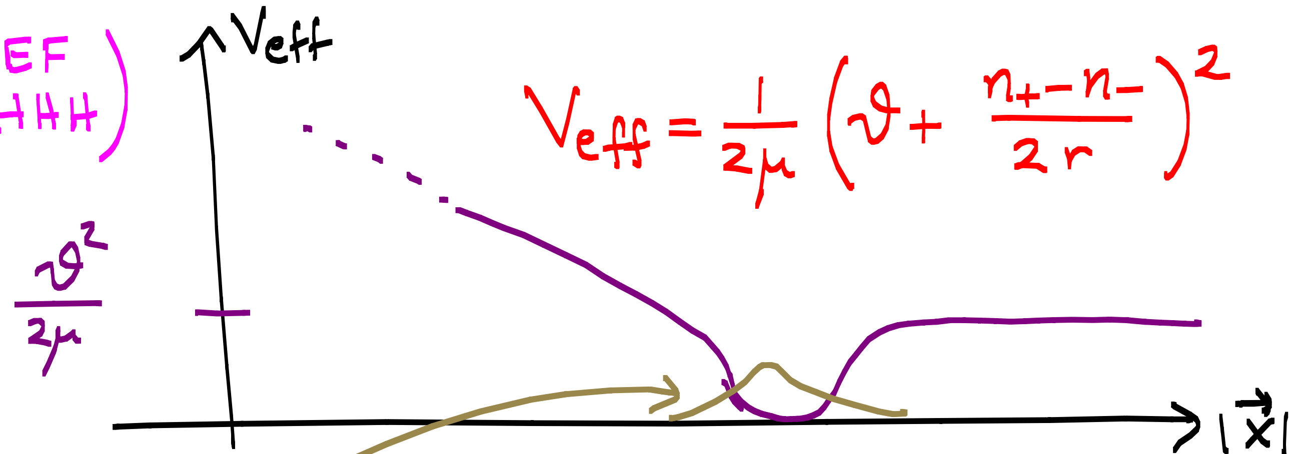
$n_\pm$ CM's $(\Phi_\pm, \Psi_\pm)$ CHARGE ± 1

SMALL $|\langle \vec{x} \rangle| \Rightarrow$ HIGGS BRANCH = MODULI OF STABLE QUIVER REP'S

LARGE $|\langle \vec{x} \rangle| \Rightarrow$ INTEGRATE OUT $\vec{\Phi}_\pm \Rightarrow$

(DENEF QQHHH)

$$V_{eff} = \frac{1}{2\mu}\left(\vartheta + \frac{n_+ - n_-}{2r}\right)^2$$



$(n_+ - n_-)$ BPS STATES OF SPIN $\frac{1}{2}(n_+ - n_- - 1)$

$\vartheta < 0$	$\vartheta = 0$	$\vartheta > 0$
n_+ HIGGS BR. BPS STATES	$(n_+ - n_-)$ COULOMB BR. $\rightarrow \infty$	n_- HIGGS BR. BPS STATES

C. DERIVATION OF SEMI-PRIMITIVE WCF

HALO STATES

SUPPOSE $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$,

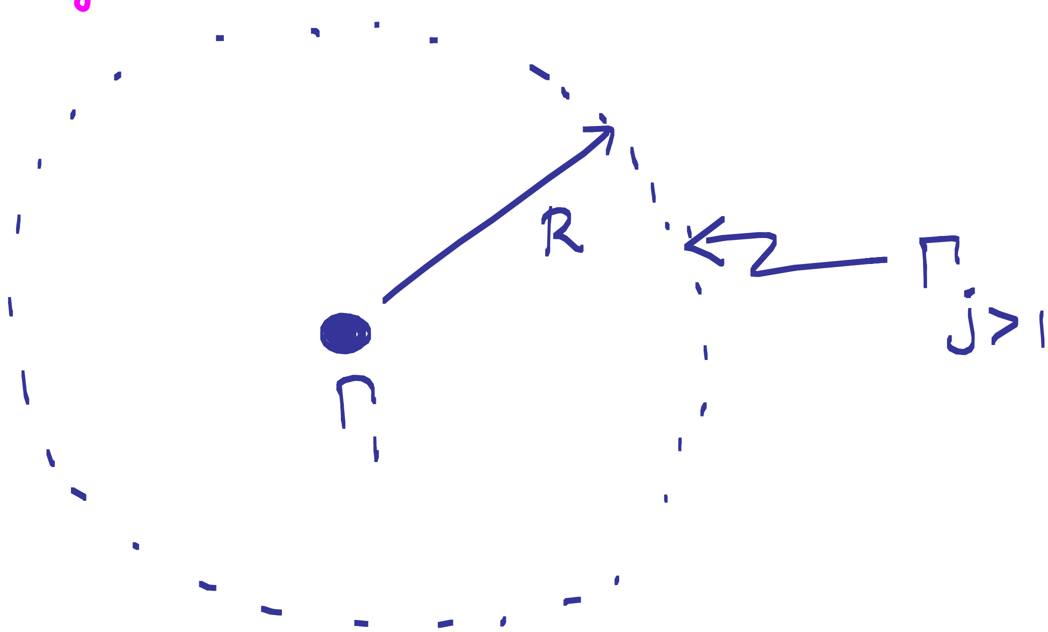
$\Gamma_j = \lambda_j \Gamma_2 \quad \lambda_j > 0, j = 2, \cdots, N$

ARE ALL MUTUALLY LOCAL

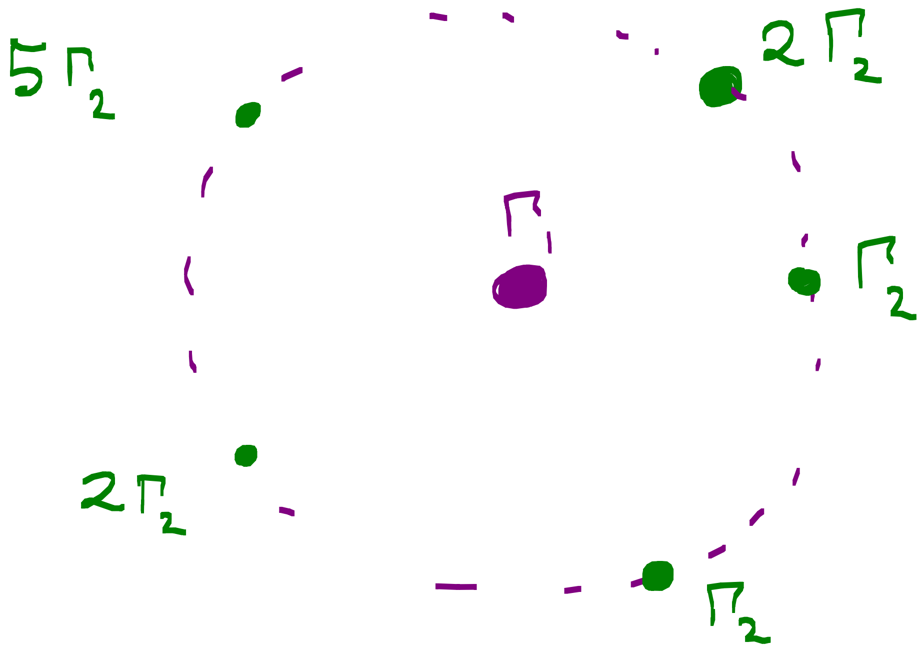
INTEGRABILITY CONDITIONS SAY

$$j \geq 2: \quad \frac{\langle \Gamma_j, \Gamma_1 \rangle}{|\vec{x}_j - \vec{x}_1|} = 2 \frac{\mathrm{Im}\left(Z(\Gamma_j)\overline{Z(\Gamma)} \right)}{|Z(\Gamma)|}$$

$\Rightarrow$ ALL $|\vec{x}_j - \vec{x}_1|$ ARE EQUAL



CROSS $MS(\Gamma_1, \Gamma_2)$: HALO RADIUS $\nearrow \infty$



THE PARTICLES IN THE HALO GENERATE A FOCK SPACE WITH

$(J_{\Gamma_1, k\Gamma_2}) \otimes \mathcal{H}(k\Gamma_2; t_{ms})$ CREATION OPERATORS OF CHARGE $k\Gamma_2$

ALL WALLS $W(\Gamma_1, N\Gamma_2)$ COINCIDE $\Rightarrow$ CROSSING A WALL WE LOSE ENTIRE FOCK SPACE:

$$\Omega(\Gamma_1) + \sum_{N \geq 1} \Delta\Omega(\Gamma \to \Gamma_1 + N\Gamma_2) u^N$$

$$= \Omega(\Gamma_1) \prod_{k>0} \left(1 - (-1)^{k\langle\Gamma_1, \Gamma_2\rangle} u^k\right)^{|\langle\Gamma_1, k\Gamma_2\rangle| \Omega(k\Gamma_2)}$$

KONTSEVICH - SOIBELMAN FORMULA

- NO PHYSICAL DERIVATION YET

EVIDENCE THAT K & S $\Omega(\Gamma;t)$'s ARE THE SAME AS PHYSICAL $\Omega(\Gamma;t)$'s:

- CAN RECOVER PRIMITIVE WCF
- CAN RECOVER SEMI-PRIMITIVE WCF
- NONTRIVIAL CHECKS FOR SU(2) SEIBERG-WITTEN WITH $N_f = 0,1,2,3$ HYPERMULTIPLETS

(LAST TWO ARE RESULTS w/ Wu-yen Chuang)

LIE ALGEBRA IS FILTERED $\Longrightarrow$

CAN RESTRICT TO

Heisenberg Algebra
$$\left\{\begin{array}{l} [e_{0,1}, e_{1,0}] = (-1)^{I_{12}-1} I_{12}\, e_{1,1} \\ e_{1,1} \ \underline{\text{CENTRAL}} \end{array}\right.$$

$$U_{0,1}^{\Omega^-(\Gamma_1)}\, U_{1,1}^{\Omega^-(\Gamma_1+\Gamma_2)}\, U_{1,0}^{\Omega^-(\Gamma_2)}$$

$$= U_{1,0}^{\Omega^+(\Gamma_2)}\, U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2)}\, U_{0,1}^{\Omega^+(\Gamma_1)}$$

$$\boxed{U_{01} U_{10} = U_{1,1}^{\pm I_{12}} \cdot U_{10} U_{01}} \Longrightarrow$$

$$U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2)-\Omega^-(\Gamma_1+\Gamma_2)} = U_{0,1}^{\Omega(\Gamma_1)} U_{1,0}^{\Omega(\Gamma_2)} U_{0,1}^{-\Omega(\Gamma_1)} U_{1,0}^{-\Omega(\Gamma_2)}$$

$$= U_{1,1}^{I_{12}\,\Omega(\Gamma_1)\,\Omega(\Gamma_2)}$$

PRIMITIVE W.C. FORMULA!

