Lie Algebras, BPS States, and String Duality

0. Introduction:

I'd like to begin with some disclaimers.

I'm not working on this subject right now - so might be hazy on some details.

My talk is based on a review I wrote hep-th/9710198

But I agreed to speak at this conference because I think the topic is important and I thought it might be useful to call attention to some ideas and open problems which have remained in a state of suspended animation for the past 8 or 9 years.

The ideas center around 3 loosely related topics:

A. Geometrical realizations of affine Lie algebras

B. Denominator products of certain GKM's and Gromov-Witten Theory

C. Denominator products + B.H. state counting

A. Geometrical Algebras

~~[scribbled out]~~ I need to remind you of some standard stuff regarding duality:

1. Narain theory
2. Heterotic / M / IIA
3. BPS states
4. BPS algebras

1. $\mathrm{Het}/T^d \times \mathbb{R}^{10-d}$ & Narain Theory (1985)

Comp's of heterotic theory on tori are based on the Narain families of CFT's

Let $II^{p,q}$ be the even unimodular lattice

embed: $$II^{p,q} \hookrightarrow \Gamma^{p,q} \subseteq \mathbb{R}^{p,q} = \mathbb{R}_-^{p,0} \oplus \mathbb{R}_+^{0,q}$$

$$\underset{P}{\Psi} \qquad = P_L \oplus P_R$$

CFT: $\Gamma^{p,q}$ specifies periodicity of cpt bosons.

Space of CFT's

$$\mathcal{N}(p,q) = O(II^{p,q}) \backslash O(p,q;\mathbb{R}) / O(p) \times O(q)$$

Heterotic string $\quad p = 16+d$
$\quad q = d.$

Now I want to discuss what is ~~[illegible]~~ the low energy gauge symmetry — as a function of these moduli:

We need to understand a little about the vertex operators and BRST cohomology

$$\text{Het} = (26\ \text{bosonic})_L \otimes (10\ \text{super})_R$$

Physical particles $\leftrightarrow$

$$V = \mathcal{P}(\partial x)\, e^{ik\cdot x}(z) \otimes \mathcal{P}(\tilde\psi, \partial \tilde x)\, e^{i\tilde k\cdot\tilde x}(\bar z)$$

$$k = (E, \vec q\,;\, P_L)$$
$$\tilde k = (E, \vec q\,;\, P_R)$$

$$P = P_L \oplus P_R \in \Gamma^{16+d,d}$$

BRST:

$$\left.\begin{aligned} N + \tfrac12 k^2 - 1 = 0 \\ \tilde N + \tfrac12 \tilde k^2 - \tfrac12 = 0 \end{aligned}\right\} \Rightarrow \quad m^2 = E^2 - \vec q^{\,2} = 2N - 2 + \vec P_L^{\,2}$$
$$= 2\tilde N - 1 + \vec P_R^{\,2}$$

Generic points in moduli space: $U(1)^{16+2d}$ gauge symmetry associated with $P=0$:

$$\partial x^i e^{ik\cdot x} \otimes \tilde\psi^\mu e^{i\tilde k\cdot \tilde x} \qquad k^2 = \tilde k^2 = 0$$
$$\partial x^\mu e^{ik\cdot x} \otimes \tilde\psi^i e^{i\tilde k \tilde x}$$

$$S = \int \sum_{a=1}^{16+d} F^a * F^a + \cdots$$

However, there are subvarieties of $W(p,q)$ where for $\mathfrak{g} = \oplus\, A, D, E \quad rk \le 16 + 2d$

$$\Gamma(\mathfrak{g}) \subset \Gamma^{p,q} \quad \text{with} \quad \Gamma(\mathfrak{g})_R = 0 .$$

$$\Rightarrow \quad P \doteq (\alpha ; 0) \qquad \alpha = \text{root vector}, \ \alpha^2 = 2$$

