Moonshine Phenomena, Supersymmetry, and Quantum Codes

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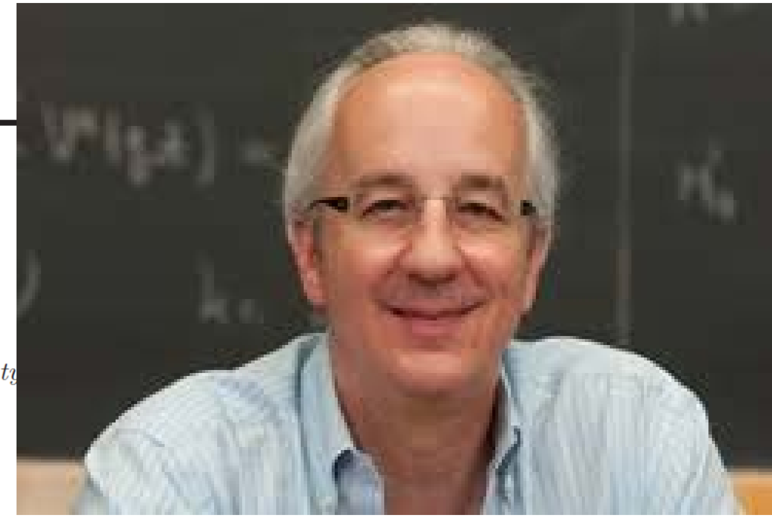
Moonshine, Superconformal Symmetry, and Quantum Error Correction



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Abstract: Special conformal field theories can have symmetry groups which are interesting sporadic finite simple groups. Famous examples include the Monster symmetry group of a $c = 24$ two-dimensional conformal field theory (CFT) constructed by Frenkel, Lepowsky and Meurman, and the Conway symmetry group of a $c = 12$ CFT explored in detail by Duncan and Mack-Crane. The Mathieu moonshine connection between the K3 elliptic genus and the Mathieu group M_{24} has led to the study of K3 sigma models with large symmetry groups. A particular K3 CFT with a maximal symmetry group preserving $(4,4)$ superconformal symmetry was studied in beautiful work by Gaberdiel, Taormina, Volpato, and Wendland [41]. The present paper shows that in both the GTVW and $c = 12$ theories the construction of superconformal generators can be understood via the theory of quantum error correcting codes. The automorphism groups of these codes lift to symmetry groups in the CFT preserving the superconformal generators. In the case of the $N = 1$ supercurrent of the GTVW model our result, combined with a result of T. Johnson-Freyd implies the symmetry group is the maximal subgroup of M_{24} known as the sextet group. (The sextet group is also known as the holomorph of the hexacode.) Building on [41] the Ramond-Ramond sector of the GTVW model is related to the Miracle Octad Generator which in turn leads to a role for the Golay code as a group of symmetries of RR states. Moreover, $(4,1)$ superconformal symmetry suffices to define and decompose the elliptic genus of a K3 sigma model into characters of the $N = 4$ superconformal algebra. The symmetry group preserving $(4,1)$ is larger than that preserving $(4,4)$.

April 1, 2020

Keywords: Conformal Field Theory, Conformal and W Symmetry, Extended Supersymmetry, Su-

Executive Summary

We had a new idea for how to solve
a long-standing interesting open problem

We worked out our idea in one nontrivial and
promising example, but it did not solve the problem.

But we found a lot of interesting things along the way.