SU(2) SEIBERG-WITTEN THEORY

Γ_1 = MONOPOLE

Γ_2 = DYON

$$[e_{a,b}, e_{c,d}] = 2(bc-ad)\, e_{a+c,\, b+d}$$

STRONG : $\pm(1,0)$, $\pm(0,1)$ $\Omega = +1$ HM

WEAK : $\pm(1,1)$ $\Omega = -2$ VM

$\pm(n, n+1)$, $\pm(n+1, n)$ $\Omega = +1$ HM

STRONG $\quad U_{1,0} \cdot U_{0,1}$

WEAK :

$$\left(U_{0,1}\, U_{1,2}\, U_{2,3} \cdots\cdots\right) U_{1,1}^{-2} \left(\cdots\cdots U_{32}\, U_{21}\, U_{10}\right)$$

EQUALITY APPEARS TO BE TRUE !

$\exists$ NEW IDENTITES FOR $N_f = 1, 2, 3$

4. D6D2D0 SYSTEM

IDENTIFY Γ WITH ITS IMAGE

$$\mathcal{E} \in K^0(X) \rightarrow \Gamma = ch\,\mathcal{E}\sqrt{\hat{A}} \in H^{ev}(X)$$

$$\underset{D6}{H^0} \oplus \underset{D4}{H^2} \oplus \underset{D2}{H^4} \oplus \underset{D0}{H^6} \ni \Gamma = (p^0, P, Q, q_0)$$

CONSIDER: $\Gamma(\beta, n) := \Gamma = (1, 0, -\beta, n)$

$\beta = P.D.[\sigma]$ $\quad \sigma \subset X$ HOLOMORPHIC CURVE

CHARGE OF (THE DUAL OF) AN IDEAL SHEAF:

$$ch\,\mathcal{I}\sqrt{\hat{A}} = 1 - \beta + n\,dV$$

CONSIDER BINDING THESE

TO D2D0 PARTICLES WITH CHARGE:

$$\Gamma_h = (0, 0, -\beta_h, n_h)$$

PLOT MARGINAL STABILITY CURVE

$$Z(\Gamma(\beta,n);t) = \lambda Z(\Gamma_h;t) \qquad \lambda \in \mathbb{R}_+$$

$$\text{IIA} \quad \langle \Gamma_1, \Gamma_2 \rangle = \int ch\, \mathcal{E}_1 \otimes \bar{\mathcal{E}}_2 \, \hat{A}$$

$$Z(\Gamma, t) = \frac{\langle \Gamma, \Omega \rangle}{\sqrt{\langle \Omega, \Omega^* \rangle}}$$

$$\text{SUGRA REGIME:} \quad \Omega = -e^{t}$$

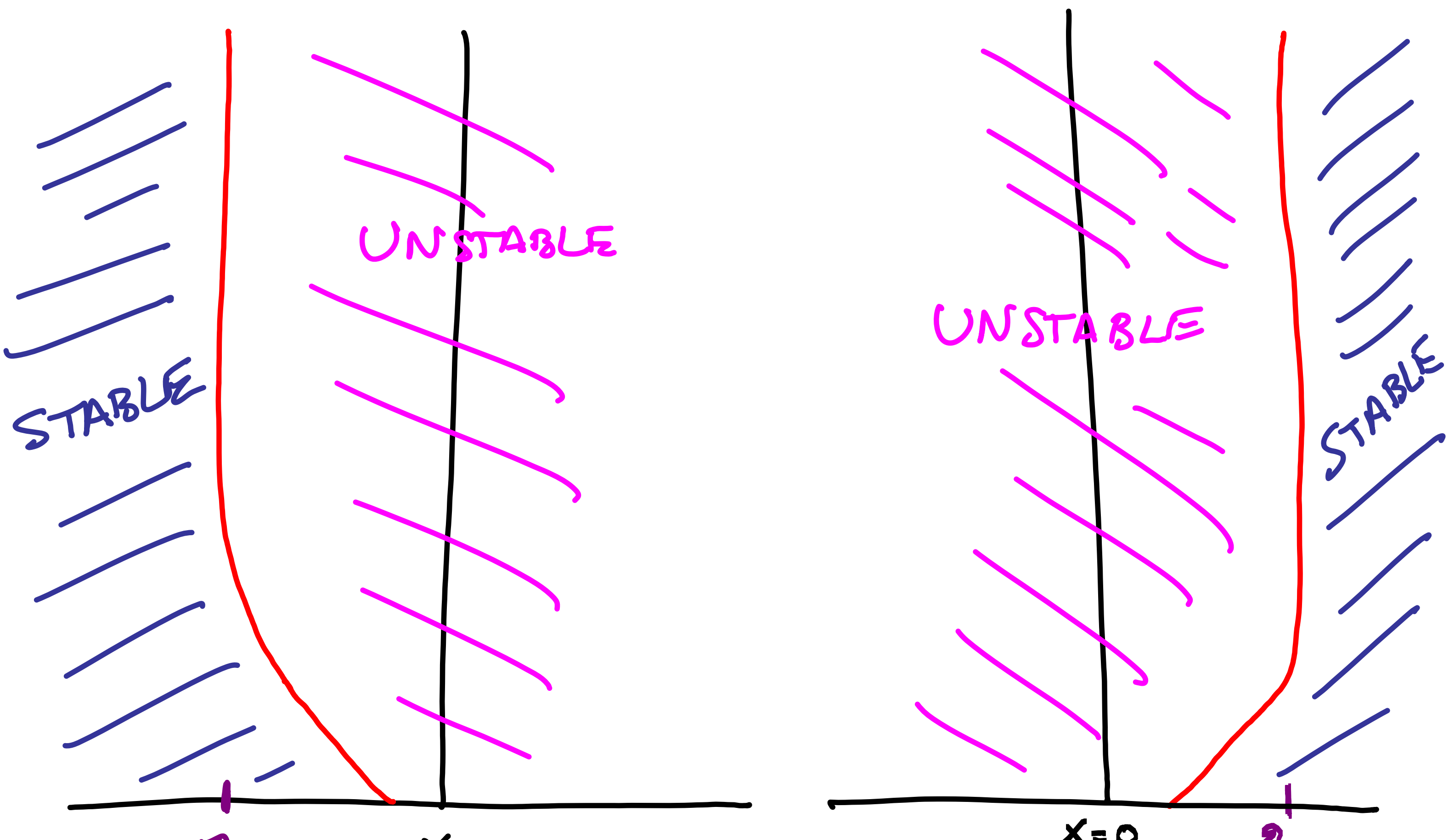
$$t = B + iJ$$

$$\boxed{\frac{t^3}{6} - \beta \cdot t - n = \lambda\left(-\beta_h \cdot t - n_h\right)} \quad \lambda \in \mathbb{R}_+$$

THESE WALLS EXTEND TO ∞ IN THE KÄHLER CONE!

SET $t = z\underline{P}$ $\quad$ $\underline{P} \in \mathcal{K}$

$\underline{z = x + iy}$



$x = \frac{n_h}{2P\cdot\beta_h}$ $\quad$ $\boxed{n_h < 0}$

$\boxed{n_h > 0}$ $\quad$ $x = \frac{n_h}{2P\cdot\beta_h}$

CONSIDER THE HALO BOUNDSTATES WITH CENTRAL PARTICLE $\Gamma(\beta, n)$ AS WE INCREASE THE B-FIELD

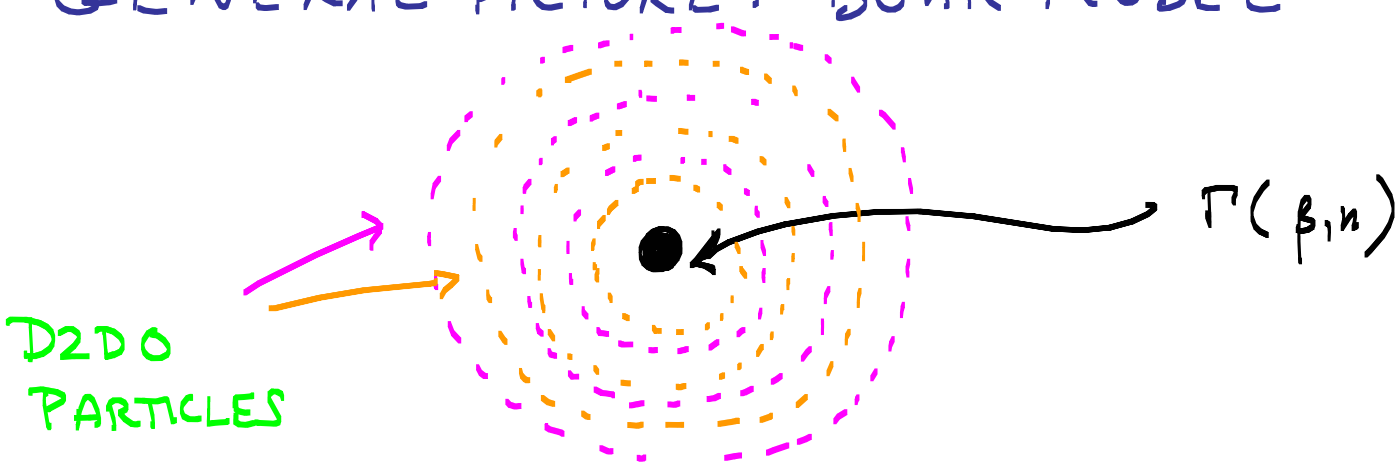
$$B = xP \qquad x \text{ INCREASES}$$

HALOS OF D2D0 PARTICLES $(0, 0, -\beta_h, n_h)$. APPEAR & DISAPPEAR.

