Über T-Dualische Transformationen: Giving Orbifold Groups A Lyft

All on side board

Aspen – June 15 – 2017

Work in progress w/ J. Harvey

Outline

1. $r \to 1/r$
2. Lifting Symmetries & Equivariant Vector Bundles
3. Toroidal CFT's
4. Weyl Groups
5. ~~General Nontrivial Involutions~~ Doomed To Fail
6. Cocycles
7. Truth & Consequences
   - Orbifolds
   - Symmetry Surfing
   - Is T-Duality A Gauge Symmetry of String Theory

Preamble:

This talk is probably relevant to the Moonshine workshop. It grew out of a project w/ Jeff Harvey where we were considering toroidal orbifolds of heterotic string theory which – by construction – have large discrete symmetries associated with subgroups of the Conway group. Then the consequences for type II on CY manifolds is potentially interesting. For example these groups would be symmetries of algebras of BPS states.

But in the course of our work we realized ~~there are some widespread and~~ we had some basic misconceptions about symmetries even in toroidal string theory. So we'll be going back to some old and well-worn stuff – a real blast from the past – but with some new twists – at least they were new to us.

(1.) $r \to 1/r$

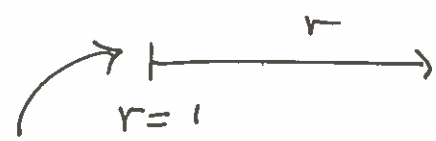
The main point is very easily demonstrated just w/ the 2D CFT of a periodic scalar field of radius r : $\mathcal{C}_r$

$$\text{Action} \sim \quad r^2 \int_{\Sigma_2} (\partial\varphi)^2 \qquad \varphi \sim \varphi + 2\pi$$

Moduli space of σ-model data: $\mathbb{R}_+$

Everybody knows

Moduli space of CFT's:



$\mathbb{Z}_2$-orbifold point.

Commonly said At $r=1$ there is enhanced $SU(2)$ symmetry and $\mathbb{Z}_2$ of $r \to 1/r$ is the Weyl group action, so T-duality is order 2!

Take a poll: How many people think you can orbifold the Gaussian model by an order 2 T-duality transformation?

Let's be a little more careful:

In the CFT we can split φ into Leftmovers + Right movers:

$$\varphi = \varphi_L + \varphi_R$$

Two symmetries:

$$\sigma_L : (\varphi_L, \varphi_R) \longrightarrow (-\varphi_L, \varphi_R)$$
$$\sigma_R : (\varphi_L, \varphi_R) \longrightarrow (\varphi_L, -\varphi_R)$$

Handling the zeromodes:

$$\{(p_L, p_R)\} = \Gamma_r = \mathbb{Z} e_r \oplus \mathbb{Z} f_r \subset \mathbb{R}^{1,1}$$

$$e_r = \frac{1}{\sqrt{2}}\left(\frac{1}{r}, \frac{1}{r}\right) \qquad f_r = \frac{1}{\sqrt{2}}(r, -r)$$

$$\sigma_L \cdot \Gamma_r = \sigma_R \cdot \Gamma_r = \Gamma_{1/r}$$

At $r = 1$ it would seem we have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of $\mathcal{C}_1$.

~~As we noted~~

Now $\mathcal{C}_1$ is the $SU(2)$ WZW model of level 1 and has

$$\widetilde{LSU(2)}_L \times \widetilde{LSU(2)}_R$$

dynamical symmetry.

Left-moving currents:

$$J^3 = \frac{1}{\sqrt{2}} \partial \varphi_L \qquad J^\pm = e^{\pm i\sqrt{2}\varphi_L}$$

Action of σ_L:

σ_L: $J^3 \to -J^3$, $J^\pm \to J^\mp$

But $J^\pm = J^1 \pm i J^2$,

J^a in the $\underline{3}$ of $SO(3)$ and this is a $180°$ degree rotation in $SO(3)$

But the Hilbert space of states $\mathcal{H}_{r=1}$ is Not a rep of $SO(3)_L$.

e.g. $V_{\pm\pm} = e^{\frac{i}{\sqrt{2}}(\pm\varphi_L \pm \varphi_R)}$ in $(2;2)$ of $SU(2)_L \times SU(2)_R$

So we must lift the $180°$ rot. to $SU(2)_L$ Where it becomes order 4, not order 2.

T-Duality of $\mathcal{C}_1$ is order 4!