$\Rightarrow$ for such P $\quad \cancel{K = (E, \vec{q}, \alpha)}$

$V_\alpha = e^{ik\cdot x} \otimes \tilde{\psi}^r e^{i\tilde{k}\cdot x} \quad \cancel{K = (E, \vec{q}, 0) \quad V_\alpha = e^{ik\cdot x} \otimes \tilde{\psi}^r e^{i\tilde{k}\cdot x}}$

$$m^2 = \vec{\alpha}^2 - 2 = 0 .$$

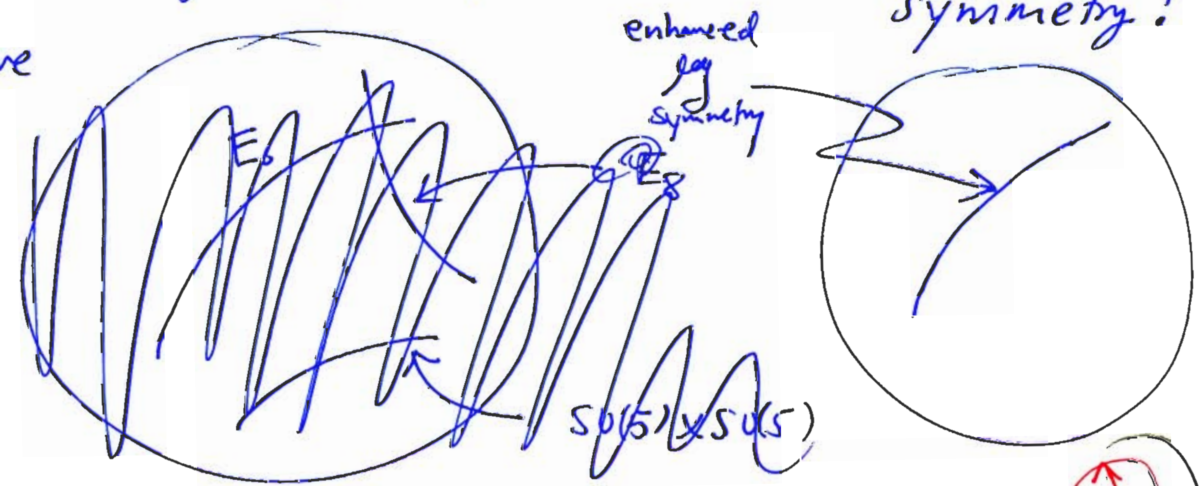
Now the interactions of these particles follow from the OPE

$$V_\alpha(z_1)\, V_\beta(z_2) \sim \frac{f_{\alpha,\beta}^{\alpha+\beta}}{z_1 - z_2} V_{\alpha+\beta}(z_2) + \cdots$$

essentially the Frenkel-Kac construction.

$\Rightarrow \quad S = \int \mathrm{Tr}(F * F) + \cdots$ Nonabelian gauge symmetry!

So we have the picture:



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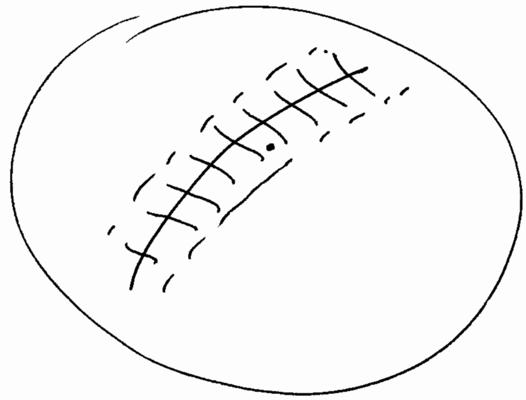
Now let us look in the neighborhood of these subvarieties:



Natural connection on the CFT - particle ~~space~~ exists but now

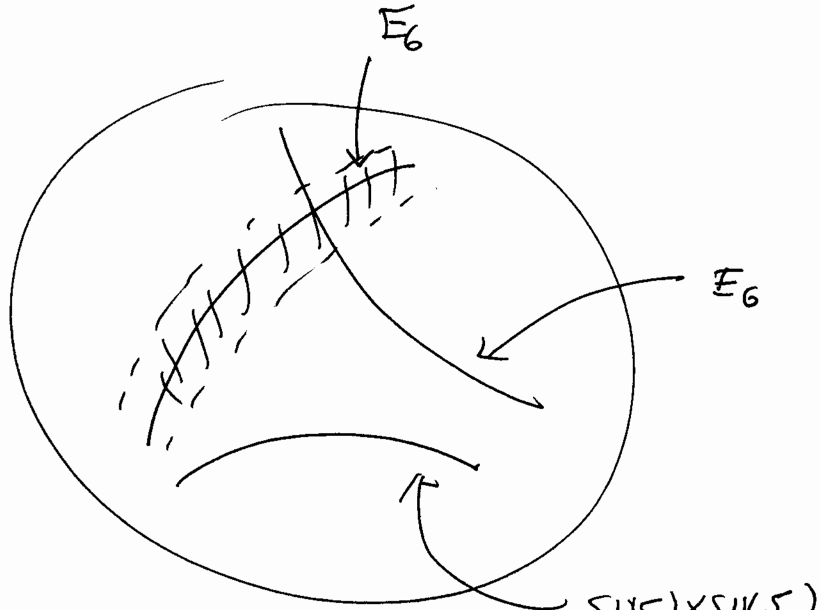
$$P = (P_L; P_R) \quad \text{with} \quad P^2 = \vec{P}_R^{\,2} - \vec{P}_L^{\,2} = -2$$

$$m^2 = \vec{P}_L^{\,2} - 2 = \vec{P}_R^{\,2} > 0$$

$\Rightarrow$ gauge bosons are massive.

An adjoint scalar field gets a vev and spontaneously breaks the gauge symmetry.

Now, of course there are many different such subvarieties



↳ question:

Is there a way to unify these gauge symmetries?

2. Heterotic/M/IIA Duality (1995)

<u>Fact</u>: $\mathcal{N}(19,3)$ = moduli of Ricci flat vol = 1 metrics on $K3$

→ M-theory / $K3 \times \mathbb{R}^7$ has moduli

$$\mathcal{N}(19,3) \times \mathbb{R}_+$$

and 16 susy's.

→ Low energy sugra coincides with Het/$T^3 \times \mathbb{R}^7$

So: " $\mathrm{Het}/T^3 \times \mathbb{R}^7 \cong M/K3 \times \mathbb{R}^7$ "

A very nontrivial isomorphism.

More precisely a host of nontrivial mathematical facts are explained by such an isomorphism and we expect it to hold if we ever really define the objects.

Let us study the gauge symmetry

"gaugefield" $C \in \Omega^3(X_{11})$ (Very rough – H. Sati will say more)

KK reduce ω^I basis for $\mathcal{H}^2(K3)$

$I = 1, \dots, 22$

$$C = A_I(x)\omega^I(y) \Longrightarrow U(1)^{22} \text{ gauge symmetry}$$

Agrees w/ Het/$T^3 \times \mathbb{R}^7$ at generic point of $\mathcal{W}(19,3)$.

What about E.S. Loci ?

$\exists$ M2 brane: solitonic object in 11-d sugra

Like a rubber sheet.

If $\Sigma \subset K3$ is a cycle you can wrap it.

Appears as a particle in $\mathbb{R}^7$

These particles have charge:

$$e^{i\int_{\mathbb{R}\times\Sigma} C} = e^{i\int_{\mathbb{R}} A_I(x)\left(\int_\Sigma \omega^I\right)} = e^{i\int A_I(x)\, q^I}$$

q^I only depends on homology class

$$H_2(K3;\mathbb{Z}) \cong \mathrm{II}^{19,3}$$

Particles have mass $= \mathrm{vol}(\Sigma)$

Witten observed:

$$\boxed{\begin{array}{l}\text{The enhanced symmetry locus on } \mathcal{W}(19,3) \\ = \text{locus of } K3 \text{ surfaces with ADE singularities!}\end{array}}$$

Local geometry near a singularity

$\mathfrak{g} \longleftrightarrow \Gamma \subset SU(2)$ DuVal - McKay

X

$\downarrow$ H.K. resolution: ALE space

$\mathbb{C}^2/\Gamma$

basis of exceptional curves $\longleftrightarrow$ $\alpha \in \Delta_s$

$$\Sigma_\alpha \cdot \Sigma_\beta = - A_{\alpha\beta}$$

Wrap M2 on Σ_α $\Rightarrow$ massless particle in limit

<u>Susy</u>: M2/Σ_α is a particle in a supermultiplet including a <u>vector particle</u>

M-Theory on an ADE singularity has ADE gauge symmetry!