1 Background On Moonshine

2 New Moonshine: Mathieu & Umbral

3 Quantum Mukai Theorem

4 GTVW Model

5 Supercurrents & Codes

6 RR States: MOG Construction Of The Golay Code

7 Concluding Remarks

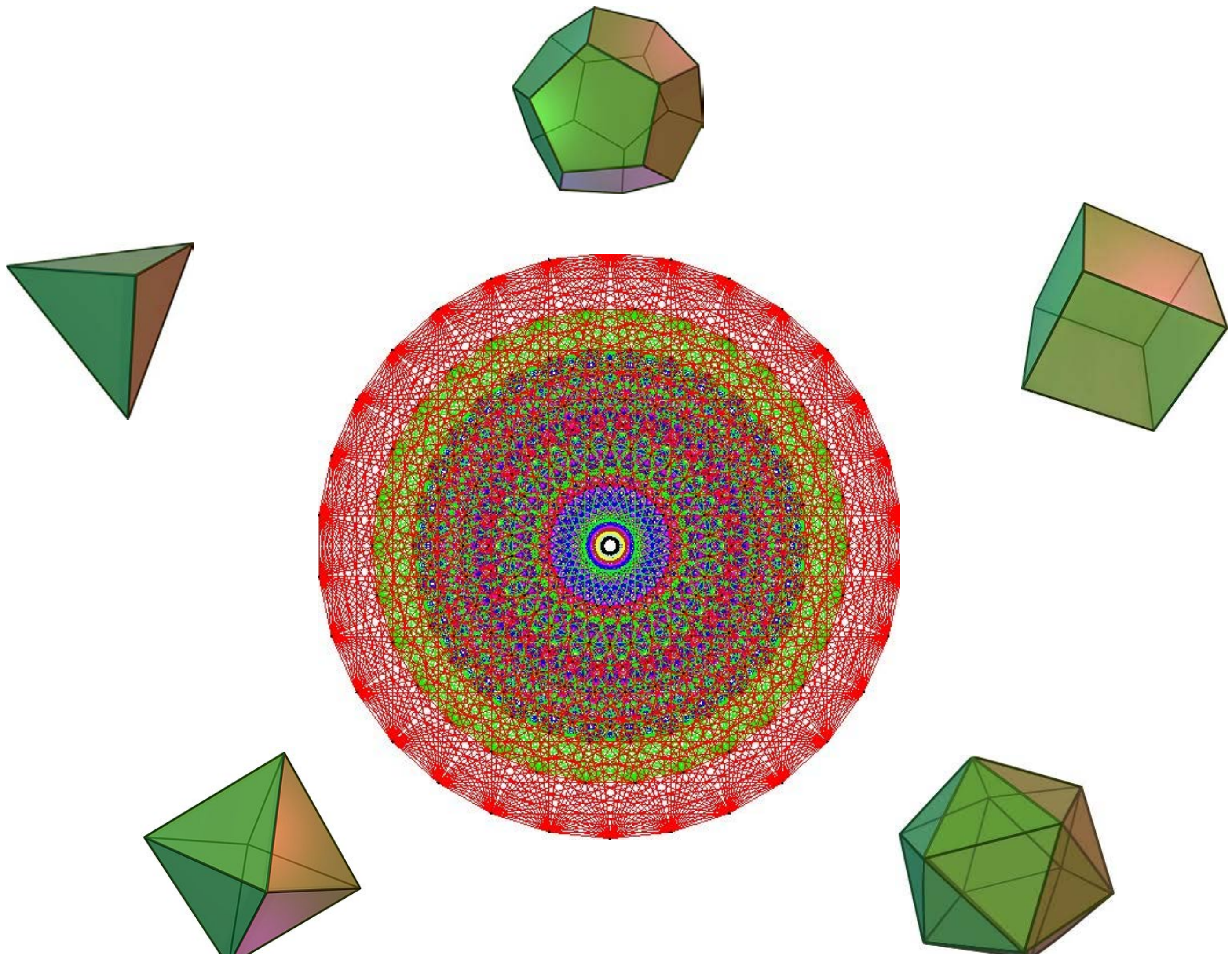
Philosophy – 1/2

We can divide physicists into two classes:

Our world is a random choice drawn from a huge ensemble:

Philosophy – 2/2

The fundamental laws of nature are based on some beautiful exceptional mathematical structure:



Background: Finite-Simple Groups

Jordan-Holder Theorem: Finite simple groups are the atoms of finite group theory.

$\mathbb{Z}_p$ p = prime A_n $n \geq 5$ $SL_n(\mathbb{F}_p)$ etc.

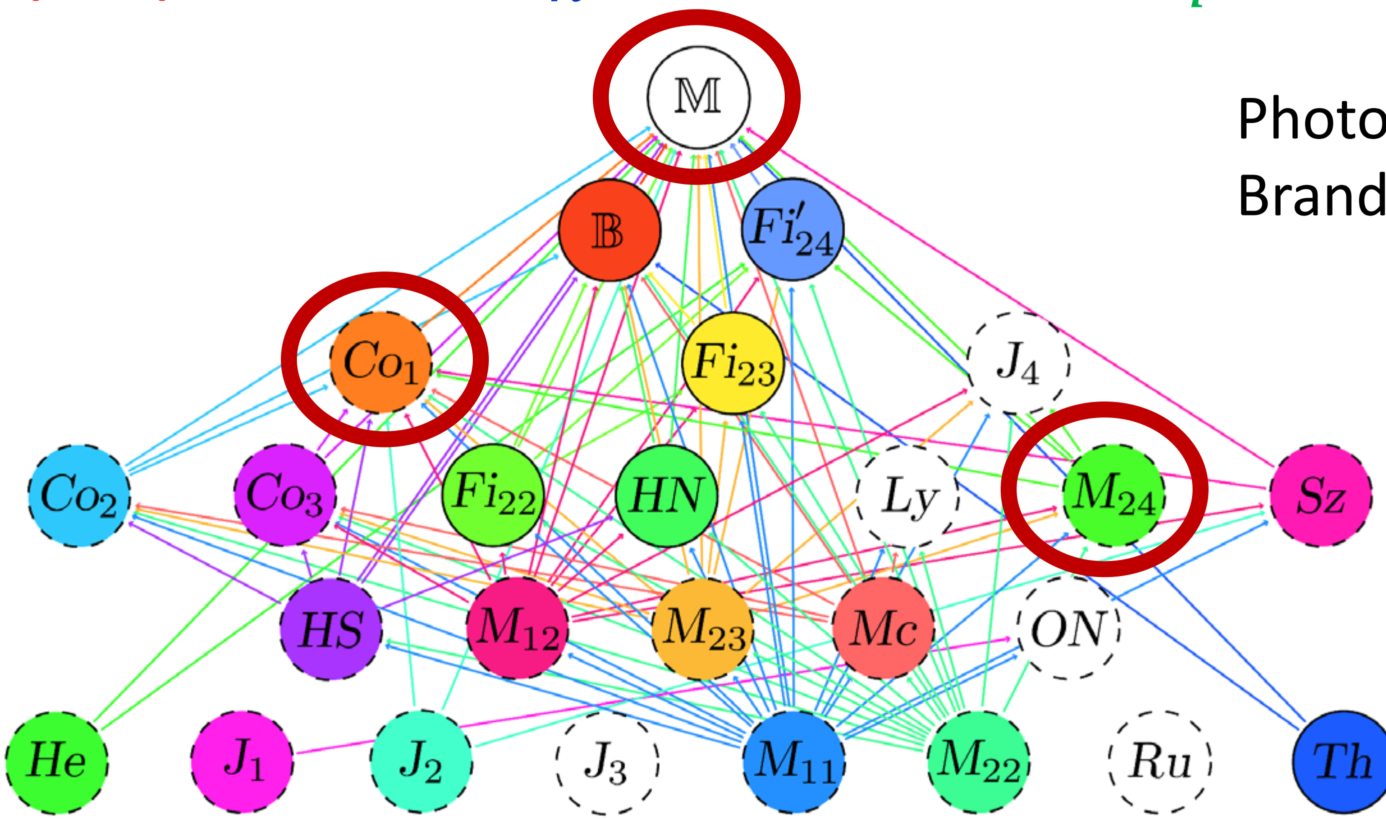


Photo credit:
Brandon Rayhaun

$$|\mathbb{M}| \cong 8 \times 10^{53}$$

$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

$$|Co_1| = 2^{21} \cdot 3^{9} \cdot 5^{4} \cdot 7^{2} \cdot 11 \cdot 13 \cdot 23 \cong 4 \times 10^{15}$$

$$|M_{24}| = 2^{10} \cdot 3^{3} \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cong 2 \times 10^{8}$$

Background: McKay & Conway-Norton 1978-1979

$$J = \sum_n J_n q^n = q^{-1} + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots$$

Now list the dimensions of irreps of $\mathbb{M}$

R_n = 1, 196883, 21296876, 842609326, 18538750076, 19360062527, 293553734298,…. , $\sim 2.6 \times 10^{26}$

$$J_{-1} = R_1 \qquad J_1 = R_1 + R_2$$

$$J_2 = R_1 + R_2 + R_3 \qquad J_3 = 2R_1 + 2R_2 + R_3 + R_4$$

A way of writing J_n as a positive linear combination of the R_j for all n is a ``***solution of the Sum-Dimension Game***.''

There are infinitely many such solutions!!

Background: Characters

Which, if any, of these solutions is interesting?