FOR $x > 0$

ALL $n_h < 0$ STATES HAVE DECAYED.
AS $x \to +\infty$ WE MOVE INTO THE STABLE REGION FOR ALL $n_h > 0$, AND EVER LARGER "ATOMS" BECOME STABLE

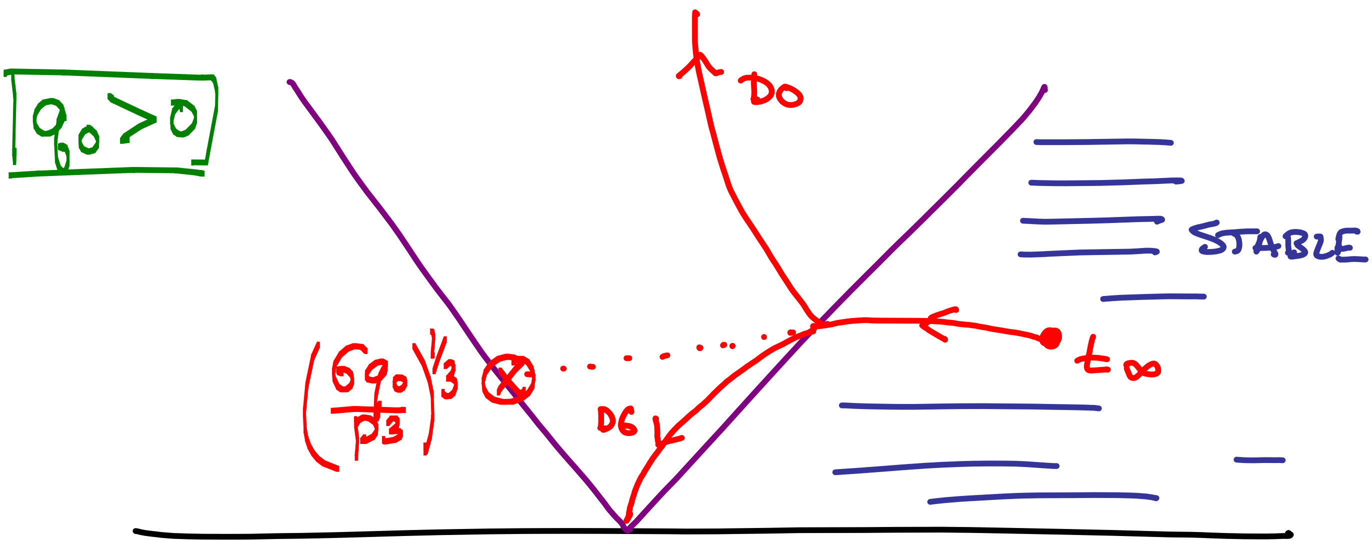
GENERAL PICTURE: BOHR MODEL

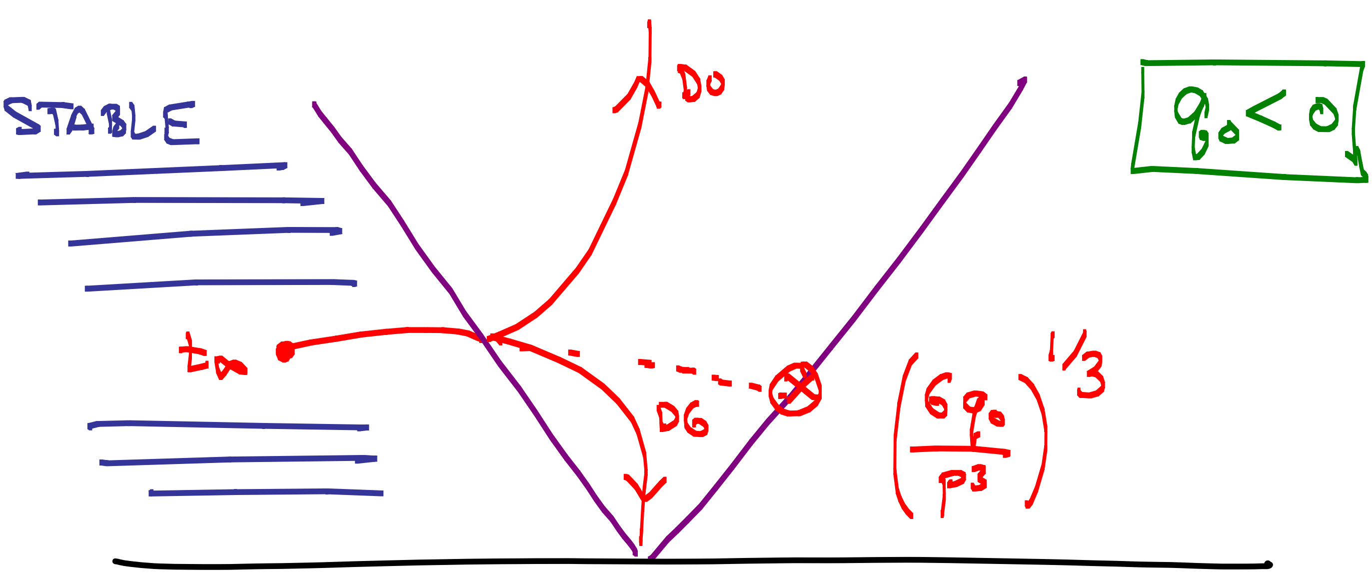


WHEN $\beta_h = 0$ WALLS LOOK DIFFERENT

$$\Gamma = \underbrace{1}_{\Gamma_1} + \underbrace{q_0\, dV}_{\Gamma_2} \qquad Z = \frac{t^3}{6} - q_0$$

SET $t = (x+iy)P \Longrightarrow$ ZERO @ $z = \left(\frac{6q_0}{P^3}\right)^{1/3} P$





INTRODUCE GENERATING FUNCTION

$$Z_{D6D2D0}(u,v;t) := \sum_{n,\beta} \Omega(\Gamma(\beta,n);t)\, u^n v^\beta$$

SEMI-PRIMITIVE WALL-CROSSING FORMULA:

CONTRIBUTION OF FOCK SPACE GENERATED BY $\Gamma_h = -\beta_h + n_h dV$ CROSSING INTO STABLE REGION:

$$Z_{D6D2D0} \longrightarrow \left(1-(-u)^{n_h} v^{\beta_h}\right)^{|n_h|\, n^0_{\beta_h}} Z_{D6D2D0}$$

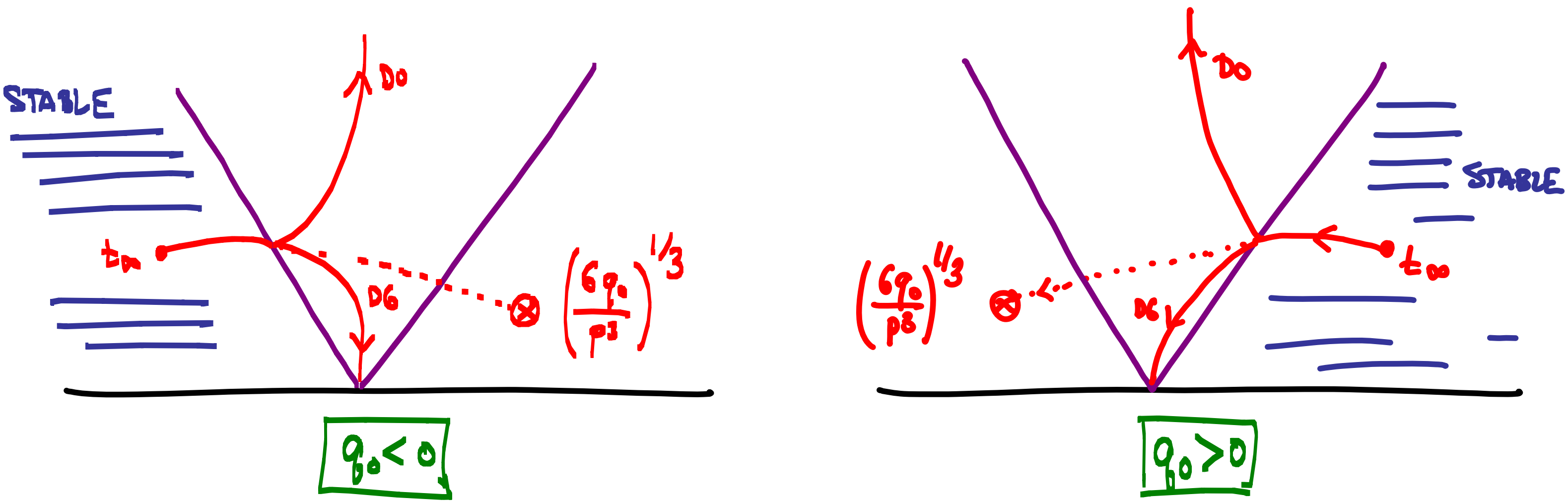
$$\Omega(-\beta_h + n_h dV) = \sum_{m_L, m_R} (-1)^{2m_L+2m_R} N^{m_L m_R}_{\beta_h}$$

$$= n^0_{\beta_h}$$

"SPIN ZERO GV INVARIANT" $(\beta_h \neq 0)$

EXAMPLE: D6D0

$$Z_{D6D0}(u) = \sum \Omega(1+q_0 dV; t)\, u^{q_0}$$



$$\Omega(q_0 dV) = -\chi(X)$$

$$Z_{D6D0}(u) = \begin{cases} \left(M(-u)\right)^{\chi(X)} & \arg z < \frac{\pi}{3} \\ 1 & \frac{\pi}{3} < \arg z < \frac{2\pi}{3} \\ \left(M(-u^{-1})\right)^{\chi(X)} & \frac{2\pi}{3} < \arg z \end{cases}$$

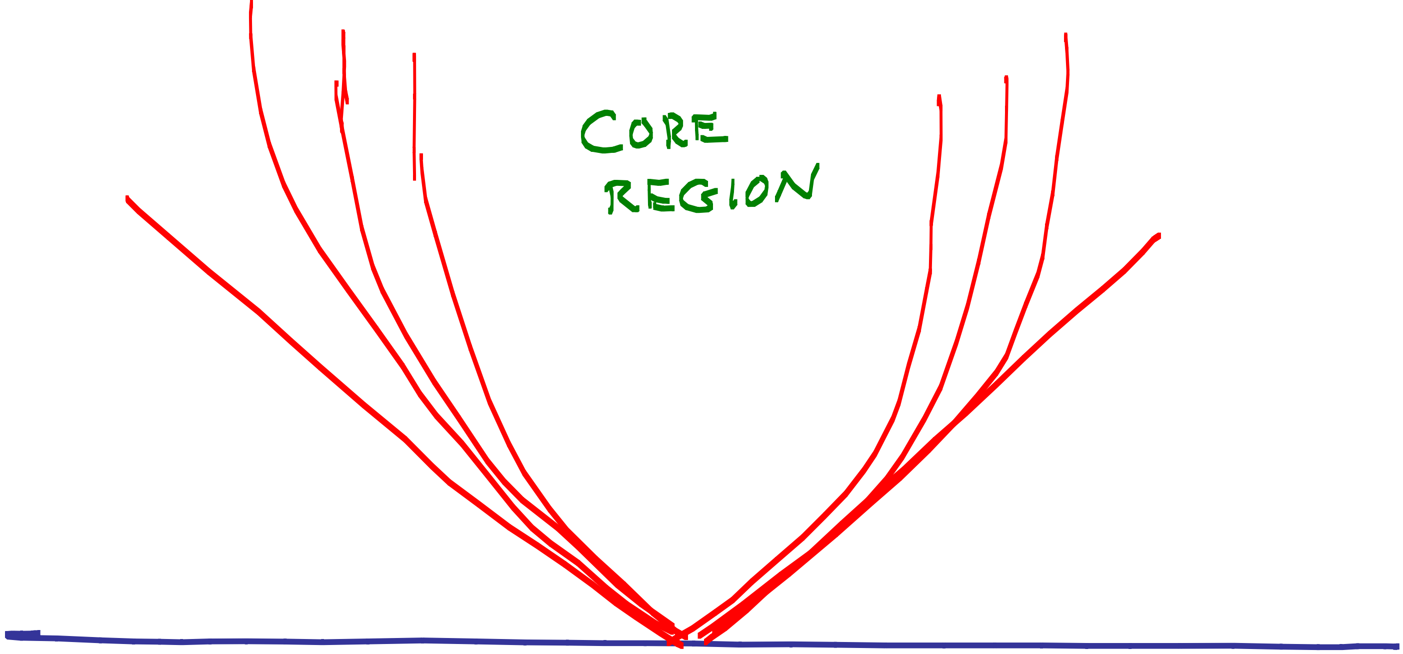
$$M(u) := \prod_{k\geq 1} (1-u^k)^{-k}$$

