Overview of the Theory of Self-Dual Fields

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Review of work done over the past few years with

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Outline

- Introduction: Review a familiar example U(1) 3D CS
- 3D Spin Chern-Simons theories
- Generalized Maxwell field and differential cohomology
- QFT Functor and Hopkins-Singer quadratic functor
- Hamiltonian formulation of generalized Maxwell and self-dual theory
- Partition function; action principle for a self-dual field
- RR fields and differential K theory
- The general self-dual QFT
- Open problems.

Introduction

Chiral fields are very familiar to practitioners of
2d conformal field theory and 3d Chern-Simons theory

I will describe certain generalizations of this mathematical structure, for the case of abelian gauge theories involving differential forms of higher degrees, defined in higher dimensions, and indeed valued in (differential) generalized cohomology theories.

These kinds of theories arise naturally in supergravity and superstring theories, and play a key role in the theory of D-branes and in the claims of moduli stabilization in string theory that have been made in the past few years.

A Simple Example

U(1) 3D Chern-Simons theory

$$\exp\left[2\pi i N\int_Y AdA\right] \qquad N\in\mathbb{Z}$$

$$F\in\Omega^2_{\mathbb{Z}}(Y) \qquad A\to A+\omega,\ \omega\in\Omega^1_{\mathbb{Z}}(Y)$$

Quantization on $Y=D\times\mathbb{R}$ gives
$\mathcal{H}(D)=$ basic representation of $\widehat{LU(1)}_{2N}$

What about the odd levels? In particular what about k=1 ?

Spin-Chern-Simons

$$\exp\left[2\pi i \frac{1}{2}\int_Y AdA\right]$$

Problem: Not well-defined.

But we can make it well-defined if we introduce a spin structure α

$$e^{2\pi i q_\alpha(A)} = \exp\left[i\pi \int_Y AdA\right] = \exp\left[2\pi i \int_Z \frac{1}{2}F^2\right]$$

Z = Spin bordism of Y.

Depends on spin structure:

$$q_{\alpha+\epsilon}(A) = q_\alpha(A) + \frac{1}{2}\int_Y \epsilon \wedge F \qquad \epsilon \in H^1(Y;\mathbb{Z}/2\mathbb{Z})$$

The Quadratic Property

We can only write $q_\alpha(A) = \frac{1}{2}\int_Y AdA \mod \mathbb{Z}$

as a heuristic formula, but it is rigorously true that

$$q_\alpha(A + a_1 + a_2) - q_\alpha(A + a_1) - q_\alpha(A + a_2) + q_\alpha(A)$$

$$= \int_Y a_1 da_2 \mod \mathbb{Z}$$

Quadratic Refinements

Let A, B be abelian groups, together with a bilinear map

$$b : A \times A \to B$$

A **quadratic refinement** is a map $q : A \to B$

$$q(x_1 + x_2) - q(x_1) - q(x_2) + q(0) = b(x_1, x_2)$$

$q(x) = \frac{1}{2} b(x, x)$ does not make sense when B has 2-torsion

As is the case for $B = \mathbb{R}/\mathbb{Z}$

So it is nontrivial to define $q_\alpha(A)$

General Principle

An essential feature in the formulation of self-dual theory always involves a choice of certain **quadratic refinements**.

Holographic Dual

Chern-Simons Theory on Y	$\cong$	2D RCFT on $M = \partial Y$

Holographic dual = ``chiral half'' of the Gaussian model

$$\pi R^2 \int_M d\phi * d\phi \qquad \phi \sim \phi + 1$$

$$\text{Conformal blocks for } R^2 = p/q$$
$$= \text{CS wavefunctions for } N = pq$$

The Chern-Simons wave-functions $\Psi(A|_M)$ are the conformal blocks of the chiral scalar coupled to an external current = A:

$$\Psi(A) = Z(A) = \langle \exp \int_M A d\phi \rangle$$

Holography & Edge States

$\mathrm{Quantization\ on}\ Y = D \times \mathbb{R}$ is equivalent to

quantization of the chiral scalar on $\partial Y = S^1 \times \mathbb{R}$

Gaussian model for $R^2 = p/q$ has level 2N = 2pq current algebra.

$$\mathrm{Quantization\ on}\ S^1 \times \mathbb{R}\ \mathrm{gives}$$
$$\mathcal{H}(S^1) = \mathrm{representations\ of}\ \widehat{LU(1)}_{2N}$$

What about the odd levels? In particular what about k=1 ?

When $R^2=2$ we can define a ``squareroot theory''

This is the theory of a self-dual scalar field.