Every solution defines an infinite-dimensional $\mathbb{Z}$-graded representation of $\mathbb{M}$

$$V = q^{-1} R_1 \oplus q(R_1 \oplus R_2) \oplus q^2(R_1 \oplus R_2 \oplus R_3) \oplus \cdots$$

Now for every $g \in \mathbb{M}$ we can compute the character:

$$\chi(q;g) := Tr_V \, g \, q^N$$

A solution of the Sum-Dimension game is *modular* if the $\chi(q;g)$ is a modular function in $\Gamma_0(m)$ where $g^m = 1$.

Amazing Fact Of Monstrous Moonshine

There is a unique modular solution of the Sum-Dimension game!

Moreover the $\chi(q;g)$ have very special properties.
(``genus zero'')

Chiral Conformal Field Theory

Massless scalar in 1+1 dimensions:

$$\mathrm{x}(\sigma, t) = x_L(\sigma + t) + x_R(\sigma - t)$$

$$\partial_z x^j = -i \sum_n \alpha_n^j e^{inz} \qquad \mathrm{j} = 1, \ldots, 24$$

$$z = \sigma + \tau \qquad \left[\alpha_n^i, \alpha_m^j\right] = n\delta^{ij}\delta_{n+m,0}$$

Periodic scalar $\Rightarrow$ $\qquad \alpha_0^j = p^j \in \Lambda$

Leech & Golay

FLM use Leech lattice Λ:

Definition:[Cohn,Kumar,Miller,Radchenko,Viazovska]

$\Lambda \subset \mathbb{R}^{24}$ is the best sphere packing in d=24

Λ can be constructed using the Golay code $\mathcal{G} \subset \mathbb{F}_2^{24}$

$\mathcal{G}$ is a special 12-dimensional subspace with nice error-correcting properties. Discovered @ Bell Labs in 1949 and used by Voyager 1&2 to send color photos

Definition: $M_{24} \subset S_{24}$ is the subgroup of permutations preserving the set $\mathcal{G}$

$$Aut(\Lambda) = Co_0 \subset SO(24)$$

Special B-field

Moreover, target space torus has a very special ``B-field''

$\frac{1}{2!} B_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \Rightarrow$ ``topological term'' in the action

$$S = \int d^2\sigma \, (\, G_{\mu\nu} \, \partial_i x^{\mu} \, \partial^i x^{\nu} + B_{\mu\nu} \epsilon^{ij} \partial_i x^{\mu} \partial_j x^{\nu})$$

Translation symmetry by half-periods converted to a magnetic translation group:

$$T\left(\frac{1}{2} v_1\right) T\left(\frac{1}{2} \, v_2\right) = (-1)^{v_1 \cdot v_2} \, T\left(\frac{1}{2} v_2\right) T\left(\frac{1}{2} v_1\right)$$

This is a discrete Heisenberg group: There is a unique irreducible representation: It is $2^{\frac{24}{2}} = 2^{12}$ dimensional.

FLM Construction – 2/3

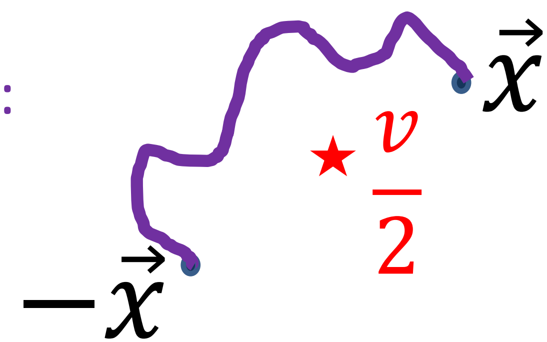
Now ``orbifold'' by $\vec{x} \to -\vec{x}$ for $\vec{x} \in \mathbb{R}^{24}/\Lambda$

''Orbifold by a symmetry G of a CFT'':

Gauge the symmetry

Symmetric twist fields: 2^{24}-dimensional space:

Basis: σ_v where v are the ``TRIM'' $\left[\frac{1}{2}v\right] \in T$



Chiral twist fields span a ``square-root'' of this representation: Very subtle quantum fields.

$\mathbb{M}$ As An Automorphism Group

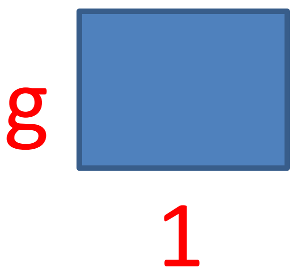
OPE of conformal fields form a VOA:

$$\mathcal{O}_i(z)\mathcal{O}_j(z) \sim z_{12}^{-\Delta_{ijk}} c_{ijk}\mathcal{O}_k(z_2) + \cdots$$

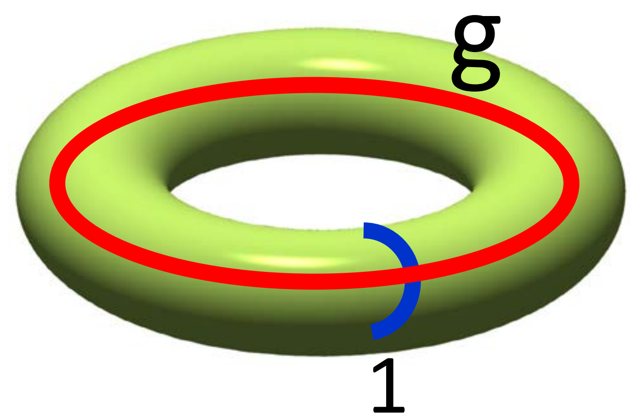
FLM & Borcherds: Automorphisms of the OPE algebra of the quotient theory = $\mathbb{M}$

Magnetic translation group of translations by TRIM + Co_1 + a ``quantum symmetry'' exchanging twisted & untwisted sectors generate the Monster.

Payoff: Conceptual Explanation of Modularity



$$:= Tr_{\mathcal{H}}\, g\, q^{L_0-\frac{c}{24}} =$$



$\Longrightarrow$ Modularity

This is the gold standard for the conceptual explanation of Moonshine-modularity

A truly satisfying conceptual explanation of genus zero properties remains elusive.

(The best attempt: Paquette, Persson, Volpato 2017)

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Why Should Physicists Care? 1/2

CFT explanation of Monstrous Moonshine by Frenkel, Lepowsky, Meurman, & Borcherds drove many developments in 2d CFT, expecially RCFT

Techniques introduced to explain moonshine – orbifolds, VOA, holomorphic CFT have played a key role in other aspects of physics as well and have led to many important advances…

e.g. modular tensor categories are a direct descendent of this research --

Why Should Physicists Care? 2/2

History repeats itself

Lightning does not strike twice

New Moonshine

Eguchi, Ooguri, Tachikawa 2010 + much interesting subsequent work.