SIMILARLY, WALL-CROSSINGS FOR THE FULL Z_{D6D2D0} AS $x \to \infty$ BUILD UP AN INFINITE PRODUCT SIMILAR TO THE INFINITE PRODUCT FORM OF $Z_{DT}(u,v)$

ON THE OTHER HAND, AN ARGUMENT FROM M-THEORY [Dijkgraaf, Verlinde, Vafa; Denef, Moore] IMPLIES:

$$\lim_{x \to +\infty} Z_{D6D2D0}(u,v;zP) = Z_{DT}(u,v)$$

$$\lim_{x \to -\infty} Z_{D6D2D0}(u,v;zP) = Z_{DT}(u^{-1},v)$$



- STATES IN CORE REGION ARE COMPLICATED BOUNDSTATES
- PRODUCT OF WALL-CROSSINGS $\Rightarrow$

$$Z_{DT}^{1,r=0}(u,v) = \prod_{\beta>0,k>0}\left(1-(-u)^k v^\beta\right)^{k n_\beta^0}$$

- LIMIT FOR $x \longrightarrow +\infty$:

$$Z'_{DT}(u,v) = \underbrace{Z_{DT}^{1,r=0}(u,v)}_{\text{HALOS}}\ \underbrace{Z_{DT}^{1,r>0}(u,v)}_{\text{CORES}}$$

$$Z_{DT}^{1,r>0}(u,v) = \prod_{\substack{\beta>0,k>0\\ r>0}}\ \prod_{\ell=0}^{2r-2}\left(1-(-u)^{r-\ell-1}v^\beta\right)^{(-1)^{r+\ell}\binom{2r-2}{\ell}n_\beta^r}$$

APPLICATION TO LOCAL CALABI-YAU

(M. AGANAGIC, G.M, D. JAFFERIS)

LOCAL LIMIT: CHOOSE CURVE CLASS β_h

CHOOSE $P, \hat{T} \in \mathcal{K}$ WITH $P \cdot \beta_h = 1$, $\hat{T} \cdot \beta_h = 0$

$$t = e^{i\phi} \Lambda \hat{T} + z \cdot P$$

$$\hat{T}^3 = 1, \quad 0 \leq \phi \leq \pi, \quad \Lambda \longrightarrow \infty$$

$$\lim_{\Lambda \to \infty} \Omega(\Gamma(\beta, n); t) = \Omega_\infty(\Gamma(\beta, n); z, \phi)$$

Ω_∞ WILL HAVE W.C. FOR DECAYS:

$$\Gamma(\beta'', n'') \longrightarrow \Gamma(\beta', n') + (-m\beta_h + n\, dV)$$

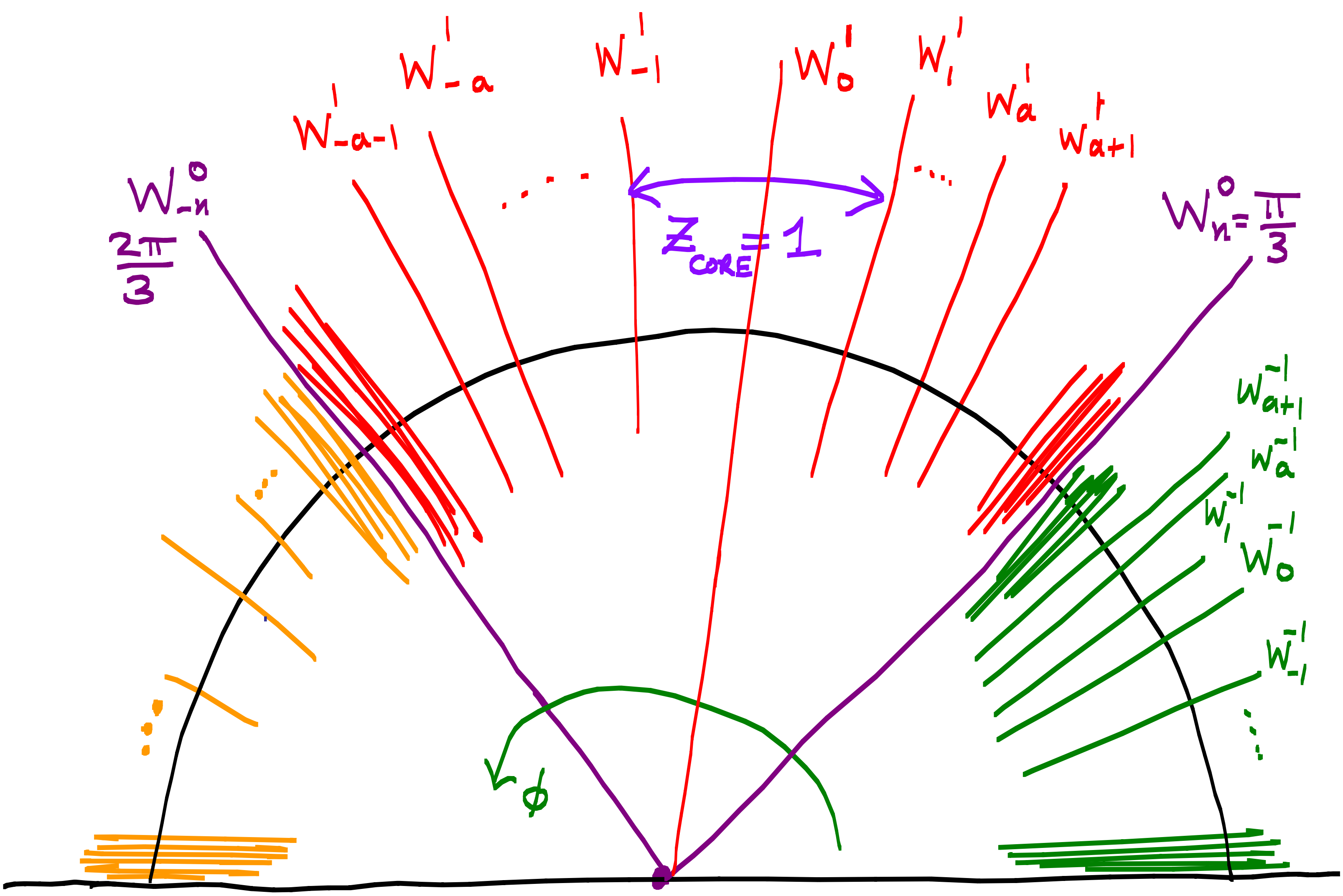
$$e^{3i\phi} \Lambda^3 = \lambda(-mz - n) \qquad \lambda \in \mathbb{R}_+$$

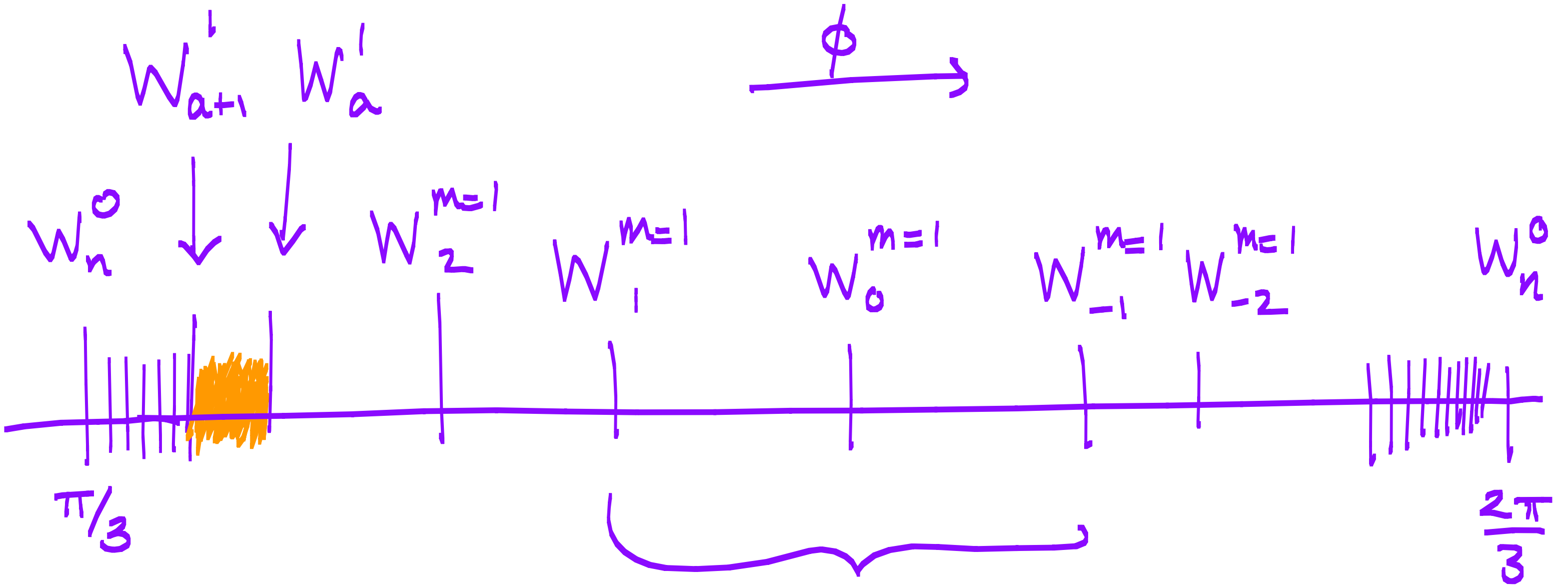
$$W_n^m = \left\{ \phi = \frac{1}{3} \arg(-mz - n) \bmod \frac{2\pi}{3} \mathbb{Z} \right\}$$