The Free Fermion

Indeed, for $R^2 = 2$ there are four reps of the chiral algebra:

$$1, e^{\pm \frac{i}{2}\phi}, e^{i\phi} \qquad \text{Free fermion: } \psi = e^{i\phi}$$

Self-dual field is equivalent to the theory of a chiral free fermion.

From this viewpoint, the dependence on spin structure is obvious.

Note for later reference:

A spin structure on a Riemann surface M is a quadratic refinement of the intersection form modulo 2 on $H^1(M,\mathbb{Z})$ This is how the notion of spin structure will generalize.

General 3D Spin Abelian Chern-Simons

3D classical Chern-Simons with compact gauge group G classified by

$$k \in H^4(BG;\mathbb{Z})$$

3D classical spin Chern-Simons with compact gauge group classified by
a different generalized cohomology theory

D. Freed

$$0 \to H^4(BG;\mathbb{Z}) \to E^4(BG;\mathbb{Z}) \stackrel{w_2}{\to} H^2(BG;\mathbb{Z}_2)$$

$$\text{Gauge group } G = U(1)^r$$

$H^4(BG;\mathbb{Z})$ Even integral lattices of rank r

$$\exp[i\pi \int k_{ij} A^i dA^j]$$

$E^4(BG;\mathbb{Z})$ Integral lattices Λ of rank r.

Classification of quantum spin abelian Chern-Simons theories

Theorem (Belov and Moore) For G= $U(1)^r$ let Λ be the integral lattice corresponding to the classical theory. Then the quantum theory only depends on

$$\text{a.)}\ \sigma(\Lambda) \mod 24$$

$$\text{b.) A quadratic refinement } q: \mathcal{D} \to \mathbb{R}/\mathbb{Z}$$

$$\text{of the bilinear form on } \mathcal{D} = \Lambda^*/\Lambda \text{ so that}$$

$$|\mathcal{D}|^{-1/2} \sum_{\gamma \in \mathcal{D}} e^{2\pi i q(\gamma)} = e^{2\pi i \sigma/8}$$

Higher Dimensional Generalizations

Our main theme here is that there is a generalization of this story to higher dimensions and to other generalized cohomology theories.

This generalization plays an important role in susy gauge theory, string theory, and M-theory

Main Examples:

- Self-dual (2p+1)-form in (4p + 2) dimensions. (p=0: Free fermion & p=1: M5 brane)
- Low energy abelian gauge theory in Seiberg-Witten solution of d=4,N=2 susy
- RR fields of type II string theory
- RR fields of type II ``orientifolds''

Generalized Maxwell Field

Spacetime = *M*, with *dim(M) =n*

Gauge invariant information:

Maxwell	$F \in \Omega^{\ell}(M)$	Fieldstrength
Dirac	$a \in H^{\ell}(M;\mathbb{Z})$	Characteristic class
Bohm-Aharonov-Wilson-'t Hooft:	$H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})$	Flat fields

All encoded in the holonomy function

$$\chi : Z_{\ell-1}(M) \to \mathbb{R}/\mathbb{Z}$$

Differential Cohomology

a.k.a. Deligne-Cheeger-Simons Cohomology

To a manifold M and degree ℓ we associate an infinite-dimensional abelian group of characters with a fieldstrength:

$$\check{H}^{\ell}(M)$$

$$\Sigma = \partial B \quad \Longrightarrow \quad \chi(\Sigma) = \exp[2\pi i \int_B F]$$

Simplest example:

$$\check{H}^{1}(M) = Map(M, U(1)) \qquad F = d\phi$$

Next we want to get a picture of the space $\check{H}^{\ell}(M)$ in general

Structure of the Differential Cohomology Group - I

Fieldstrength exact sequence:

F

$$0 \to \overbrace{H^{\ell-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{H}^{\ell}(M) \xrightarrow{\text{fieldstrength}} \Omega^{\ell}_{\mathbb{Z}}(M) \to 0$$

Characteristic class exact sequence:

A

$$0 \to \underbrace{\Omega^{\ell-1}(M)/\Omega^{\ell-1}_{\mathbb{Z}}(M)}_{\text{Topologically trivial}} \to \check{H}^{\ell}(M) \xrightarrow{\text{char.class}} \underbrace{H^{\ell}(M;\mathbb{Z})}_{\text{Topological sector}} \to 0$$

Connected!