Now generalize in two ways:

Generalize the target space torus T to sigma model with target $\mathcal{X}$

Make the theory worldsheet supersymmetric: (x^{μ}, ψ^{μ})

Get a CFT if $\mathcal{X}$ is a complex manifold that solves Einstein's equations: $R_{\mu\nu} = 0$

A K3 surface IS a solution of the Euclidean signature Einstein equations that is also compact and simply connected.

Now CFT has (4,4) superconformal symmetry.

(Super-) Conformal Symmetry:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \qquad n, m \in \mathbb{Z}$$

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

Superconformal symmetry ⇒ supercurrent:

$$T_F(z) = \sum_r G_r z^{-r-\frac{3}{2}} \qquad T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

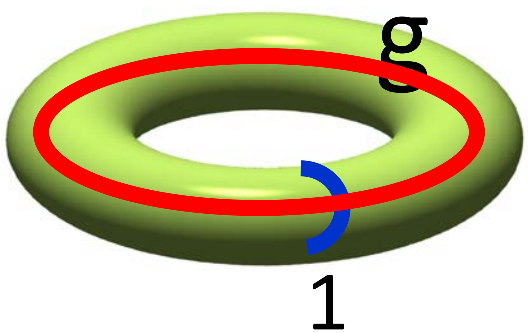
(p, q) superconformal symmetry ⇒

p holomorphic $T_F^i(z)$ and q anti-holomorphic $T_F^a(\bar{z})$

Elliptic Genus (Witten index) for K3

for any symmetry group G of the CFT, if $g \in G$ and g commutes with a right-moving susy

$$\mathcal{E}_{\mathcal{C}}^{g}(z,\tau) = Tr_{\mathcal{H}_{RR}}(-1)^F \cdot \; g \; \cdot \; e^{2\pi\, i\, \tau\left(L_0 - \frac{c}{24}\right) + 2\pi i\, z\, J_0^3 \, - 2\pi i\, \bar{\tau}\left(\tilde{L}_0 - \frac{c}{24}\right)}$$

$=$ 

Modularity

Example:

$$\mathcal{E}_{\mathcal{C}}^{g=1}(z,\tau) = \quad 8\left[\left(\frac{\vartheta_2(z)}{\vartheta_2(0)}\right)^2 + \left(\frac{\vartheta_3(z)}{\vartheta_3(0)}\right)^2 + \left(\frac{\vartheta_4(z)}{\vartheta_4(0)}\right)^2\right]$$

The New Moonshine Phenomena Remain Unexplained – 1/2

Remarkably one can also define functions $\mathbb{E}^g(z,\tau)$ for all $g \in M_{24}$ with the ``right'' modular properties,

AS IF there were an M24 symmetry of the K3 sigma model.....

$$g \in Aut(\mathcal{C}) \Rightarrow \quad \mathbb{E}^g(z,\tau) = \mathcal{E}_{\mathcal{C}}^g(z,\tau)$$

But there is no obvious M24 action on the K3 sigma model !!

New Moonshine: Mathieu Moonshine

Model has (4,4) susy so consider isotypical decomposition:

$$\mathcal{H}_{RR} = \oplus_{h,\ell;\tilde{h},\tilde{\ell}} D_{h,\ell;\tilde{h},\tilde{\ell}} \; R_{h,\ell} \otimes \tilde{R}_{\tilde{h},\tilde{\ell}}$$

$R_{h,\ell}$: Unitary highest weight irrep of N=4 with $L_0 v = h\, v$ and $J_0^3 v = \ell\, v$

$$ch_{h,\ell}(z,\tau) \coloneqq Tr_{R_{h,\ell}}\, e^{2\pi i\,\tau\left(L_0 - \frac{c}{24}\right) + 2\pi\, i\, z\, J_0^3}$$

g commutes with (4,4) $\Rightarrow$ $\quad \mathcal{E}_{\mathcal{C}}^{g}(z,\tau) =$

$$\sum_{n\geq 0,\ell} \left(Tr_{D_{n+\frac{1}{4},\ell;\frac{1}{4},0}}(g) \; - 2\, Tr_{D_{n+\frac{1}{4},\ell;\frac{1}{4},\frac{1}{2}}}(g) \right) ch_{n+\frac{1}{4},\ell}(z,\tau)$$

Statement Of Mathieu Moonshine

[EOT 2010, M. Cheng 2011, Gaberdiel, Hohenegger, Volpato 2011; Gannon 2012]

There exist an infinite set of representations of the group $M24$

$$H_{0,0}, \qquad H_{0,\frac{1}{2}}, \qquad H_n, \qquad n \geq 1$$

$$\mathbb{E}^g(z,\tau) := Tr_{H_{0,0}}(g)\, ch_{\frac{1}{4},0} + Tr_{H_{0,\frac{1}{2}}}(g)\, ch_{\frac{1}{4},\frac{1}{2}} + \sum_{n=1}^{\infty} Tr_{H_n}(g)\, ch_{n+\frac{1}{4},\frac{1}{2}}$$

Has suitable modular behavior for ALL $g \in M24$

IF $g \in Aut(\mathcal{C})$ **then** $\mathcal{E}_{\mathcal{C}}^{g}(z,\tau) = \mathbb{E}^g(z,\tau)$

Why Is It Moonshine?

There is no obvious action of M24 on $\mathcal{C}$ nor on the highest weight states $D_{n+\frac{1}{4},\ell;\frac{1}{4},\tilde{\ell}}$

Why should $\mathbb{E}^{g}(z,\tau)$ have good modular properties when we don't know how g acts on the CFT?

The New Moonshine Phenomena Remain Unexplained – 2/2

There is no known analog of the FLM construction revealing M24 symmetry. .

Despite 10 years of intense effort by a small, but devoted, community of physicists and mathematicians….

We don't understand something about symmetries of 2d conformal field theories.

It might be something important. Or maybe not.

Umbral Moonshine: This is only the first of a series of similar examples.

A nontrivial generalization of this statement: There is one example for each of the 23 Niemeier lattices based on root systems.

[Cheng, Duncan, Harvey, 2012]

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Quantum Mukai Theorem

Most obvious approach is to find a K3 surface $\mathcal{X}$ with a lot of symmetry, so that the σ $-$model also has a lot of symmetry.

Important no-go theorem:

There is a 1-1 correspondence between

(a.) Symmetry groups of K3 sigma-models commuting with (4,4) supersymmetry.