STUDY THE PRODUCTS AS WE CHANGE ϕ AT CONSTANT Z

EXAMPLE OF CONIFOLD: $H_2 = \beta \cdot \mathbb{Z}$

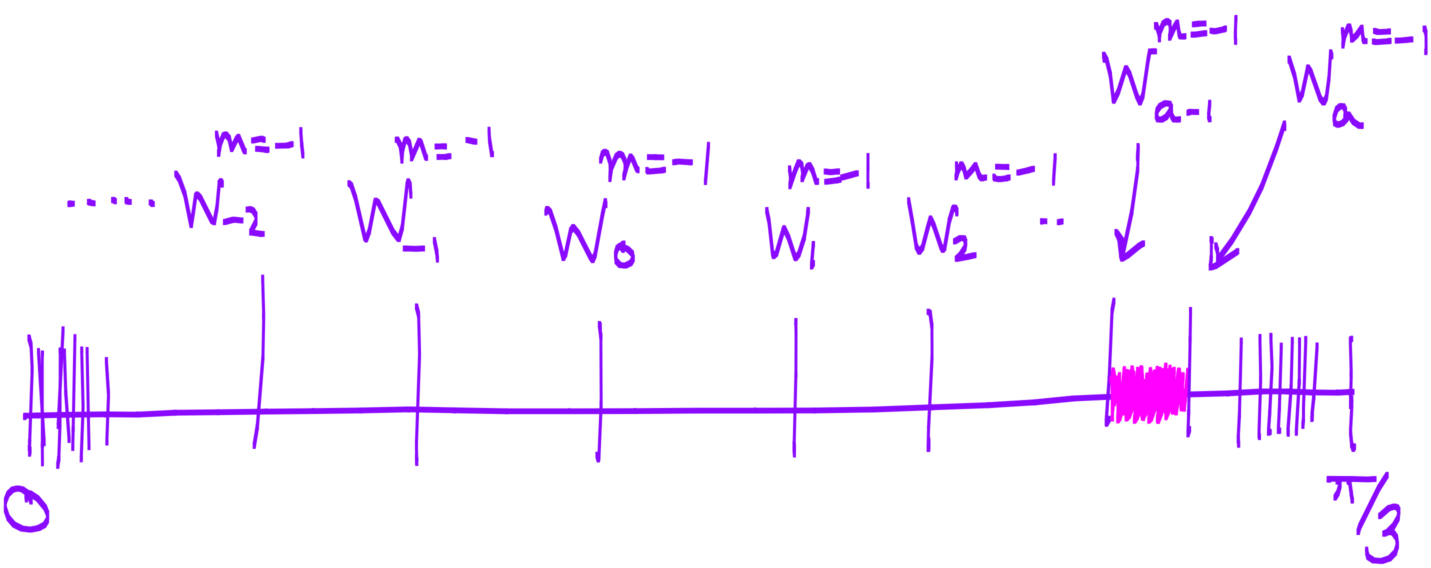
$$\Omega(-m\beta + n\,dV) = \begin{cases} -2 & m = 0 \\ 1 & m = \pm 1 \\ 0 & \text{else} \end{cases}$$



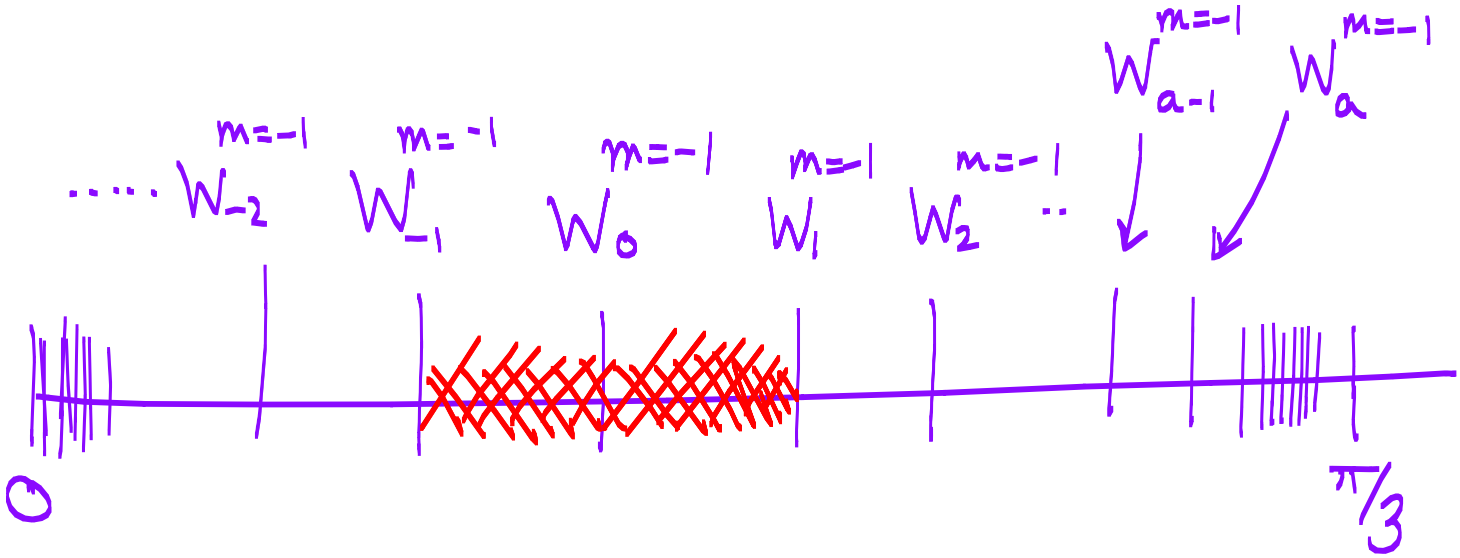


$$Z = Z_{\text{core}} = 1$$

$$Z = \prod_{k=1}^{a} \left(1-(-u)^k v\right)^k \qquad W^{m=1}_{a+1} < \phi < W^{m=1}_{a}$$



$$Z = M(-u)^2 \cdot \prod_{k=1}^{\infty}\left(1-(-u)^k v\right)^k \cdot \prod_{k=a}^{\infty}\left(1-(-u)^k v^{-1}\right)^k$$



$$Z = \left(M(-u)\right)^2 \prod_{k=1}^{\infty}\left(1-(-u)^k v\right)^k \prod_{k=1}^{\infty}\left(1-(-u)^k v^{-1}\right)^k$$

INTERESTINGLY – PRECISELY SUCH ∞-PRODUCTS HAVE APPEARED RECENTLY IN WORK OF B. SZENDROI

$\exists$ RELATED RESULTS FOR $\mathbb{C}^2/ADE \times \mathbb{C}$ CONNECTING TO B. YOUNG & J. BRYAN.

5. THE D4-D2-D0 SYSTEM: MODULARITY

NOW CONSIDER $p^0 = 0$

$$\Gamma = P + Q + q_0 dV$$

REGULAR ATTRACTOR POINT:

$$\boxed{\begin{array}{c} P \text{ IN KÄHLER CONE } \& \quad \hat{q}_0 < 0 \\ \hat{q}_0 := q_0 - \frac{1}{2}\left(D_{ABC}P^C\right)^{-1} Q_A Q_B \end{array}}$$

THESE ARE BLACK HOLES:

HORIZON AREA $= 4 \; S(\Gamma) = 4\pi \, |Z_*(\Gamma)|^2$

$$S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\hat{q}_0 \, \chi(P)}$$

$$\chi(P) := P^3 + c_2 \cdot P > 0 \text{ FOR } P \in \text{KÄHLER CONE}$$

EXPECT: $\log \Omega(\Gamma; t) \sim S(\Gamma)$

FOR "LARGE" Γ AND "LARGE" $\mathrm{Im}\, t$

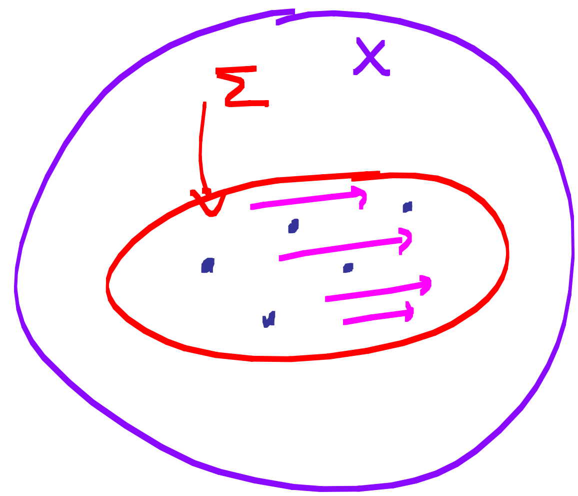
A. ROUGH MICROSCOPIC DESCRIPTION

FOR LARGE J: SINGLE D4 WRAPS $\Sigma \in |P|$

$\chi(P) = P^3 + c_2 \cdot P$ = EULER CHARACTER OF Σ

FLUX $F \in H^2(\Sigma, \mathbb{Z})$

AND N $\overline{D0}$'s



COMPUTE INDUCED RR CHARGES:

D2: $Q = (\iota_\Sigma)_*(F)$

D0: $\hat{q}_0 = \frac{\chi(P)}{24} + \frac{1}{2}(F^-)^2 - N$

SUSY $\Rightarrow$ $N \geq 0$, $F^{2,0} = 0 \Rightarrow (F^-)^2 \leq 0 \Rightarrow$

$$\hat{q}_0 \leq (\hat{q}_0)_{max} = \frac{\chi(P)}{24}$$

$\mathcal{M}(P,F,N) :=$ MODULI OF SUCH D4'S

ROUGHLY:

$$
\begin{tikzcd}
\mathrm{Hilb}^N(\Sigma) \arrow[r, hook] \arrow[d] & \mathcal{M}(P,F,N) \arrow[d] \\
\underset{\text{Smooth}}{\Sigma} \arrow[r, hook] & \{\Sigma \in |P| \mid F \in H^{1,1}(\Sigma)\}
\end{tikzcd}
$$

"MODULI OF STABLE OBJECTS $\mathcal{E}$ IN THE DERIVED CATEGORY WITH SPECIFIED CHERN CLASSES"

$$ch\,\mathcal{E}\sqrt{\hat{A}} = P + Q + q_0 \quad (*)$$

$$= \bigcup_{\substack{F,N \\ \text{s.t.} \\ (*)}} \mathcal{M}(P,F,N)$$

B. INDEX OF BPS STATES

$$\text{"}\ \Omega(\Gamma)_\infty := \lim_{J\to\infty} \Omega(\Gamma; B+iJ)\ \text{"}$$

$$d(F,N) := (-1)^{\dim \mathcal{M}} \chi\left(\mathcal{M}(P,F,N)\right)$$

$\Omega(\Gamma)_\infty$ = FINITE SUM OF $d(F,N)$

SURPRISE: WHEN $h^{1,1}(X) > 1$ THERE ARE SPLITTINGS @ ∞ :

$$\Gamma = P + Q + q_0 dV$$

$$= (P' + Q' + q_0' dV) + (P'' + Q'' + q_0'' dV)$$

WITH: $\sqrt{-\hat{q}_0''(P'')^3} > \sqrt{-\hat{q}_0 P^3}$

$\Longrightarrow$ EVEN THE LEADING ORDER ENTROPY IS CHAMBER DEPENDENT

[E. ANDRIYASH + G.M.]