Structure of the Differential Cohomology Group - II

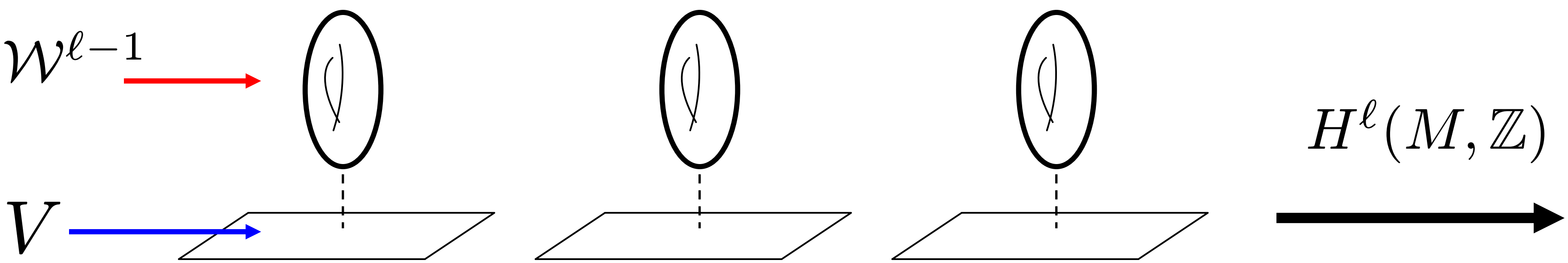
The space of differential characters has the form: $\check{H}^\ell = T \times \Gamma \times V$

T: Connected torus of topologically trivial flat fields:

$$\mathcal{W}^{\ell-1}(M) = H^{\ell-1}(M,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$$

Γ: Discrete (possibly infinite) abelian group of topological sectors: $H^\ell(M,\mathbb{Z})$.

V: Infinite-dimensional vector space of "oscillator modes." $V \cong \mathrm{Im} d^\dagger$.



Example 1: Loop Group of U(1)

Configuration space of a periodic scalar field on a circle:

$$\check{H}^1(S^1) = Map(S^1, U(1)) = LU(1)$$

Topological class = Winding number: $w \in H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$

Flat fields = Torus $\mathbb{T}$ of constant maps: $H^0(S^1, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$

Vector Space: $V = \Omega^0/\mathbb{R}$ Loops admitting a logarithm.

$$\check{H}^1(S^1) = \mathbb{T} \times \mathbb{Z} \times V$$

This corresponds to the explicit decomposition:

$$\varphi(\sigma) = \exp\left[2\pi i\phi_0 + 2\pi i w\sigma + \sum_{n\neq 0} \frac{\phi_n}{n} e^{2\pi i n\sigma}\right]$$

More Examples

$$\check{H}^0(pt) = \mathbb{Z} \qquad \check{H}^1(pt) = \mathbb{R}/\mathbb{Z}$$

$\check{H}^2(M)$ Group of isomorphism classes of line bundles with connection on M.

$\check{H}^3(M)$ Group of isomorphism classes of gerbes with connection on M: c.f. B-field of type II string theory

$\check{H}^4(M)$ Home of the abelian 3-form potential of 11-dimensional M-theory.

Multiplication and Integration

There is a ring structure:

$$\check{H}^{\ell_1}(M) \times \check{H}^{\ell_2}(M) \to \check{H}^{\ell_1+\ell_2}(M)$$

Fieldstrength and characteristic class multiply in the usual way.

Family of compact oriented n-folds

$$\begin{array}{ccc} M_s & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ s & \in & S \end{array} \quad \Longrightarrow \quad \int_{\mathcal{X}/S} : \check{H}^{\ell}(\mathcal{X}) \to \check{H}^{\ell-n}(S)$$

Recall: $\check{H}^1(pt) = \mathbb{R}/\mathbb{Z}$

Poincare-Pontryagin Duality

M is compact, oriented, dim(M) = n

There is a very subtle ***PERFECT PAIRING*** on differential cohomology:

$$\check{H}^{\ell}(M) \times \check{H}^{n+1-\ell}(M) \rightarrow \mathbb{R}/\mathbb{Z}$$

$$\langle [\check{A}_1], [\check{A}_2] \rangle := \int_M^{\check{H}} [\check{A}_1] * [\check{A}_2]$$

On topologically trivial fields:

$$\langle [A_1], [A_2] \rangle = \int_M A_1 dA_2 \mod \mathbb{Z}$$

Example:Cocycle of the Loop Group

Recall $\check{H}^1(S^1) = LU(1)$:

$$\langle \varphi, \tilde{\varphi} \rangle = ??$$

$$\varphi = \exp(2\pi i \phi) \qquad\qquad \phi : \mathbb{R} \to \mathbb{R}$$

$$\phi(s+1) = \phi(s) + w$$

$w \in \mathbb{Z}$ is the winding number.

$$\langle \varphi, \tilde{\varphi} \rangle = \int_0^1 \phi \frac{d\tilde{\phi}}{ds} ds - w\tilde{\phi}(0) \qquad \text{mod } \mathbb{Z}$$

Note! This is (twice!) the cocycle of the basic central extension of $LU(1)$.