(b.) Subgroups of Co_0 fixing sublattices of Λ of rank ≥ 4.

M. Gaberdiel, S. Hohenegger, R. Volpato 2011

Why The Leech Lattice?

{Hyperkahler volume 1 metrics on a K3 surface }
= { Positive 3-planes in $H^2(K3;\mathbb{R})$}

$H^2(K3;\mathbb{Z})$ is even unimodular of signature (3,19)

Linear span of the three hyperkahler forms ω_I span $\Sigma \subset H^2(K3;\mathbb{R})$

The full K3 lattice $H^*(K3,\mathbb{Z})$ is
even unimodular, of signature (4,20)

Why The Leech Lattice ?

Using the sigma-model data $(G_{\mu\nu}, B_{\mu,\nu})$
construct a 4d positive definite space $\Pi \subset H^*(K3;\mathbb{R})$

{Space of sigma models on a K3 surface}
= {Positive 4-planes $\Pi \subset H^4(K3;\mathbb{R})$}

[Aspinwall-Morrison 1994]

$\Pi^\perp$ has signature (0,20) with G-action.
With ingenuity it can be embedded into Λ
with same G-action

With Anindya Banerjee, we recently used similar methods to classify all the hyperkahler isometry groups of K3 surfaces – there is an explicit list of 40 cases:

All of these theorems are generalizations of the famous Mukai result relating symplectic automorphisms of K3 surfaces to certain subgroups of M_{23}

#	Rank	Order	G	Index	α	Type
1	5	960	$2^4 : A_5$	#11357	2	M_{23}^*
2	5	384	$4^2 : S_4$	#18135	2	M_{23}
3	5	360	A_6	#118	3	M_{23}^*
4	5	288	$A_{4,4}$	#1026	2	M_{23}^*
5	5	192	T_{192}	#1493	2	M_{23}^*
6	5	192	H_{192}	#955	2	M_{23}
7	5	168	$L_2(7)$	#42	3	M_{23}^*
8	5	120	S_5	#34	3	M_{23}
9	5	72	M_9	#41	2	M_{23}
10	5	72	$N_{72} \cong 3^2 D_8$	#40	2	M_{23}
11	5	48	$T_{48} \cong Q_8 \rtimes S_3$	#29	2	M_{23}^*
12	6	192	$4^2 A_4$	#1023	2	M_{23}^*
13	6	96	$2^4 D_6$	#227	2	M_{23}
14	6	72	$A_{4,3}$	#43	3	M_{23}^*
15	6	64	$\Gamma_{25} a_1$	#138	2	M_{23}
16	6	60	A_5	#5	4	M_{23}^*
17	6	48	$2 \times S_4$	#48	2	M_{23}
18	6	36	$3^2 Z_4$	#9	3	M_{23}
19	6	36	$S_{3,3}$	#10	2	M_{23}
20	6	21	F_{21}	#1	3	M_{23}^*
21	6	20	$\mathrm{Hol}(\mathbb{Z}_5)$	#3	3	M_{23}
22	6	16	SD_{16}	#8	2	M_{23}
23	7	48	$2^4 \times 3$	#50	2	M_{23}^*
24	7	32	$2^4 Z_2$	#27	2	M_{23}
25	7	32	$Q_8 * Q_8$	#49	2	M_{23}
26	7	24	S_4	#12	4	M_{23}
27	7	8	Q_8	#4	2	M_{23}
28	8	18	$A_{3,3}$	#4	3	M_{23}^*
29	8	16	$D_8 \times 2$	#11	2	M_{23}
30	8	12	D_{12}	#4	4	M_{23}
31	8	12	A_4	#3	4	M_{23}^*
32	8	10	D_{10}	#1	4	M_{23}^*
33	9	16	2^4	#14	2	M_{23}
34	9	8	D_8	#3	4	M_{23}
35	10	8	2^3	#5	2	M_{23}
36	10	6	S_3	#1	5	M_{23}
37	10	4	4	#1	4	M_{23}
38	12	4	2^2	#2	4	M_{23}
39	12	3	3	#1	6	M_{23}^*
40	16	2	2	#1	8	M_{23}

From the viewpoint of explaining Mathieu Moonshine, the QMT is a huge disappointment:

M24 is not a subgroup of any quotient of any GHV group.

We need a new idea

Moonshine is about the elliptic genus.

Only (4,1) susy is needed to define the elliptic genus

Stab(4,1) is much bigger than Stab(4,4).

So Jeff and I studied one example where we can compute Stab(4,1) exactly.

It did not solve the problem.

Nevertheless, we found something interesting along the way.

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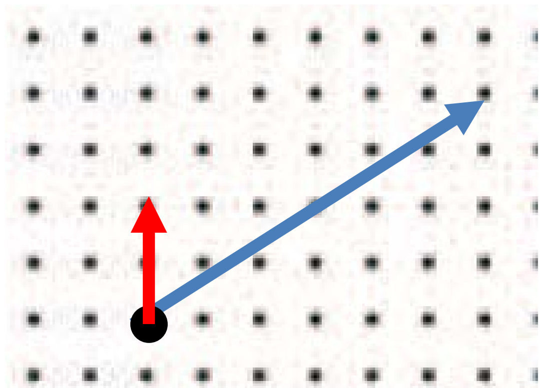
Symmetries Preserving Sublattices

Given a symmetric lattice what sublattices fixed by some nontrivial subgroup of the point group ?

In general, a sublattice preserves none of the (nontrivial) crystal symmetries of the ambient lattice.

Consider, e.g., the lattice generated by (p,q) in the square lattice in the plane.