- FOR $\Gamma = P + Q + q_0\, dV$,

$P \in$ KÄHLER CONE, $\exists$ DISTINGUISHED CHAMBER:

$$\boxed{\Omega(\Gamma)_\infty := \lim_{\lambda \to \infty} \Omega(\Gamma; B + i\lambda P)}$$

CLAIM: LIMIT EXISTS & IS B-INDEPENDENT

(FINITENESS OF ATTRACTOR FLOW TREES)

HENCEFORTH WORK IN THIS CHAMBER.

C. MODULARITY

$$\tau \in \mathcal{H} \quad \& \quad C \in \iota^*_\Sigma\left(H^2(X,\mathbb{C})\right)$$

$$Z(\tau,\bar{\tau},C) :=$$

$$\sum_{F,N} d(F,N) \exp\left\{-2\pi i \tau \hat{q}_0 - 2\pi i \bar{\tau} \tfrac{1}{2}(F^+)^2 - 2\pi i F\cdot\left(C+\tfrac{P}{2}\right)\right\}$$

SUSY PARTITION FUNCTION OF D3 INSTANTON

U-DUALITY $\Longrightarrow$

$Z(\tau,\bar{\tau},C)$ IS A JACOBI FORM $\Longrightarrow$

$$Z(\tau,\bar{\tau},C) = \sum_{\mu \in L^*/L} H_\mu(\tau) \underbrace{\Theta_{\mu,L}(\tau,\bar{\tau},C)}_{\text{SIEGEL-NARAIN}}$$

$$L := \iota^*_\Sigma\left(H^2(X,\mathbb{Z})\right) \subset \underbrace{H^2(\Sigma;\mathbb{Z})}_{\text{SELF-DUAL}}$$

$\ell \in L$ IS ALWAYS IN $H^{1,1}(\Sigma) \Rightarrow$

$$d(F+\ell,N) = d(F,N) \quad \forall \ell \in L$$

- $H_\mu(\tau)$ IS A VECTOR-VALUED NEARLY HOLO. MODULAR FORM OF WEIGHT $w = -1 - \frac{h^{1,1}(X)}{2}$ AND MULTIPLIER SYSTEM M^* DUAL TO THAT OF $\Theta_{\mu,L}$

THIS HAS TWO INTERESTING CONSEQUENCES

- $w < 0 \Longrightarrow H_\mu$ IS DETERMINED BY ITS POLAR TERMS.

SUPPRESS μ-INDEX FOR SIMPLICITY:

$$H(\tau) = \sum_{\hat{q}_0} \Omega(\Gamma)_\infty \, e^{-2\pi i \hat{q}_0 \tau}$$

$$= \underbrace{\sum_{0 < \hat{q}_0 \leq \frac{\chi(P)}{24}} (\cdots\cdots)}_{\text{POLAR}} + \underbrace{\sum_{-\infty < \hat{q}_0 \leq 0} (\cdots\cdots)}_{\text{NONPOLAR}}$$

1. FOURIER COEFF'S OF CUSP FORMS WEIGHTED BY POLAR DEGENERACIES VANISH.

$\mathcal{P}$ = VECTOR SPACE OF POLYNOMIALS IN $e^{-2\pi i \hat{q}_0 \tau}$ FOR

$$0 < \hat{q}_0 \leq (\hat{q}_0)_{max} = \frac{\chi(P)}{24}$$

THEN [D. NIEBUR; J. MANSCHOT + G.M.]

$$0 \to M_w(\Gamma, \mathcal{M}^*) \to \mathcal{P} \to S_{2-w}(\Gamma, \mathcal{M})$$

$$e(-\hat{q}_0\tau) \longrightarrow G^{(\hat{q}_0)}(\tau)$$

$$G^{(\hat{q}_0)}(\tau) = \sum_{\Gamma_\infty \backslash \Gamma} (c\tau+d)^{-3-h/2}\, e^{+2\pi i \hat{q}_0 \gamma(\tau)}$$

$\Rightarrow$ PREDICTION OF MODULARITY:

$$\boxed{\sum_{0<\hat{q}_0\leq\frac{\chi(P)}{24}} \Omega(\Gamma)_\infty\, G^{(\hat{q}_0)}(\tau) = 0}$$

2. POLAR DEGENERACIES DETERMINE ALL OTHER DEGENERACIES THROUGH AN EXPLICIT FORMULA – THE RADEMACHER FORMULA.

THE NONPOLAR DEGENERACIES ARE OF PHYSICAL INTEREST FOR BLACK HOLE ENTROPY.

LEADS TO

$$\Omega(\Gamma)_\infty \sim \oint d\tau \, e^{-2\pi i \hat{q}_0 \tau} \, H^{\text{POLAR}}(-1/\tau)$$

$$+ \cdots$$

SO WE WANT TO COMPUTE THE POLAR DEGENERACIES...

C. MACROSCOPIC POLAR STATES

IF $\Gamma = (0, P, Q, q_0) = P + Q + q_0 dV$

IS POLAR: $0 < \hat{q}_0 \leq (\hat{q}_0)_{max}$

THEN $Z(\Gamma; t)$ HAS A ZERO.

INDEED $S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\hat{q}_0 \chi(P)}$

SO NO SINGLE-CENTERED SOLUTION

BUT $H(\tau)$ HAS $w < 0 \Rightarrow$ SOME POLAR DEGENERACIES ARE NONZERO

$\Rightarrow$ THESE MUST BE REALIZED AS SPLIT ATTRACTOR STATES.

SIMPLE EXAMPLE

PURE D4 : $\Gamma = P + q_0 dV$

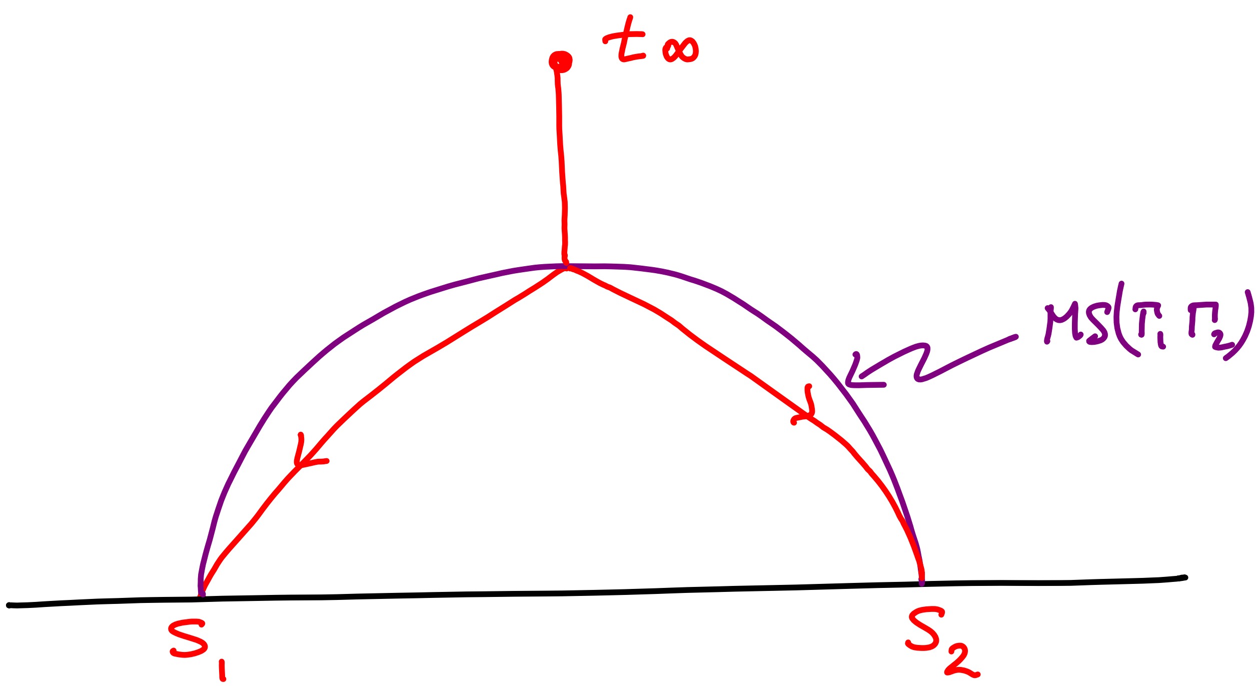
WITH $q_0 = \hat{q}_0 = (\hat{q}_0)_{max} = \dfrac{\chi(P)}{24}$

FIND ONLY ONE SPLITTING

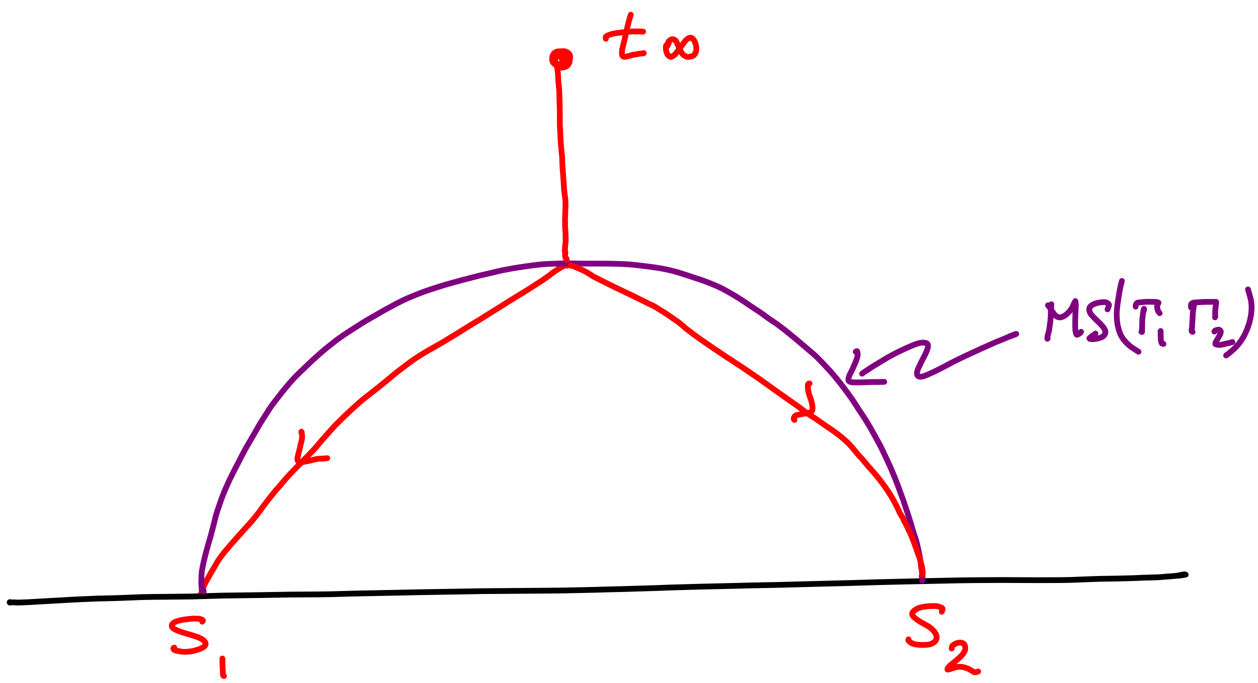
$$\Gamma = P + q_0 dV = \Gamma_1 + \Gamma_2$$

$$= \underbrace{e^{S_1}\left(1 + \frac{c_2(X)}{24}\right)}_{\text{1 D6 WITH FLUX} = S_1} - \underbrace{e^{S_2}\left(1 + \frac{c_2(X)}{24}\right)}_{1\,\overline{\text{D6}}\text{ W/ FLX } S_2}$$

$$S_1 - S_2 = \underline{P}$$



MOREOVER - YOU CAN COMPUTE THE POLAR DEGENERACY:



$$\Omega(\Gamma, t_\infty) = (-1)^{I_{12}-1} |I_{12}| \Omega(\Gamma_1) \Omega(\Gamma_2) = (-1)^{I_{12}-1} |I_{12}|$$

$$I_{12} = \langle \Gamma_1, \Gamma_2 \rangle = \frac{P^3}{6} + \frac{c_2(X) \cdot P}{12}$$