QFT Functor

For generalized Maxwell theory the physical theory is a functor from a geometric bordism category to the category of Hilbert spaces and linear maps.

$$\mathrm{Action} = \pi R^2 \int_M F * F + \mathrm{sources}$$

To get an idea of the appropriate bordism category consider the presence of electric and magnetic currents (sources):

$$dF = j_m \in \Omega^{\ell+1}(M)$$

$$d * F = j_e \in \Omega^{n-\ell+1}(M)$$

Bordism Category

Objects: Riemannian (n-1)-manifolds equipped with electric and magnetic currents

$$\check{j}_m \in \check{H}^{\ell+1}(M) \qquad \check{j}_e \in \check{H}^{n-\ell+1}(M)$$

Morphisms are bordisms of these objects.

Not that for fixed $\check{j}_m, \check{j}_e$ the generalized Maxwell field lies in a torsor for $\check{H}^{\ell}(M)$

Partition Functions

$$\check{j}_m \in \check{H}^{\ell+1}(M) \qquad \check{j}_e \in \check{H}^{n-\ell+1}(M)$$

Family of closed spacetimes:

$$\begin{array}{ccc} M_s & \rightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ s & \in & S \end{array}$$

⇒ Partition functions

$$Z(\check{j}_m, \check{j}_e; M_s)$$

The theory is anomalous in the presence of both electric and magnetic current: The partition function is a section of a line bundle with connection:

$$\int_{\mathcal{X}/S} \check{j}_e \cdot \check{j}_m \in \check{H}^2(S)$$

Freed

Hilbert Spaces

Similarly, for families of spatial (n-1)-manifolds:

$$\begin{array}{ccc} X_s & \rightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ s & \in & S \end{array}$$

We construct a bundle of projective Hilbert spaces with connection over S. Such bundles are classified by gerbes with connection. In our case:

$$\int_{\mathcal{X}/S} \check{j}_e \cdot \check{j}_m \in \check{H}^3(S)$$

Self-Dual Case

Now suppose $\dim M = 4p+2$, and $\ell = 2p+1$.

We can impose a (Lorentzian) self-duality condition ***F = *F***.

Self-duality implies $\check{j}_e = \check{j}_m \in \check{H}^{2p+2}(M)$

Self-dual theory is a ``square-root'' of the non-self-dual theory so

anomalous line bundle for partition function is heuristically $\frac{1}{2}\int \check{j}\cdot\check{j}$

Interpret this as a quadratic refinement of $\int \check{j}_1 \cdot \check{j}_2 \in \check{H}^2(S)$

Hopkins-Singer Construction

$$\begin{array}{ccc} \mathcal{F}_s & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ s & \in & S \end{array}$$

Family of manifolds of relative dimension 4p+4-i , i=0,1,2,3

Family comes equipped with $\check{j} = \check{j}_e = \check{j}_m \in \check{H}^{2p+2}(\mathcal{X})$

H&S construct a quadratic map (functor) which refines the bilinear map (functor)

$$\int \check{j}_1 \cdot \check{j}_2 \in \check{H}^i(S)$$

depending on an integral lift λ of a Wu class (generalizing spin structure)

Physical Interpretation

$$q(\check{j}) \in \check{H}^i(S)$$

$i = 0,. \ \dim \mathcal{F}_s = 4p + 4:$

Basic topological invariant: The signature of $\mathcal{F}_s$

$i = 1, \dim \mathcal{F}_s = 4p + 3:$ Chern-Simons action $q(\check{j})$:

$i = 2, \dim \mathcal{F}_s = 4p + 2:$

Anomaly line bundle for partition function $Z(\check{j})$

$i = 3, \dim \mathcal{F}_s = 4p + 1:$

Gerbe class for Hilbert space $\mathcal{H}(\mathcal{F}_s)$

Important subtlety: Actually $q(\check{j}) \in \check{I}^i(S)$

Example: Construction of the quadratic function for i=1

$$\text{Extend } \check{j} \in \check{H}^{2p+2}(\mathcal{F}_s) \text{ to } \check{H}^{2p+2}(Z_s)$$

$$\partial Z_s = \mathcal{F}_s \qquad \dim Z_s = 4p+4$$

$$e^{2\pi i q_\lambda(\check{j})} := \exp\left[2\pi i \tfrac{1}{2} \int_Z F(\check{j}) \wedge (F(\check{j}) - \lambda_Z)\right]$$

λ_Z lift of the Wu class $\nu_{2s+2}(Z)$ Hopkins & Singer

Construction of the Self-dual Theory

That's where the Hilbert space and partition function should live….