Closely related, but more convenient:

$$M/X_{10}\times S^1 = IIA/X_{10} \quad \rightarrow$$

$$Het/T^4\times\mathbb{R}^6 = IIA/K3\times\mathbb{R}^6$$

Now $\mathcal{N}(20,4)\times\mathbb{R}_+$

Instead of M2 we have D2 + other things, I will come to.

Before leaving this topic - ~~I have always~~ beautiful as it is, I have always found this discussion to be incomplete and unsatisfactory.

On the M/IIA side we never computed the interactions of gauge particles.

We would like to compute the nonabelian interactions geometrically. ~~and understand better where this nonabelian gauge symmetry "comes from".~~

3. BPS States

These theories have susy. So $\exists$ hermitian op's Q_α on the Hilbertspace transforming as spinors. In fact many

$$\{ Q^i_\alpha, Q^j_\beta \} = M_{\alpha\beta}(\text{momentum, charge})$$

$$= \delta^{ij} (C\Gamma^\mu)_{\alpha\beta} P_\mu + J_{\alpha\beta} Z^{ij}$$

1 particle Hilbert spaces = induced repn's from the little superalgebra

$\therefore$ induced from repn $\mathcal{H}$ of a Clifford algebra.

For some P_μ, Z^{ij} the quadratic form is <u>degenerate</u>

$\Longrightarrow$ These lead to "small" or "BPS" repn's of susy

They are exceptional – hence more rigid
– existence is often independent of small, ~~and~~ and even large changes of physical parameters.

They play a distinguished role.

Exple 1 : Consider $Het/T^d \times \mathbb{R}^{10-d}$

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^\mu)_{\alpha\beta} P_\mu \qquad \alpha = 1, \cdots, 16$$

Typical $P^2 = -m^2 \neq 0$

$$\dim \mathcal{H} = 2^8 = 128_B + 128_F$$

If $P^2 = 0, P \neq 0$ $\not{P}$ degenerate on ½ space

$$\dim \mathcal{H} = 2^4 = 8_B + 8_F = $$ YM multiplet for R-moving gndstate, so

These states correspond to V.O.'s of the form keep

$$\boxed{\mathcal{P}_I(\partial x) e^{ik.x} \otimes \tilde{\zeta}\cdot\tilde{\psi}\, e^{i\tilde{k}\cdot\tilde{x}}(\bar{z})}$$

$$\tilde{k}^2 = 0, \qquad \text{so} \qquad k^2 = 2 - 2N$$

where $\mathcal{P}_I$ is at level $N \Rightarrow I = 1, \cdots, P_{24}(N)$

$$\frac{1}{\eta^{24}} = q^{-1} \sum_{N \geq 0} P_{24}(N) q^N$$

States are massive $\qquad m^2 = \vec{P}_R^{\,2}$

Charged $\qquad P \in II^{16+d,d}$

Note: $\dim \mathcal{H}_{BPS}(P) = d(P^2)$ since $P^2 = \vec{P}_R^{\,2} - \vec{P}_L^{\,2} = 2N - 2$

as required by T-duality.

Exple 2: ~~[crossed out]~~ $IIA/K3 \times \mathbb{R}^6 = Het/T^4 \times \mathbb{R}^6$

It is interesting to compare with the dual description under $Het/T^4 = IIA/K3$.

- Charge lattice $\cong II^{20,4} \cong H^*(K3;\mathbb{Z})$ now involves H^0, H^2, H^4

- BPS states are identified with cohomology classes on the moduli space of coherent sheaves $\mathcal{E}$ on K3

$$P = v(\mathcal{E}) = ch\,\mathcal{E}\,\sqrt{\hat{A}} = (r, c_1, ch_2 + r)$$

$$\mathcal{H}_{BPS} \cong H^*(\mathcal{M}(v)) \otimes \pi$$

minimal repⁿ of the little superalgebra

Amazing mathematical fact:

$$\dim \mathcal{H}_{BPS}(P) = 16\, p_{24}(N) \quad !$$

Example: $v = (1, 0, 1-L)$

$$\mathcal{M}(v) = Hilb^L(K3) \longrightarrow Sym^L(K3)$$

and for any surface we have Göttsche's formula

$$\sum q^L \chi(\mathrm{Hilb}^L S) = \frac{\prod(1+q^n)^{b_{odd}}}{\prod(1-q^n)^{b_{ev}}} \overset{K3}{=} \frac{q}{\eta^{24}}$$