Fixed Sublattices Of The Leech Lattice

The culmination of a long line of work is the classification by Hohn and Mason of the 290 isomorphism classes of fixed-point sublattices of the Leech lattice:

99	4	245760	$2^8{:}M_{20}$	$2_{II}^{-2}4_{II}^{-2}$	0	1	1	1	1	-	Mon_a*
100	4	30720	$[2^9].A_5$	$2_{II}^{-4}5^{-1}$	0	1	1	1	1	-	Mon_a*
101	4	29160	$3^4{:}A_6$	$3^{+2}9^{+1}$	1	1	1	1	1	-	S*
102	4	20160	$L_3(4)$	$2_{II}^{-2}3^{-1}7^{-1}$	2	1	1	1	2	1	M_{23}*
103	4	12288	$[2^{12}3]$	$2_{II}^{+2}4_3^{+1}8_1^{+1}$	0	1	2	1	1	-	Mon_a
104	4	9216	$[2^{10}3^2]$	$2_{II}^{+4}3^{+2}$	0	1	2	1	1	-	Mon_a*
105	4	6144	$[2^{11}3]$	$2_{II}^{-2}4_6^{+2}3^{-1}$	0	1	4	1	1	-	Mon_a
106	4	5760	$2^4{:}A_6$	$4_5^{-1}8_1^{+1}3^{+1}$	2	1	1	1	2	1	M_{23}*
107	4	4096	$2^{1+8}{:}2^3$	4_4^{+4}	0	1	8	1	1	-	Mon_a
108	4	2520	A_7	$3^{+1}5^{+1}7^{+1}$	3	1	1	1	2	1	M_{23}*
109	4	1944	$3^{1+4}{:}2.2^2$	$2_2^{+2}3^{+3}$	1	1	1	1	1	-	S
110	4	1920	$2^4{:}S_5$	$4_3^{-1}8_1^{+1}5^{-1}$	2	1	2	1	3	1	M_{23}
111	4	1344	$2^3{:}L_2(7)$	$4_2^{+2}7^{+1}$	2	1	1	1	3	1	M_{23}*
112	4	1152	$Q(3^2{:}2)$	$8_6^{-2}3^{-1}$	2	1	2	1	2	1	M_{23}*

GTVW Model

Largest group $Stab(4,4) \cong 2^8 : M20$ associated with a distinguished K3 sigma model investigated by Gaberdiel, Taormina, Volpato, Wendland.

2d susy sigma model with target:

$\mathcal{X} = T/\mathbb{Z}_2$ $\quad T = \mathbb{R}^4/L$ $\quad L$: 4d bcc lattice

$$\mathcal{X} = \frac{T(Spin8)}{\mathbb{Z}_2}$$

Special B-field

$$B(v,w) = g(v,w)\ mod\ 2$$

$$v, w \in \pi_1(T)$$

Equivalence To A WZW Model

Amazing result of GTVW:

This model is isomorphic to the product of 6 copies of the ***bosonic*** k=1 SU(2) WZW model !

WZW with $G = SU(2)^6$ and each factor has WZW term with $k = 1$

$SU(2)$ current algebra with level $k = 1$ has 2 unitary hw irreps: V_0 and V_1

Spin CFT vs. Bosonic CFT

To a CFT and a non-anomalous $\mathbb{Z}_2$ $-$symmetry one can construct a ``spin lift'' by coupling to the Arf invertible TQFT and gauging.

To a spin CFT one can associate a GSO-projected bosonic CFT

GSO projection of the GTVW model is the bosonic ``level 1'' $SU(2)^6$ WZW model

Nonabelian Bosonization

("Witten's nonabelian bosonization" or "FKS construction")

Gaussian model: $S = \frac{R^2}{4\pi}\int \partial x \tilde{\partial} x \qquad x \sim x + 2\pi$

$$e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}+w\,R\right)x}(z) \otimes e^{\frac{i}{\sqrt{2}}\left(\frac{n}{R}-w\,R\right)\tilde{x}}(\tilde{z})$$

At R=1 we have a theory equivalent to the $SU(2)_1$ WZW model

$$J^3(z) = \frac{1}{\sqrt{2}}\partial x(z), J^{\pm}(z) = e^{\pm i\sqrt{2}x}(z)$$

$$\tilde{J}^3(\tilde{z}) = \frac{1}{\sqrt{2}}\partial \tilde{x}(\tilde{z}), \tilde{J}^{\pm}(\tilde{z}) = e^{\pm i\sqrt{2}\tilde{x}}(\tilde{z})$$

Gives an $su(2)_L \oplus su(2)_R$ current algebra.

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Supersymmetry From A ***Bosonic*** WZW Model?

We need to find a holomorphic current $T_F(z)$ of dimension $\frac{3}{2}$

With OPE:

$$T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

Chiral Fields Of Dimension 3/2

Introduce product of six holomorphic fields in the spin ½

$$V_{\epsilon_1,\epsilon_2,\ldots,\epsilon_6} := \exp\left(\frac{i\sqrt{2}}{2}(\epsilon_1 X_1 + \epsilon_2 X_2 + \cdots + \epsilon_6 X_6)\right) \quad \epsilon_i \in \{\pm 1\}$$

Conformal dimension = $\left(\frac{1}{4}\right) \times 6 = \frac{3}{2}$

$V_{\epsilon_1,\epsilon_2,\ldots,\epsilon_6}$ span a 2^6 dimensional vector space of holomorphic (3/2,0) operators.

Identify this space with the space of states in a system of 6 Qbits. For any $s \in (\mathbb{C}^2)^{\otimes 6}$ write V_s

Which Ones Are Supercurrents?

The V_s have OPE's:

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \frac{\bar{s}s}{z_{12}}T(z_2) + \frac{\bar{s}\Sigma^A s}{z_{12}^2}J^A(z_2) + \frac{\bar{s}\Sigma^{AB}s}{z_{12}}J^A J^B(z_2) + \cdots$$

J^A : generators of $SU(2)^6$ affine Lie algebra, $A = 1, \ldots, 3 \cdot 6 = 18$

Σ^A, Σ^{AB} generate 1- and 2- Qbit errors

$$T_F(z)T_F(w) \sim \frac{\frac{\hat{c}}{4}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{z-w} + \cdots$$

N=1 Generator

Up to global symmetry there is a unique N=1 generator.

Using results of GTVW it is V_{Ψ} for

$$\Psi = [\emptyset] + i\,[123456] + ([1234] + [3456] + 1256]) + i([12] + [34] + [56]) + ([135] + [245] + [236] + [146]) - i([246] + 235] + [136] + [145])$$

$$[135] \coloneqq |\, -,+,-,+,-,+\, \rangle$$

Obtained by meticulous translation from the susy for the K3 sigma model....

Is there a <u>*code*</u> governing this quantum state?

Yes!! It is the ``hexacode''

$\mathbb{F}_4$ And The Hexacode

Finite field of $4 = 2^2$ elements: $\mathbb{F}_4 = \{ 0, 1, \omega, \bar{\omega} \}$

Addition: $1 + \omega = \bar{\omega}$ $\quad 1 + \bar{\omega} = \omega$ $\quad \omega + \bar{\omega} = 1$

Multiplication: $\omega \cdot \omega = \bar{\omega}$ $\quad \omega \cdot \bar{\omega} = 1$

Hexacode: $\mathcal{H}_6 \subset \mathbb{F}_4^6$

$$w = \left(a, b, c, \Phi_{abc}(1), \Phi_{abc}(\omega), \Phi_{abc}(\bar{\omega})\right)$$

$$\Phi_{abc}(x) := a\,x^2 + b\,x + c$$

Relation To Quaternion Group

$$Q \cong \{ \pm 1, \pm i\,\sigma^1 , \pm i\,\sigma^2 , \pm i\,\sigma^3 \} \subset SU(2)$$

Group of special unitary bit-flip and phase-flip errors in theory of QEC.