INDEED = THE CORRECT ANSWER FOR χ (MODULI OF PURE D4) $= \chi(|P|)$

DESCRIBING THE SPLIT ATTRACTOR FLOWS FOR $0 < \hat{q}_0 < \frac{\chi(P)}{24}$ IS MUCH MORE COMPLICATED...

IN GENERAL, POLAR STATES CAN BE VERY COMPLICATED SPLIT ATTRACTORS, REALIZED IN MANY DIFFERENT WAYS....

BUT IN THE LIMIT $P \to \infty$ WE CAN SAY SOMETHING

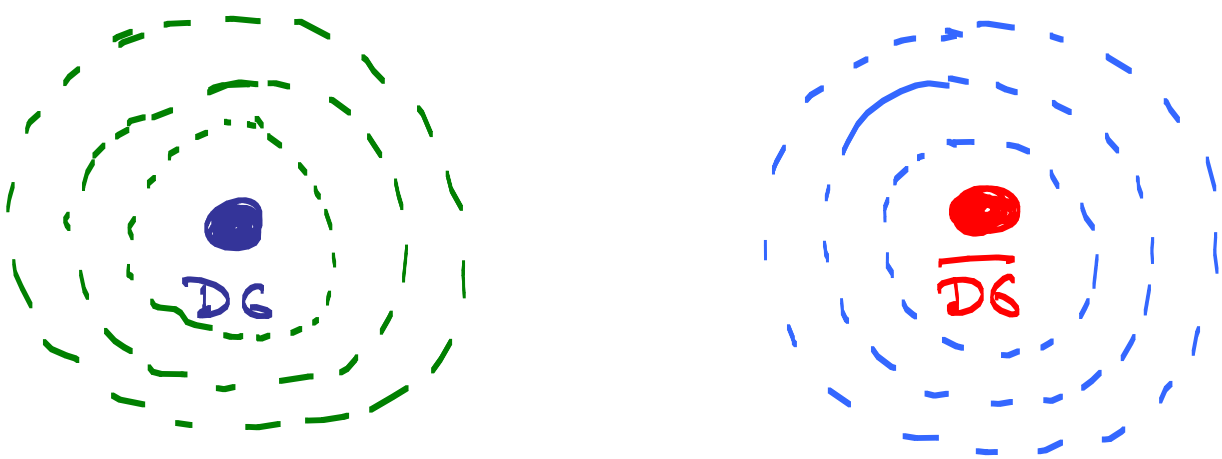
EXTREME POLAR STATES

$$H^{POLAR}(\tau) = \underbrace{|I_p|\, e^{-2\pi i \tau \frac{\chi(p)}{24}} + \cdots}_{\text{"EXTREME POLAR"}} \quad \underbrace{+\mathcal{O}\left(e^{\frac{-2\pi i\tau}{|P|}}\right)}_{\text{"BARELY POLAR"}}$$

E.P.S. CONJECTURE: $\exists\ \epsilon < 1$ SO THAT

$$\frac{\hat{q}_0^{max} - \hat{q}_0}{\hat{q}_0^{max}} < \epsilon \quad \Longrightarrow$$

POLAR STATES SPLIT AS $D6\overline{D6}$ + HALOS:



$$\Gamma_1 = e^{S_1}(1 - \beta_1 + n_1 dV) \qquad \Gamma_2 = -e^{S_2}(1 - \beta_2 + n_2 dV)$$

SO, BY THE W.C.F. TOGETHER WITH RESULTS ON Z_{D6D2D0} THE EXTREME POLAR DEGENERACIES ARE RELATED TO:

$$|Z_{DT}|^2 \overset{\text{MNOP}}{=\!=} |Z_{TOP}|^2$$

SUGGESTING A RELATION LIKE THE OSV CONJECTURE

$$\Omega(\Gamma)_\infty = \int d\phi \; |Z_{top}(g_{top}, t)|^2 e^{-2\pi q \cdot \phi}$$

- $\exists$ STRONG ARGUMENTS FOR $|\hat{q}_0| \gg p^3$
- $\exists$ POTENTIAL COUNTEREXAMPLES FOR $|\hat{q}_0| \lesssim p^3$: "ENTROPY ENIGMA"

6. CONCLUSION

WITH MORE TIME I WOULD GO ON TO DESCRIBE THE APPLICATION TO OSV.

MOST IMPORTANT OPEN PROBLEM IS THE BEHAVIOR OF BARELY POLAR DEGENERACIES.

TURNS OUT TO BE RELATED TO A SHARP MATHEMATICAL QUESTION:

$$\lim_{\lambda \to \infty} \frac{\log \log |N_{DT}(\lambda^2 \beta, \lambda^3 n)|}{\log \lambda} = k(\beta, n)$$

EXPECT: $\quad 2 \leq k(\beta, n) \leq 3$

6. ROUGH SKETCH OF OSV

OUR VERSION OF OSV:

If $\mathrm{ch}\,\Gamma\sqrt{\hat{A}} = P + Q + q_0$ with P in the Kähler cone, then:

$$\boxed{\Omega(\Gamma)_\infty := \lim_{\lambda\to\infty} \Omega(\Gamma; t = B + i\lambda P)}$$

LIMIT IS WELL-DEFINED & B-INDEPENDENT.

Then

$$\Omega(\Gamma)_\infty = \int d\phi\, \mu(\phi) \left| Z^{\epsilon}_{top}(g_{top}, t) \right|^2 e^{-2\pi q \cdot \phi} \cdot \left(1 + O(e^{-\Delta})\right)$$

Where:

1. $g_{top} = \frac{2\pi}{\phi^0}$ $\quad t^A = \frac{1}{\phi^0}\left(\phi^A + i\frac{P^A}{2}\right)$

2. $Z_{top}(g,t)$ = top. string p.f.

$$= \sum_{\beta,n} N_{DT}(\beta,n)\left(-e^{-g}\right)^n e^{2\pi i \beta\cdot t}$$

3. $$Z^{\epsilon}_{top}(g,t) = \sum_{\substack{\beta\cdot P < \epsilon P^3 \\ |n| < \epsilon P^3}} \left(\cdots\cdots\right)$$

4. $$\mu(P,\phi) = \frac{1}{g^2_{top}} \mathrm{Re}\left(X^\Lambda \frac{\partial F^{\epsilon}_{top}}{\partial X^\Lambda}\right)$$

$$= \frac{1}{g^2_{top}} e^{-K} \qquad \left(b_1(X) = 0\right)$$

5. Δ = FUNCTION OF: ϵ, P, ϕ^0:

If $\left(g_{top}\right)^{s.p.} \approx \sqrt{\frac{-\hat{q}_0}{P^3}} \gg 1$

Then $$e^{-\Delta} = \exp\left(-\frac{\pi}{12\mu}\frac{\epsilon}{\phi^0}P^3\right)$$

- ABOVE ASSUMES THE TRUTH OF THE "EPS CONJECTURE"

- MOREOVER, THE PHENOMENON OF "SWING STATES" $\Rightarrow$ WE MUST TAKE

$$\epsilon = \delta \, |P|^{-\xi_{c.d.}}$$

ξ_{cd} = "Core dump exponent"
KNOWN TO BE ≤ 3.

- BUT FOR W.S. INSTANTONS TO BE RELEVANT WE NEED $\xi_{cd} \leq 2$.

$\Rightarrow$ <u>OPEN</u> <u>PROBLEM</u>.

x

1. FAREYTAIL:

$$Z_{D4D2D0} = \sum \text{MOD. TMNS. OF } Z^{POLAR}_{D4D2D0}$$

2. $$Z^{POLAR}_{D4D2D0} = Z^{\epsilon}_{D6\overline{D6}}(t_{ms}) + ET(\epsilon)$$

↑ EPS CONJECTURE

3. $$Z^{\epsilon}_{D6\overline{D6}}(t_{ms}) = Z^{\epsilon}_{D6D2D0}(t_{ms})\, Z^{\epsilon}_{\overline{D6D2D0}}(t_{ms})$$

↑ W.C. Formula

4. $$Z^{\epsilon}_{D6D2D0}(t_{ms}) = Z^{\epsilon}_{DT}$$

$$\epsilon \sim |P|^{-\xi_{cd.}} \qquad P \longrightarrow \infty$$

SWING STATE CONJECTURE: $\xi_{cd} < 2$

5. $Z_{DT} = Z_{GW} = Z_{TOP}$ MNOP CONJ.

STEPS 1 & 2 MAKE IMPORTANT APPROXIMATIONS

7. PROBLEMS AT WEAK COUPLING

IN THE CHARGE REGIME

$$g_{top} \sim \sqrt{\frac{-\hat{q}_0}{p^3}} \lesssim \mathcal{O}(1)$$

THE ABOVE DERIVATION BREAKS DOWN

- MODIFICATIONS OF THE "MODERN F.T." BECOME IMPORTANT
- BARELY POLAR DEGENERACIES BECOME LARGE

TOY MODEL: $\chi \sim p^3/24$

$$\frac{1}{\eta^\chi} = \sum_{n=0}^{\infty} p_\chi(n)\, e^{2\pi i\left(n - \chi/24\right)}$$

$$p_\chi\left(\frac{\chi}{24}+\ell\right) \sim \exp\left[\, k\cdot\chi + k'\cdot\ell \,\right]$$

AND THERE IS GOOD REASON THE DERIVATION BREAKS DOWN ...