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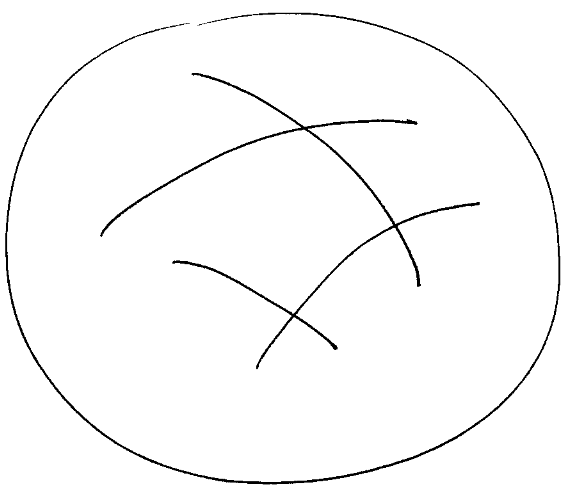
4. <u>BPS Algebras</u>: Now I start to make some more speculative comments.

Let us return to the Narain picture



At different loci different VO's become massless.

Is there a unified symmetry?

Now recall a standard fact from CFT

$\{V^i(z)\}$ <u>mutually local</u>, dim = 1, Vir primaries

Then $V^i(z_1)V^j(z_2) \sim \cdots + \frac{f_k^{ij}}{z_{12}} V^k(z_2)$

f_k^{ij} define structure constants of a <u>Lie algebra</u>.

Uses contour deformation + holomorphy.

This was formalized by Lian + Zuckerman

Thm: Gh #1 BRST coho. of a V.O.A. is canonically a Lie algebra.

For heterotic BPS states:

$$V_{I,k,\tilde{k}} = \mathcal{P}_I e^{ikx} \otimes e^{i\tilde{k}\tilde{x}} \qquad \cancel{\tilde{k}^2=0}$$

~~involve left and right movers.~~

~~Moreover the OPE of two such generally contains no BPS state.~~

Then $\mathcal{P}_I e^{ik\cdot x}$ are dim = 1 Vir. primaries

However given two states V^1, V^2 in general they are not mutually local

Observe:

leftmoving currents mutually local $\Longleftrightarrow (\tilde{k}_1+\tilde{k}_2)^2=0$

$\Longleftrightarrow$ OPE contains a nonzero BRST state.

$\Longleftrightarrow \exists$ holomorphic pole $\frac{1}{z_{12}}$

For such pairs

it makes sense to set:

$$R\left(V_{k_1\tilde{k}_1}, V_{k_2\tilde{k}_2}\right) := \oint_{z_2} dz_1\, V_1 \ V_2$$

You can use this observation to define a product (in different ways) on BPS states. None are really satisfactory.

For example, given a pair of states you can boost them to be mutually local $\Rightarrow$ algebraic structure on $\bigoplus_P \mathcal{H}_{BPS}(P)$

1. defined everywhere on $\mathcal{N}(p,q)$
2. Coincides with Lie product on massless states
3. Extends to ∞-dim'l Lie algebra on <u>some</u> subsets of BPS states

<u>Example</u>:

$$\boxed{\begin{array}{l} V_{L,\xi} = \xi\cdot\partial X\, e^{i(t+x)\frac{L}{\sqrt{2}}} \qquad L \in \mathbb{Z} \\ R(V_{L,\xi}, V_{L'\xi'}) = \xi\cdot\xi' \frac{L}{\sqrt{2}}\, i(\partial t+\partial X)\, \delta_{L+L',0} \end{array}}$$