For each $x \in \mathbb{F}_4$ associate a Pauli operator $h(x)$

$$h(x)h(y) = \pm\, h(x+y)$$

$$h(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$h(\omega) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad h(\overline{\omega}) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

But the sign cannot be removed by redefinitions.

$$\xleftarrow{\;h\;}$$

$$1 \to \{\pm 1\} \to Q \to \mathbb{F}_4^+ \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0$$

Associate Pauli operators $h(x)$ to $x \in \mathbb{F}_4$

$$h(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$h(\omega) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad h(\overline{\omega}) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$h(x)h(y) = c_{x,y}\, h(x+y)$$

$c_{x,y}$: A nontrivial cocycle

N=1 Generator And The Hexacode

For $w = (x_1, x_2, \ldots, x_6) \in \mathbb{F}_4^6$ define

$$h(w) := h(x_1) \otimes h(x_2) \otimes \cdots \otimes h(x_6)$$

$$h(w_1)h(w_2) = \chi(w_1, w_2)h(w_1 + w_2)$$

For general $w_1, w_2 \in \mathbb{F}_4^6$ cannot remove signs χ .

Nontrivial fact: The cocycle is trivial when restricted to $\mathcal{H}_6$!

$$h(w_1)h(w_2) = h(w_1 + w_2) \qquad w_1, w_2 \in \mathcal{H}_6 \subset \mathbb{F}_4^6$$

$$P = 2^{-6} \sum_{w \in \mathcal{H}_6} h(w)$$

One dimensional projection operator

$$\Psi \in Im(P)$$

Consequences: 1/2

V_Ψ generates an N=1 superconformal symmetry:

$$V_s(z_1)V_s(z_2) \sim \frac{\bar{s}s}{z_{12}^3} + \frac{\bar{s}s}{z_{12}}T(z_2) + \frac{\bar{s}\Sigma^A s}{z_{12}^2}J^A(z_2) + \frac{\bar{s}\Sigma^{AB} s}{z_{12}}J^A J^B(z_2) + \cdots$$

Σ^A, Σ^{AB} generate 1- and 2- qubit errors

$$\bar{\Psi}\Sigma^A\Psi = 0 \quad \& \quad \bar{\Psi}\Sigma^{AB}\Psi = 0$$

$\boxed{\text{\underline{\textit{Because}}}\ \Psi \text{ is in a QEC.} \qquad \Rightarrow \quad T_F = V_\Psi}$

Consequences: 2/2

$$Stab(\Psi) = \{g \in SU(2)^6 \times S_6 : \quad g \cdot \Psi = \Psi \}$$

The group of error operators that leaves the message Ψ invariant

It is a finite group

Again follows from the error-correcting properties of the hexacode because the generators of $SU(2)^6$ are the Σ^A

The Answer:
Holomorph Of The Hexacode

$$Hol(G) \coloneqq G\colon Aut(G)$$

Example: $Euc(\mathbb{A}^n) = Hol(\mathbb{R}^n) = \mathbb{R}^n\colon O(n)$

$$1 \to \mathbb{Z}_2^5 \to Stab(\Psi) \to Hol(\mathcal{H}_6) \to 1$$

$Hol(\mathcal{H}_6)$ = Sextet (Sextad) group:
A distinguished maximal subgroup of M_{24}

This leads to the conclusion that the symmetries of the GTVW model that commute with (4,1) supersymmetry is NOT ``large enough'' to explain M24 Moonshine.

Stab(4,4) & Stab(4,1)

$$Hol(\mathcal{H}_6) \cong 2^6{:}\,3 \cdot S_6$$

$$Stab(4,4) \cong 2^9{:}\,M_{20}$$

$$M_{20} \cong \; 2^4{:}\,A_5$$

$$1 \to Stab(4,4) \to Stab(4,1) \to (\mathbb{Z}_2 \times \mathbb{Z}_2){:}\,\mathbb{Z}_3 \to 1$$

$$|Stab(4,1)| = 2^{17} \cdot 3^2 \cdot 5$$

$$|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$$

Digression: Relation To Other Quantum Codes

This [[6,0,4]] quantum code is related to a well known QEC constructed from a unique [[5,1,3]] code.

It is related to Ψ_{GTVW} by a local unitary transformation $u \in SU(2)^6$

We realized this with TOM MAINIERO.

Mainiero has shown how to formulate a cohomology theory associated to ANY quantum state in a multipartite system $\mathcal{H} = \otimes_{i \in I} \mathcal{H}_i$

The Poincare polynomial is a surrogate for von Neumann entropy. Tom computed: $P(y) = 432\, y^2$ for both states.

Digression:
Mainiero's Entanglement Homology

$\mathcal{H} = \bigotimes_{i \in I} \mathcal{H}_i$ with state ρ

For all $J \subset I$ define the partial trace ρ_J

Algebras $\mathcal{A}_J$ $\qquad\qquad$ $\mathcal{H}_J := GNS(\rho_J)$

$$\underbrace{\mathrm{GNS}(\rho_\emptyset)}_{\mathbb{C}} \longrightarrow \prod_{|T|=1} \mathrm{GNS}(\rho_T) \rightrightarrows \prod_{|T|=2} \mathrm{GNS}(\rho_T) \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow\end{smallmatrix} \prod_{|T|=N-1} \mathrm{GNS}(\rho_T) \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow\end{smallmatrix} \mathrm{GNS}(\rho_P)$$

Is a SIMPLICIAL SET: $\Rightarrow$ Homology theory,

noncommutative geometry,...