We now explain to what extent the theory has been constructed.

- (Partial) construction of the Hilbert space.
- (Partial) construction of the partition function.

Hamiltonian Formulation of Generalized Maxwell Theory

Spacetime: $M = X \times \mathbb{R}$.

Generalized Maxwell fields: $[\check{A}] \in \check{H}^\ell(M)$.

$$\text{Action} = \pi R^2 \int_M F * F$$

Canonical quantization: $\mathcal{H}(X) = L^2(\check{H}^\ell(X))$

There is a better way to characterize the Hilbert space.

Above formulation breaks manifest electric-magnetic duality.

Group Theoretic Approach

Let K be any (locally compact) abelian group (with a measure)

$\mathcal{H} = L^2(K)$ is a representation of K: $\forall k_0 \in K$

$$(T_{k_0}\psi)(k) := \psi(k + k_0).$$

Let $\hat{K}$ be the Pontryagin dual group of characters of K

$\mathcal{H} = L^2(K)$ is also a representation of $\hat{K}$: $\forall \chi \in \hat{K}$

$$(M_\chi \psi)(k) := \chi(k)\psi(k)$$

But! $\boxed{T_{k_0} M_\chi = \chi(k_0) M_\chi T_{k_0}}.$

So $\mathcal{H} = L^2(K)$: is a representation of the Heisenberg group central extension:

$$1 \to U(1) \to \text{Heis}(K \times \hat{K}) \to K \times \hat{K} \to 1$$

Heisenberg Groups

Theorem A Let G be a topological abelian group. Central extensions, $\tilde{G}$, of G by $U(1)$ are in one-one correspondence with continuous bimultiplicative maps $s : G \times G \to U(1)$ which are alternating (and hence skew).

- s is *alternating*: s(x,x) = 1
- s is *skew*: s(x,y) = s(y,x) $^{-1}$
- s is *bimultiplicative*:

$$s(x_1 + x_2, y) = s(x_1, y)s(x_2, y) \quad \& \quad s(x, y_1 + y_2) = s(x, y_1)s(x, y_2)$$

If $x \in G$ lifts to $\tilde{x} \in \tilde{G}$ then $s(x, y) = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$.

Definition: If s is nondegenerate then $\tilde{G}$ is a *Heisenberg group*.

Theorem B: (Stone-von Neuman theorem). If $\tilde{G}$ is a Heisenberg group then the unitary irrep of $\tilde{G}$ where $U(1)$ acts canonically is unique up to isomorphism.

Heisenberg group for generalized Maxwell theory

$$\text{If } K = \check{H}^{\ell}(X), \text{ then PP duality} \Rightarrow \hat{K} = \check{H}^{n-\ell}(X):$$

$$\tilde{G} := \mathrm{Heis}\big(\check{H}^{\ell}(X) \times \check{H}^{n-\ell}(X)\big)$$

via the group commutator:

$$s\left(([\check{A}_1],[\check{A}_1^D]),([\check{A}_2],[\check{A}_2^D])\right) = \exp\left[2\pi i\Big(\langle[\check{A}_2],[\check{A}_1^D]\rangle - \langle[\check{A}_1],[\check{A}_2^D]\rangle\Big)\right].$$

The Hilbert space of the generalized Maxwell theory *is* the unique irrep of the Heisenberg group $\tilde{G}$

N.B! This formulation of the Hilbert space is *manifestly electric-magnetic dual*.

Flux Sectors from Group Theory

Electric flux sectors diagonalize the flat fields $H^{\ell-1}(X,\mathbb{R}/\mathbb{Z})$

Electric flux = dual character: $e \in H^{n-\ell}(X;\mathbb{Z})$

Magnetic flux sectors diagonalize dual flat fields $H^{n-\ell-1}(X,\mathbb{R}/\mathbb{Z})$

Magnetic flux = dual character: $m \in H^{\ell}(X;\mathbb{Z})$

These groups separately lift to commutative subgroups of $\tilde{G} := \mathrm{Heis}(\check{H}^{\ell} \times \check{H}^{n-\ell})$.

However they do not commute with each other!

$$\mathcal{U}_E(\eta_e) := \text{translation operator by } \eta_e \in H^{\ell-1}(X,\mathbb{R}/\mathbb{Z})$$

$$\mathcal{U}_M(\eta_m) := \text{translation operator by } \eta_m \in H^{n-\ell-1}(X,\mathbb{R}/\mathbb{Z})$$

$$[\mathcal{U}_e(\eta_e),\mathcal{U}_m(\eta_m)] = T(\eta_e,\eta_m) = \exp\left(2\pi i \int_X \eta_e \beta \eta_m\right)$$

T: torsion pairing, β= Bockstein: $\beta(\eta_m) \in \mathrm{Tors}(H^{n-\ell}(X,\mathbb{Z}))$.