keep

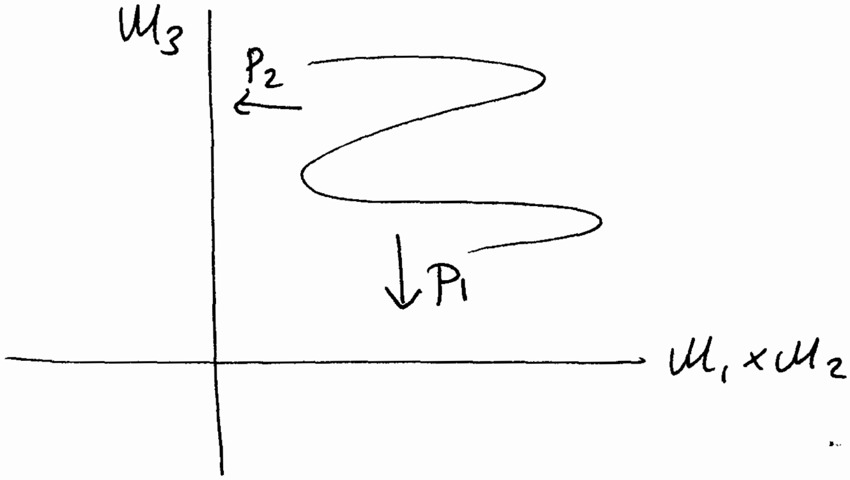
Now we have a natural question: What would the type II description of this product be?

A natural guess/proposal (HM 96)

$$H^*(\mathcal{M}(v_1)) \otimes H^*(\mathcal{M}(v_2)) \longrightarrow H^*(\mathcal{M}(v_1+v_2))$$

$$\mathcal{C} \subset \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3$$

$$:= \{ (\mathcal{E}_1\ \mathcal{E}_2\ \mathcal{E}_3) \mid 0 \to \mathcal{E}_2 \to \mathcal{E}_3 \to \mathcal{E}_1 \to 0 \}$$



$$R(\omega_1 \otimes \omega_2) = (p_2)_* p_1^*(\omega_1 \otimes \omega_2)$$

This construction was inspired by Nakajima's work and also occurs in Ringel-Hall algebras.

Nakajima has two examples:

- $\bigoplus_L H^*(\mathrm{Hilb}^L S)$ module for Heisenberg algebra

- Rank r vector bdles on $\widetilde{\mathbb{C}^2/\Gamma} \rightsquigarrow$ Highest wt modules for $\widehat{\mathfrak{g}(\Gamma)}$

Use duality to translate this into statements about BPS states — supports the idea that these products are indeed dual descriptions.

$$V_{L,\xi} \longleftrightarrow \text{D0 branes}$$

$$e^{iEt}\rho_I e^{\frac{i}{\sqrt{2}}\left(\frac{L-r}{V} - rV\right)x} e^{i\alpha\cdot y} \longleftrightarrow \text{D0D2D4 boundstates}$$

$$\bigoplus_{\ell} H^*(\mathcal{M}(r, c_1, r-\ell))$$

algebra of D0 branes acts as a Heisenberg algebra on this module.

One can do something similar for ALE singularities. Finite volume K3 surface $\rightarrow$ deformation of N's construction.

D0D2D4 with $v = (r, 0, r-L)$

corresponds to

$$e^{iEt}\, \rho_I\, e^{\frac{i}{\sqrt{2}}\left(\frac{L-r}{V} - rV\right)X} \otimes \tilde{\zeta}\tilde{\varphi}\, e^{\frac{i}{\sqrt{2}}\left(\frac{L-r}{V} + rV\right)\tilde{X}}$$

Our $V_{L,\zeta}$ correspond to D0 branes. So

$$\bigoplus_{\ell} H^*\left(\mathcal{M}(r, 0, r-\ell)\right)$$

is a module for the action of the subalgebra of D0 branes.

Note that it is not a Heisenberg algebra but

$$\frac{i}{\sqrt{2}V}(\partial t + \partial X)$$

acts on this module as the c-number r!

One can do something similar for K3 developing an ALE singularity. Finite $V \Rightarrow$ deformation of Nakajima's construction.

(Even $\vec{c}_1 \neq 0 \Rightarrow$ deformation of Heisenberg module.)

Two remarks

1. Gritsenko + Nikulin constructed ∞-dim'l holomorphic GKM's ~~assoc~~ associated to different families of K3 surfaces using automorphic forms. Closely related.

2. Obviously you can generalize the type II construction to CY 3-folds. Morrison + Moore computed for curves of ADE sing's + recovered fd. Lie alg's. ABCDEFG

~~3. Recent work of Joyce~~

3. Recent work of Joyce

Now, we've never quite known what to do with these algebras. In the remainder of the talk I am going to mention some ~~[illegible]~~ evidence that they should play an important role in string duality.