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RR Sector And The Golay Code

The RR groundstates of GTVW in the WZW description form a rep of $(SU(2)_L \times SU(2)_R)^6$

$$e^{\pm\frac{i}{\sqrt{2}}X_L^{(\alpha)}} \otimes e^{\pm\frac{i}{\sqrt{2}}X_R^{(\alpha)}} \in \left(\frac{1}{2}\right)_L^{(\alpha)} \otimes \left(\frac{1}{2}\right)_R^{(\alpha)}$$

$$\alpha = 1,2,3,4,5,6$$

There is a distinguished basis of RR groundstates:

$$\mathbb{H} \cong \left[\left(\frac{1}{2}\right)_L \otimes \left(\frac{1}{2}\right)_R \right]_{\mathbb{R}}$$

as $SU(2)_L \times SU(2)_R$ representations

The usual basis $1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ of quaternions corresponds to 4 distinguished spin states:

$$1 \leftrightarrow \frac{1}{\sqrt{2}}(\,|+,-\rangle - |-,+\rangle) := |1\rangle$$

$$\mathfrak{i} \leftrightarrow \frac{1}{\sqrt{2}}(\,|+,+\rangle + |-,-\rangle) := |2\rangle$$

$$\mathfrak{j} \leftrightarrow \frac{i}{\sqrt{2}}(\,|+,+\rangle - |-,-\rangle) := |3\rangle$$

$$\mathfrak{k} \leftrightarrow \frac{i}{\sqrt{2}}(\,|+,-\rangle + |-,+\rangle) := |4\rangle$$

In this basis the action of $h(x)_L \otimes h(x)_R$ is diagonal action by signs, e.g $h(1)_L \otimes h(1)_R$ takes:

$|1\rangle \to |1\rangle$, $|2\rangle \to |2\rangle$, $|3\rangle \to -|3\rangle$, $|4\rangle \to -|4\rangle$

Column Interpretations Of Hexacode Digits

$|1\rangle \to |1\rangle$, $|2\rangle \to |2\rangle$, $|3\rangle \to -|3\rangle$, $|4\rangle \to -|4\rangle$

$$\begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$h(x)_L \otimes h(x)_R \quad x \in \mathbb{F}_4$$

$$\begin{array}{ccccc} x = & 0 & 1 & \omega & \bar{\omega} \\ \begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{array} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

So the subgroup $\mathcal{H}_6 \subset Stab(\Psi)$ acts diagonally on the distinguished bases for the RR sector as a 4×6 array of 0's and 1's

	$h(x_1)$	$h(x_2)$	$h(x_3)$	$h(x_4)$	$h(x_5)$	$h(x_6)$
$\vert 1\rangle$						
$\vert 2\rangle$						
$\vert 3\rangle$						
$\vert 4\rangle$						

Example:

	$h(1)$	$h(1)$	$h(\omega)$	$h(\omega)$	$h(\bar{\omega})$	$h(\bar{\omega})$
$\vert 1\rangle$	0	0	0	0	0	0
$\vert 2\rangle$	0	0	1	1	1	1
$\vert 3\rangle$	1	1	0	0	1	1
$\vert 4\rangle$	1	1	1	1	0	0

Golay Code & The MOG

Nontrivial statement: The length 24 codewords generated from this $\mathcal{H}_6$-action = Golay code words!

This gives half the Golay code $\mathcal{G}^+$

To get the full Golay code include worldsheet parity (exchanging left- and right-moving dof). This acts as the parity operator in $O(4)$

$$\begin{bmatrix} - \\ + \\ + \\ + \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ ``odd interpretations of hexacode digits ''

Golay Code & The MOG

The action of the stabilizer of $\Psi_L - \Psi_R$ within $\langle P, Q^6 \rangle \subset \left(Pin(4)\right)^6$ in the canonical basis of RR states defines the full Golay code.

This presentation of the Golay code is the <u>M</u>iracle <u>O</u>ctad <u>G</u>enerator of Curtis and Conway.

Result: A clean physical interpretation of the MOG.

So What?

The Golay code can be found in this action of symmetries commuting with $N = 1$ supersymmetry.

By definition, the automorphism group of the Golay code is $M24$

So M24 is a symmetry group
OF
the group of symmetries…

Is this the long-sought explanation of Mathieu Moonshine?

Not yet: We do not understand why the ``symmetry group OF the group of symmetries'' should imply symmetry properties of the Witten index.

However, along the way we have found some intriguing relations between quantum codes, supersymmetry and Moonshine.

We can ask if that relation persists in other examples exhibiting Moonshine.

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Other Moonshine Examples

Interestingly, a similar pattern emerges for the other two moonshine examples for Co_1 and $\mathbb{M}$

Co_1 based on $Ising^{\otimes 24}$: There is a unique supercurrent based on a quantum code (related to Golay): Reinterpretation of work of John Duncan on Conway Moonshine.

Beauty and the Beast: Superconformal Symmetry in a Monster Module

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Abstract. Frenkel, Lepowsky, and Meurman have constructed a representation of the largest sporadic simple finite group, the Fischer–Griess monster, as the automorphism group of the operator product algebra of a conformal field theory with central charge $c = 24$. In string terminology, their construction corresponds to compactification on a $\mathbf{Z}_2$ asymmetric orbifold constructed from the torus $\mathbf{R}^{24}/\Lambda$, where Λ is the Leech lattice. In this note we point out that their construction naturally embodies as well a larger algebraic structure, namely a super-Virasoro algebra with central charge $\hat{c} = 16$, with the supersymmetry generator constructed in terms of bosonic twist fields.

Important gap: What is the actual supercurrent?

The above ideas will probably allow us to fill this gap.

More Examples?

Theo Johnson-Freyd: Classified N=1 supercurrents in a wide variety of super-VOA's.

SUPERSYMMETRY AND THE SUZUKI CHAIN

THEO JOHNSON-FREYD

ABSTRACT. We classify $N{=}1$ SVOAs with no free fermions and with bosonic subalgebra a simply connected WZW algebra which is not of type E. The latter restriction makes the classification tractable; the former restriction implies that the $N{=}1$ automorphism groups of the resulting SVOAs are finite. We discover two infinite families and nine exceptional examples. The exceptions are all related to the Leech lattice: their automorphism groups are the larger groups in the Suzuki chain (Co_1, Suz:2, $\mathrm{G}_2(4)$:2, J_2:2, $\mathrm{U}_3(3)$:2) and certain large centralizers therein (2^{10}:M_{12}:2, M_{12}:2, $\mathrm{U}_4(3)$:D_8, M_{21}:2^2). Along the way, we elucidate fermionic versions of a number of VOA operations, including simple current extensions, orbifolds, and 't Hooft anomalies.

Do they all have connections to QEC?

Conclusions

1. New approach to Mathieu Moonshine based on Stab(4,1)

In the GTVW model it does not work.
Almost nothing is known about other points in the full moduli space of (4,1) models

2. Interesting connections between QEC and 2d N=1 superconformal symmetry – raises many questions.