Example: Maxwell theory on a Lens space

$$S^3/\mathbb{Z}_k \times \mathbb{R} \qquad H^1(L_k;\mathbb{R}/\mathbb{Z}) \cong H^2(L_k;\mathbb{Z}) = \mathbb{Z}_k$$

Acting on the Hilbert space the flat fields generate a Heisenberg group extension

$$0 \to \mathbb{Z}_k \to \mathrm{Heis}(\mathbb{Z}_k \times \mathbb{Z}_k) \to \mathbb{Z}_k \times \mathbb{Z}_k \to 0$$

This has unique irrep ***P*** = clock operator, ***Q*** = shift operator

$$PQ = e^{2\pi i/k} QP$$

States of definite electric and magnetic flux $|e\rangle = \frac{1}{\sqrt{k}} \sum_m e^{2\pi i e m/k} |m\rangle$

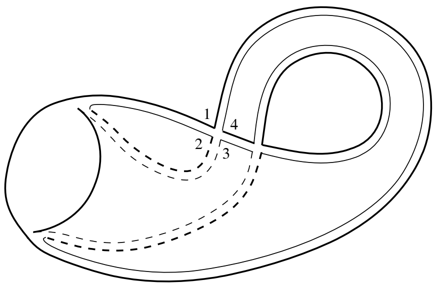
This example already appeared in string theory in Gukov, Rangamani, and Witten, hep-th/9811048. They studied AdS5xS5/Z3 and in order to match nonperturbative states concluded that in the presence of a D3 brane one cannot simultaneously measure D1 and F1 number.

An Experimental Test

Since our remark applies to Maxwell theory: Can we test it experimentally?

Discouraging fact: No region in $\mathbb{R}^3$ has torsion in its cohomology

With A. Kitaev and K. Walker we noted that using arrays of Josephson Junctions, in particular a device called a ``superconducting mirror,'' we can ``trick'' the Maxwell field into behaving as if it were in a 3-fold with torsion in its cohomology.



To exponentially good accuracy the groundstates of the electromagnetic field are an irreducible representation of $\mathrm{Heis}(\mathbb{Z}_n \times \mathbb{Z}_n)$

See arXiv:0706.3410 for more details.

Hilbert Space for Self-dual fields

Now return to $\dim M = 4p+2$, and $\ell = 2p+1$.

For the non-self-dual field we represent $\mathrm{Heis}(\check{H}^{\ell}(X) \times \check{H}^{\ell}(X))$

Proposal: For the self-dual field we represent: $\mathrm{Heis}(\check{H}^{\ell}(X))$

Attempt to define this Heisenberg group via

$$s_{\mathrm{trial}}([\check{A}_1],[\check{A}_2]) = \exp 2\pi i \langle [\check{A}_1],[\check{A}_2]\rangle .$$

It is skew and and nondegenerate, but *not* alternating!

$$s_{\mathrm{trial}}([\check{A}],[\check{A}]) = (-1)^{\int_X \nu_{2p} \; a(\check{A})}$$

Gomi 2005

$\mathbb{Z}_2$ -graded Heisenberg groups

Theorem A': Skew bimultiplicative maps classify $\mathbb{Z}_2$ -graded Heisenberg groups.

$\mathbb{Z}_2$ grading in our case:

$$\epsilon([\check{A}]) = 0 \quad \text{if} \quad \int \nu_{2p}\ a(\check{A}) = 0 \bmod 2$$

$$\epsilon([\check{A}]) = 1 \quad \text{if} \quad \int \nu_{2p}\ a(\check{A}) = 1 \bmod 2$$

Theorem B': A $\mathbb{Z}_2$-graded Heisenberg group has a unqiue $\mathbb{Z}_2$ -graded irreducible representation.

This defines the Hilbert space of the self-dual field

Example: Self-dual scalar: $p = 0$.
The $\mathbb{Z}_2$-grading is just *fermion number*!