B. Gromov-Witten Theory

In some examples Borcherds product, denominator products of GKM. are related to GW invts

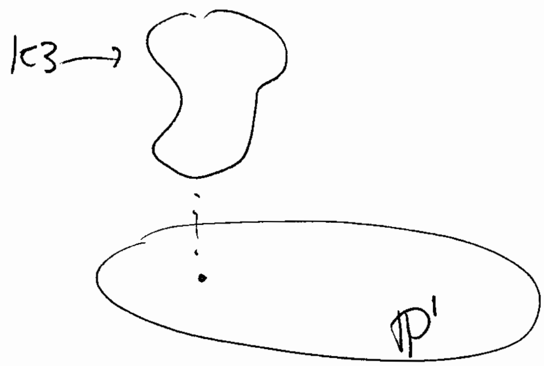
- X C.Y. 3-fold.
- Gromov-Witten potential:

$$F_{GW}(\lambda, t) = \sum_{h \geq 0} \sum_{\beta \in H_2(X, \mathbb{Z})} N_{h,\beta} \, \lambda^{2h-2} e^{-t \cdot \beta}$$

$N_{h,\beta}$ = (rational) Gromov-Witten invariants.

K3-fibered family

$$\begin{tikzcd} K3 \arrow[r] & X \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$



Duality: $IIA / X \times \mathbb{R}^4 \cong Het / K3' \times T^2 \times \mathbb{R}^4$

- $Vol(\mathbb{P}^1) \to \infty$ leaves behind the Kähler classes on K3

Consider

$$F_1 = \sum N_{1,\beta}\, q^{\beta} \qquad \text{turnsout} \;=\; \log \Phi$$

Φ denominator product of GKM

Comes about: F_1 related to heterotic 1-loop amplitude:

$$I = \int_{\mathcal{F}} \frac{d^2\tau}{y}\, \mathbf{f}(q)\, \overline{\Theta}_{\Gamma^{2,s+2}}(q,\bar{q})$$

$$\mathbf{f}(q) = \sum_{n=-1}^{\infty} c(n)\, q^n \qquad \text{wt } -s/2 \quad \text{related to elliptic genus}$$

$$\Gamma^{2,s+2} = II^{1,1} \oplus \Gamma^{1,s+1}, \qquad \Gamma^{1,s+1} = \text{Picard lattice of K3 fiber.}$$

Function on $O(s+2,2)/O(s+2)\times O(2) = \mathbb{R}^{s+1,1} + i V_+^{s+1,1} \subseteq \mathbb{C}^{s+1,1}$

$$I = -\log \| \Phi(z) \|^2$$

$$\Phi(z) = e^{2\pi i \rho\cdot z} \prod_{\substack{r>0 \\ \text{in } \Gamma^{s+1,1}}} \left(1 - e^{2\pi i r\cdot z}\right)^{c(-r^2/2)}$$

(Proves Borcherd's thm.

Φ automorphic for $O(\Gamma^{s+2,2})$ of weight $c(0)/2$)

Example :

$$X = (S \times E)/\mathbb{Z}_2$$

$$S/\mathbb{Z}_2 = \text{Enriques.}$$

$$E = \text{elliptic curve}$$

$$\Pi^{1,s+1} = II^{1,9}$$

$$f(q) = \cancel{q^{-1}\frac{\prod(1+q^{2n+1})^8}{\prod(1-q^n)^8}}\; q^{-1}\frac{1}{\prod(1-q^n)^8(1+q^{2n})^8} = \left(\frac{\eta(2\tau)}{\eta(\tau)\eta(4\tau)}\right)^8$$

Φ denominator product "Fake Monster Lie superalgebra"

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2 Remarks / Open Problems

1. What is relation between these GKM's and the algebra of BPS states?

Known that: $F_1 = \sum_{\mathcal{H}_{BPS}^{\lambda m}} \log m^2 - \sum_{\mathcal{H}_{BPS}^{\lambda m}} \log m^2$

2. For general C.Y. Gopakumar-Vafa rewrite

$$F_{GW}(\lambda, q) = \sum_{d\geq 1} \sum_{s\geq 0} \sum_{\beta} n_\beta^{(s)} \frac{1}{d}\left(2\sin\frac{d\lambda}{2}\right)^{2s-2} q^{d\beta}$$

D0D2

$n_\beta^{(s)}$ = integer GV invariants ~ dimen's of BPS states

$$\Rightarrow e^{F_{GW}} = \underbrace{\prod_{\substack{k\geq 1\\ \beta\in H_2}} \left(1 - e^{2\pi i k\lambda} q^\beta\right)^{k n_\beta^{(0)}} \cdot \prod \left(1 - q^\beta\right)^{n_\beta^{(1)}}} \cdot \underbrace{\prod_{s>1} (\cdots)^{n_\beta^{(s)}}}$$

Work with Harvey ⇒ essentially Borcherds products

don't converge

Suggestive...