ENTROPY ENIGMA

NOW CHOOSE $q_0 < 0$, P AMPLE SO

$$\Gamma = (0, P, 0, q_0)$$

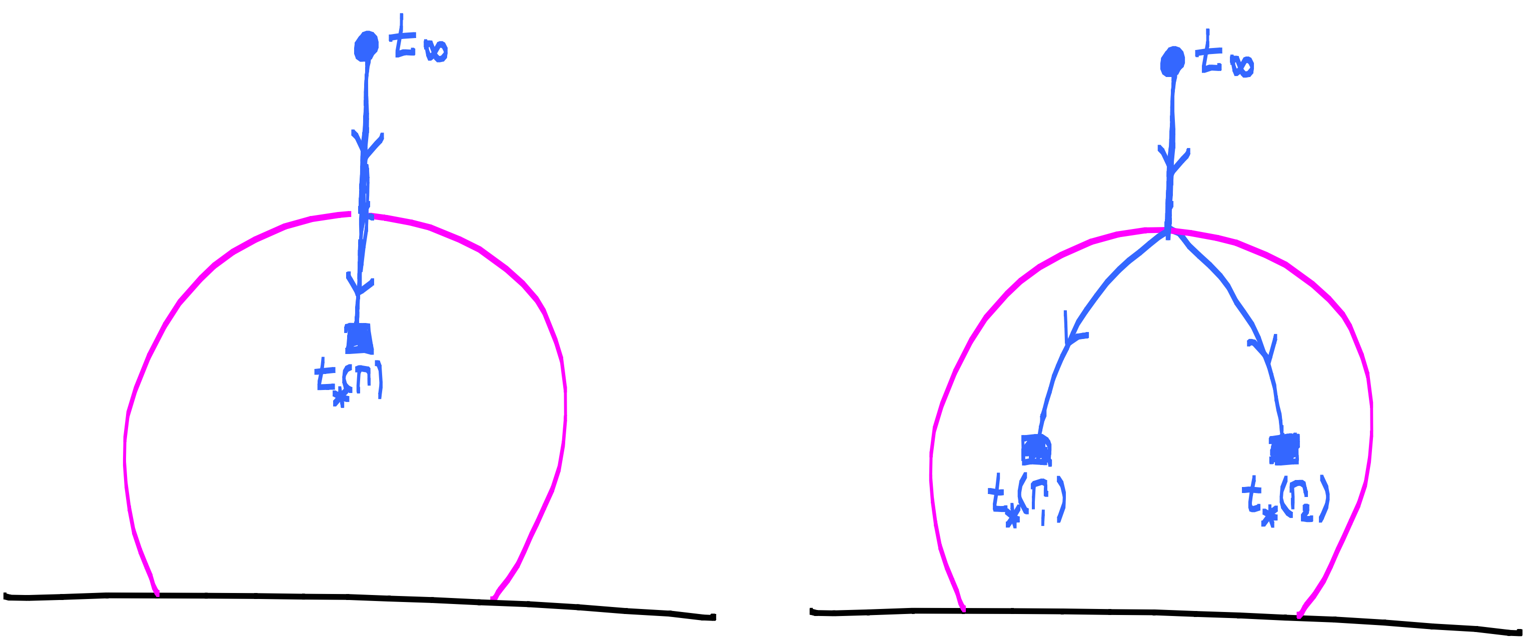
HAS A REGULAR ATTRACTOR POINT

NEVERTHELESS! WE CAN CHOOSE

q_0, Q_A SO THAT $\exists$ A TWO-CENTERED

SOLUTION WITH $\Gamma = \Gamma_1 + \Gamma_2$

$$\Gamma_1 = \left(r, \tfrac{1}{2}P, Q, \tfrac{1}{2}q_0\right) \quad \Gamma_2 = \left(-r, \tfrac{1}{2}P, -Q, \tfrac{1}{2}q_0\right)$$



BOTH SOLUTIONS EXIST

SO... COMPARE ENTROPIES

$$S(\Gamma) \quad \text{vs.} \quad S(\Gamma_1) + S(\Gamma_2)$$

IN FACT,

$\exists$ FAMILY OF CHARGES

$$\lambda\Gamma = \lambda(0, P, 0, q_0) = \Gamma_1^\lambda + \Gamma_2^\lambda$$

$$\Gamma_1^\lambda = \left(r, \tfrac{\lambda}{2}P, \lambda^2 Q, \tfrac{\lambda}{2}q_0\right) \quad \Gamma_2^\lambda = \left(-r, \tfrac{\lambda}{2}P, -\lambda^2 Q, \tfrac{\lambda}{2}q_0\right)$$

SCALING OF ENTROPIES:

$$S(\lambda\Gamma) = \lambda^2 S(\Gamma)$$

BUT!

$$\boxed{S(\Gamma_1^\lambda) = S(\Gamma_2^\lambda) \sim \frac{(\lambda P)^3}{r} \sim \lambda^3}$$

$\Rightarrow$ MANY IMPLICATIONS FOR PHYSICS & MATHEMATICS

8. SOME OPEN PROBLEMS

a.) PHYSICAL DERIVATION OF THE KS FORMULA

b.) HOW TO COMPUTE POLAR DEGENERACIES EFFECTIVELY?

c.) RESOLVE THE QUESTION OF THE ENTROPY ENIGMA: ARE THERE CANCELLATIONS BRINGING $e^{\lambda^3} \to e^{\lambda^2}$?

d.) IS THERE AN OSV-LIKE RELATION FOR $\Omega(\Gamma; t_*(\Gamma))$? DO THESE ENJOY AUTOMORPHY PROPERTIES?

e.) OUR PROOF OF (STRONG COUPLING) OSV SUGGESTS A NONPTVE DEFINITION OF Z_{TOP}. IS IT PHYSICALLY NATURAL?

f.) ARE BPS ALGEBRAS USEFUL TO THESE IDEAS?

B. MATH APPLICATIONS
(WITH E. DIACONESCU)

CONSIDER THE CASE WHERE D4 WRAPS A <u>RIGID</u> SURFACE S IN C.Y.

P IS <u>NOT</u> IN KÄHLER CONE

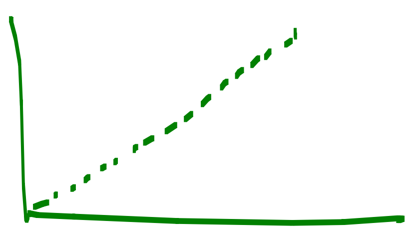
EXAMPLE $r=2$ BUNDLE ON S WITH $b^{11}(S)>1$

AS A FUNCTION OF J CAN HAVE MATHEMATICAL ("SLOPE") INSTABILITY:

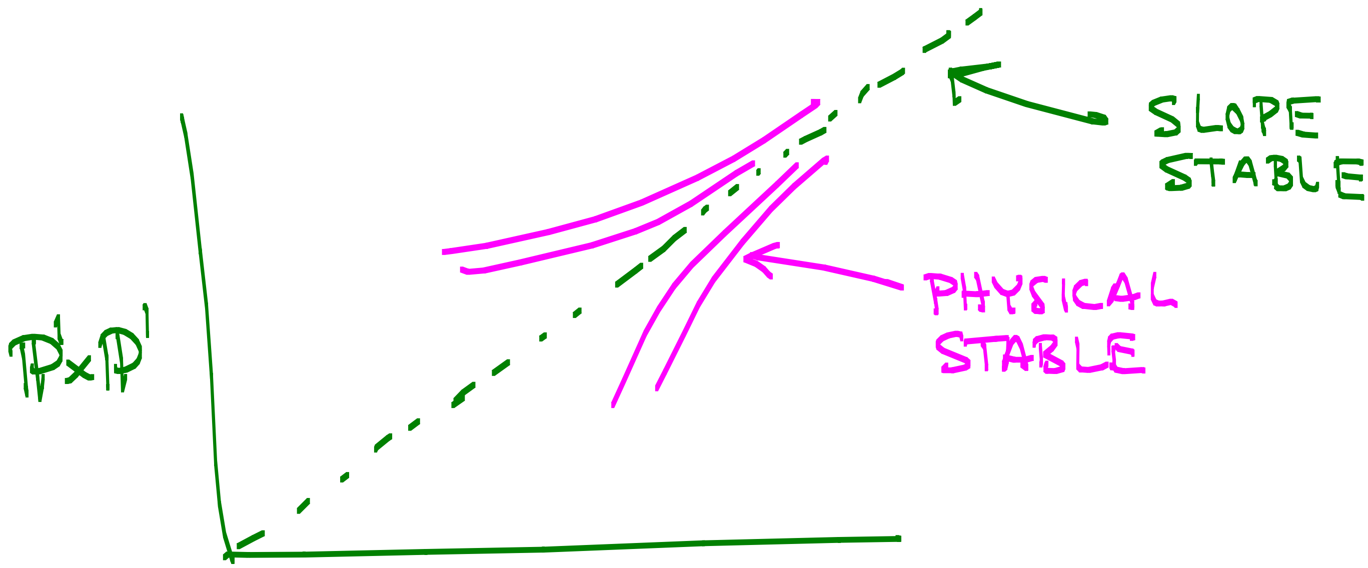
$$0 \to \mathcal{I}_{Z_2}(n_2) \to \mathcal{E} \to \mathcal{I}_{Z_1}(n_1) \to 0$$

SUCH BUNDLES BECOME UNSTABLE ACROSS WALLS IN KÄHLER CONE

Exple $\mathbb{P}^1 \times \mathbb{P}^1$



WALLS OF PHYSICAL STABILITY ASYMPTOTE TO WALLS OF MATHEMATICAL (SLOPE) STABILITY:



REFINE $\Omega(y;\Gamma;t) = \mathrm{Tr}_{\mathcal{H}(\Gamma;t)} (-y)^{2J_3}$

$$\mathcal{H}(\Gamma;t) = H^* \{ \text{MODULI OF BRANES} \}$$

$$\Omega(y;\Gamma;t) = \text{POINCARÉ POLYNOMIAL}$$

$$\Delta\Omega = (-y)^{-\langle\Gamma_1,\Gamma_2\rangle+1} \frac{1-y^{2\langle\Gamma_1,\Gamma_2\rangle}}{1-y^2} \Omega(y;\Gamma_1)\Omega(y;\Gamma_2)$$

$\Longrightarrow$ REPRODUCE RESULTS OF GÖTTSCHE AND YOSHIOKA ON MODULI OF BUNDLES,

$\Longrightarrow$ MODULI OF D4

WRAPPING A RIGID HOLO. SURFACE

IS NOT MODULI SPACE OF (SLOPE STABLE)

SHEAVES!! EVEN WHEN W.S. INST.

CORRECTIONS ARE NEGLECTED.

CONTRADICTS STATEMENTS FOUND IN THE LITERATURE. (INCLUDING MY PAPERS)

PRESUMABLY THE RIGHT CONCEPT IS THE MODULI SPACE OF STABLE OBJECTS IN THE DERIVED CATEGORY (YET TO BE CONSTRUCTED)