Holographic Approach to the self-dual partition function

Identify the self-dual current with the boundary value of a Chern-Simons field in a dual theory in 4p+3 dimensions $\check{j} \in \check{H}^{2p+2}(Y_{4p+3})$

Identify the ``spin'' Chern-Simons action with the HS quadratic refinement:

$$e^{2\pi i q_\lambda(\check{j})} \in U(1) \text{ if } \partial Y = \emptyset$$

$$\text{If } \partial Y = M,\ e^{2\pi i q_\lambda(\check{j})} \text{ is a section of}$$

$$\mathcal{L}_{CS} \to \check{H}^{2p+2}(M)$$

⇒ So is the Chern-Simons path integral

Quantization on $Y_{4p+3} = M_{4p+2} \times \mathbb{R}$:

Two ways to quantize: Constrain, then quantize or **Quantize, then constrain.**

1. Groupoid of ``gauge fields'' $\check{Z}^{2p+2}(Y)$ isomorphism classes give $\check{H}^{2p+2}(Y)$

2. ``Gauge transformations'' $g_{\check{A}} \cdot \check{j} = \check{j} + F(\check{A})$ $[\check{A}] \in \check{H}^{2p+1}(Y)$

(boundary values of bulk gauge modes are the dynamical fields !)

3. A choice of Riemannian metric on M gives a Kahler structure on $\check{Z}^{2p+2}(M)$
$\mathcal{L}_{CS}$ is the pre-quantum line bundle.

4. Quantization: $\Psi(\check{j})$ a holomorphic section of $\mathcal{L}_{CS}$

5. Gauss law: $(g_{\check{A}} \cdot \Psi)(\check{j}) = \Psi(g_{\check{A}} \cdot \check{j})$

Quantizing the Chern-Simons Theory -II

6. Lift of the gauge group to $\mathcal{L}_{CS}$ uses ∇_{CS} **and** a quadratic refinement

$$q : H^{2p+1}(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z} \quad \text{of} \quad \int_M a_1 a_2 \mod 2$$

(generalizes the spin structure!)

7. Nonvanishing wavefunctions satisfying the Gauss law only exist for

$$a(\check{j}) + \mu = 0$$

8. On this component Ψ is unique up to normalization (a theta function),

and gives the self-dual partition function as a function of external current:

$$\Psi(\check{j}) = \langle \exp[2\pi i \int_M \check{j} \cdot \check{A}] \rangle_{\mathrm{self-dual\ theory}}$$

Partition Function and Action

We thus recover Witten's formulation of the self-dual partition function from this approach

Moreover, this approach solves two puzzles associated with self-dual theory:

P1. There is no action since

$$F = *F \quad \Rightarrow \quad \int F * F = 0$$

P2. $F = *F$ Incompatible with $F \in \Omega_{\mathbb{Z}}^{2p+1}(M)$

The Action for the Self-Dual Field

$$V := \Omega^{2p+1}(M) \text{ has symplectic structure}$$

$$\omega(f_1, f_2) = \int f_1 \wedge f_2$$

Bianchi dF=0 implies F in a Lagrangian subspace V_1 =ker d

<u>Choose</u> a transverse Lagrangian subspace

$$V_{el} \subset V = V_{el} \oplus *V_{el} := V_{el} \oplus V_{mg}$$

$$S = \int F^{el} * F^{el} + F^{el} F^{mg}$$

Equation of motion: $d\mathcal{F} = 0 \quad \mathcal{F} = F^{el} + *F^{el}$

Relation to Nonselfdual Field

One can show that the nonself-dual field at a special radius, $R^2 = 2,$ decomposes into

$$\mathcal{H}_{nsd} \cong \oplus_\alpha \mathcal{H}_{sd,\alpha} \otimes \mathcal{H}_{asd,\alpha}$$

The subscript α is a sum over

a torsor for 2-torsion points in $H^{2p}(X;\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$.

For the self dual scalar α labels R and NS sectors.

The sum on α generalizes the sum on spin structures.

Similarly: $Z_{nsd}(M_s) = \sum_q Z_{sd}(q)\overline{Z_{asd}(q)}$

Remark on Seiberg-Witten Theory

(D. Gaiotto, G. Moore, A. Neitzke)

1. Witten discovered six-dimensional superconformal field theories $\mathcal{C}_N$ with ``U(N) gauge symmetry.''

2. Compactification of $\mathcal{C}_N$ on $R^{1,3}$ x C gives d=4,N=2 U(N) gauge theories

3. The IR limit of $\mathcal{C}_N$ is the abelian self-dual theory on $R^{1,3}$ x Σ

4. The IR limit of the d=4, N=2 theory is compactification of the abelian self-dual theory on $R^{1,3}$ x Σ.

5. Σ is the Seiberg-Witten curve.