Can be understood as a sum over BPS states

C. Black Hole State Counting

* When charges are large, BPS states are very massive – one might expect them to be related to supersymmetric B.H.'s

* ~~[illegible]~~ One can sometimes argue that they are continuously connected to BH's – basis for S+V's ~~accounting for [illegible]~~ Statistical derivation of BH entropy for certain 5D susy BH's.
But generalizations to 4D state counting harder. Recent activity

* Consider Het/$T^6 \times \mathbb{R}^4$

BPS states have charge lattice $II^{22,6}$

But this only the lattice of electric charges. There are electromagnetic duals and the ~~[illegible]~~ full electric + magnetic lattice is

$$(q_e, q_m) \in II^{22,6}_{el} \oplus II^{22,6}_{mag}$$

By T~~[illegible]~~ duality

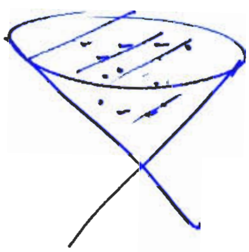
$$\dim \mathcal{H}_{BPS}(q_e, q_m) = D\left(\tfrac{1}{2} q_e^2, \tfrac{1}{2} q_m^2, q_e \cdot q_m\right)$$

Now 9 years ago DVV proposed

$$\sum_{N,M,L} D(N,M,L)\, \overset{p^N}{\cancel{e^{2\pi i N\sigma}}}\, \overset{q^M}{\cancel{e^{2\pi i M\rho}}}\, \overset{y^L}{\cancel{e^{2\pi i L v}}} = \Phi^{-1}$$

$$\Phi = \overset{p q y}{\cancel{e^{2\pi i(\rho+\sigma+v)}}} \prod_{(n,m,\ell)>0} \left(1 - \overset{p^n q^m y^\ell}{\cancel{e^{2\pi i(n\rho + m\sigma + \ell v)}}}\right)^{c(nm,\ell)}$$

$\mathbb{R}^{1,2}$



$$\sum c(n,\ell) q^n y^\ell = \text{K3 elliptic genus.}$$

$\Phi = \Delta_5^2$, $\Delta_5 =$

Gritsenko + Nikulin $\Rightarrow$ denominator product for a GKM superalgebra

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} \dashrightarrow \text{"automorphic correction"}$$

$SO(2,3;\mathbb{Z})$

Argument used 6D strings + 1 loop formula mentioned above. Never clear if it was true

Recently Strominger, Shih, Yin gave an independent argument!

$$\mathrm{Het}/T^6 = \mathrm{IIA}/K3 \times T^2$$

D0 D2 D4 $\overline{D6}$ branes : Interpret as KK monopoles. Define an index.
Strong coupling limit:

$$\mathrm{IIA}/K3 \times S^1$$

But 5D black hole degeneracies are computed by $\mathcal{E}ll(q, y; \mathrm{Sym}^N K3)$

$$\sum_{k \geq 0} p^k \, \mathcal{E}ll(q, y; \mathrm{Sym}^N K3) = \prod \frac{1}{(1 - p^k q^n y^\ell)^{c(nk, \ell)}}$$

puzzles remain....

Conclusion

~~Infinite/dimel~~

BPS states seem to be associated with some ∞-dim'l algebraic structures. They have nontrivial realizations in algebraic geometry and are connected with GW theory and black hole entropy. But the role of the algebraic structure and even the precise definition are obscure. Lot's to do. Lot's of open problems.

Conclusion

Except for the recent work of Strominger et.al. ~~all ever~~ almost everything I said was known about 10 years ago.

Nobody ever made the connections, or explained precisely why these GKM denominator products appear.

There is quite a lot to do!