6. So, the SW IR effective field theories are self-dual gauge theories.

Type II String Theory RR-Fields

Type II string theory has excitations in the RR sector which are bispinors

$$\Psi = g_{\mu\nu}(k)\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|k\rangle + \cdots + \psi_{\alpha\beta}(k)|k;\alpha\beta\rangle + \cdots$$

Type II supergravity has fieldstrengths

$$F \in \oplus_{k=0}\Omega^{2k+\epsilon}(M_{10}) \qquad \epsilon = 0(1) \quad IIA(IIB)$$

Classical supergravity must be supplemented with

- Quantization law
- Self-duality constraint

Differential K-theory

For many reasons, the quantization law turns out to use a generalized cohomology theory different from classical cohomology. Rather it is K-theory and the gauge invariant RR fields live in **differential K-theory**

$$0 \to \overbrace{K^{\ell-1}(M;\mathbb{R}/\mathbb{Z})}^{\text{flat}} \to \check{K}^{\ell}(M) \xrightarrow{\text{fieldstrength}} \Omega^{\ell}(M;\mathcal{R}) \to 0$$

$$0 \to \underbrace{\Omega^{\ell-1}(M;\mathcal{R})/\Omega^{\ell-1}_{\mathbb{Z}}(M;\mathcal{R})}_{\text{Topologically trivial}} \to \check{K}^{\ell}(M) \xrightarrow{\text{char.class}} \underbrace{K^{\ell}(M)}_{\text{Topological sector}} \to 0$$

$$\mathcal{R} = \mathbb{R}[u, u^{-1}]$$

Self-Duality of the RR field

Hamiltonian formulation:

Define $\mathrm{Heis}(\check{K}(X))$ via a skew symmetric function:

$$s([\check{C}_1],[\check{C}_2]) = \int_X^{\check{K}} [\check{C}_1]\cdot\overline{[\check{C}_2]}$$

Leading to a $\mathbb{Z}_2$-graded Heisenberg group with a unique $\mathbb{Z}_2$-graded irrep.

Partition function: Formulate an 11-dimensional CS theory

$\check{j} \in \check{K}(Y)$ & $CS(\check{j}) = \int_Y^{\check{K}O} [\check{j}]\cdot\overline{[\check{j}]}$

➡ Derive an action principle for type II RR fields.

Twisted K-theory and Orientifolds

(with J. Distler and D. Freed.)

Generalizing the story to type II string theory orientifolds

Key new features:

1. RR fields now in the **differential** **KR theory of a stack**.

2. The differential KR theory must be **twisted**. The B-field is the twisting: This organizes the zoo of orientifolds nicely.

3. Self-duality constraint leads to topological consistency condition on the twisting (B-field) leading to new topological consistency conditions for Type II orientifolds: ``twisted spin structure conditions.''

The General Construction

Looking beyond the physical applications, there is a natural mathematical generalization of all these examples:

1. We can define a generalized abelian gauge theory for any multiplicative generalized cohomology theory E.

2. Self-dual gauge theories can only be defined for Pontryagin self-dual generalized cohomology theories. These have the property that there is an integer s so that for any E-oriented compact manifold M of dimension n:

$$E^{n-s-j}(M) \otimes E^j(M; \mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$

Given by $\int_M^E x_1 x_2$ is a perfect pairing.

General Construction – II

3. We require an isomorphism (for some integer d – the degree):

$$\theta : E^d(\cdot) \to E^{n+2-s-d}(\cdot)$$

which is the isomorphism between electric and magnetic currents.

4. Choose a quadratic refinement q, of

$$b(x_1, x_2) = \int_M^E \theta(x_1) x_2$$

Conjecture (Freed-Moore-Segal): There exists a self-dual quantum field theory associated to these data with the current

$$\check{j} \in \check{E}^d(M)$$

Open Problems and Future Directions – I

- We have only determined the Hilbert space up to isomorphism.

- We have only determined the partition function as a function of external current. We also want the metric dependence.

- A lot of work remains to complete the construction of the full theory

A second challenging problem is the construction of the nonabelian theories in six dimensions. These are the proper home for understanding the duality symmetries of four-dimensional gauge theories. On their Coulomb branch they are described by the above self-dual theory, which should therefore give hints about the nonabelian theory.

For example: Is there an analog of the Frenkel-Kac-Segal construction?

Open Problems and Future Directions – II

A third challenging open problem is to understand better the compatibility with M-theory. The 3-form potential of M-theory has a cubic ``Chern-Simons term''

$$\int_{M_{11}} CdCdC$$

When properly defined this is a cubic refinement of the trilinear form

$$\check{H}^4(M) \times \check{H}^4(M) \times \check{H}^4(M) \to \mathbb{R}/\mathbb{Z}$$

Many aspects of type IIA/M-theory duality remain quite mysterious .

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