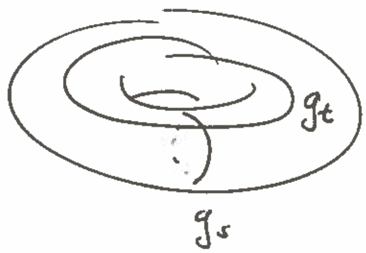
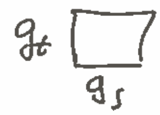
Credits:

1// Something similar is in Nahm-Wendland: Mirror symmetry for $T^4/\mathbb{Z}_2$ is order 4. Not quite same: They have an order 4 tmn on the σ-model data. We have an order 2 tmn on σ-model data but the lift to the CFT is order 4.

2// Above discussion comes from conversations w/ N. Seiberg. We had stumbled on this another way — I'll now explain

Our original route to this observation was via <u>modular covariance</u> – let us define this term

Quite generally, suppose G acts (as a group of aut's) on a CFT $\mathcal{C}$.
Then we can consider the torus amplitude with twisted b.c.'s

$Z(g_t, g_s; \tau) :=$  $:=$  if $[g_s, g_t] = 1$

In the Hamiltonian viewpoint there is a twisted statespace $\mathcal{H}^{g_s}$ and g_t acts on it and

$$Z(g_t, g_s; \tau) = \mathrm{Tr}_{\mathcal{H}^{g_s}}\, g_t\, q^{H} \bar{q}^{\tilde{H}} \qquad H = L_0 - c/24$$

<u>Def:</u> Modular Covariance says
$\forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

$$Z(g_t, g_s; \gamma\cdot\tau) = e^{i\phi(\gamma)}\, Z(\underbrace{g_s^{-b} g_t^{d}\, ; g_s^{a} g_t^{-c}}_{\text{action of large diffeo on b.c.'s}}\, ; \tau)$$

Returning to our Gaussian model $\mathcal{C}_1$
<u>Suppose</u> (counterfactually) σ_L acts on $\mathcal{C}_1$ with order 2.

Then we can compute $\sigma_L \underset{1}{\Box}$ from trace:

$$\sigma_L \,|\{n_i\}, \{\tilde{n}_i\}; p\rangle \overset{?}{=} (-1)^N \,|\{n_i\}, \{\tilde{n}_i\}; \sigma_L \cdot p\rangle$$

↑ # left-moving oscillators

This is the formula assumed in the literature!

$$\sigma_L \underset{1}{\Box} = B_- \overline{B}_+ \overline{\vartheta_3(2\tau)}$$

$$B_\pm = \frac{1}{q^{1/24}\prod(1 \mp q^n)}, \quad \overline{\vartheta_3(2\tau)} = (\#)_{\Gamma^{\sigma_L}}$$

Now act w/ modular transformations:

$$\sigma_L \underset{1}{\Box} \xrightarrow{S} 1 \underset{\sigma_L}{\Box} \xrightarrow{T} \sigma_L \underset{\sigma_L}{\Box} \xrightarrow{T} \sigma_L^2 \underset{\sigma_L}{\Box} \xrightarrow{S} \sigma_L \underset{\sigma_L^2}{\Box}$$

<u>Compute</u>:

$$\sigma_L \underset{\sigma_L^2}{\Box} = e^{-i\pi/4} B_- \overline{B}_+ ~~\vartheta_4(2\tau)~~ \overline{\vartheta_2(2\tau)}$$

IS NOT proportional to $\sigma_L \underset{1}{\Box}$!!

Assuming σ_L has order 4 lift $\tilde{\sigma}_L$ —

Can construct modular covariant $h \underset{g}{\Box}$.

There is a 3^{rd} way to understand this phenomenon in terms of cocycles for vertex operators – we'll come back to that later. That method shows the correct action:

$$\tilde{\sigma}_L \,|\, \{n_i\}, \{\tilde{n}_i\}; p \rangle = e^{-\frac{i\pi}{2}(n+w)^2} (-1)^N \,|\, \{n_i\}, \{\tilde{n}_i\}; \sigma_L \cdot p \rangle$$

(2) Equivariant Bundles & Liftings

We want to generalize the above, and put it into proper geometrical context.

Review some standard geometrical notions

X is a (left) G-space if we have transformations

$$x \xrightarrow{\;g\;} g\cdot x$$

s.t.

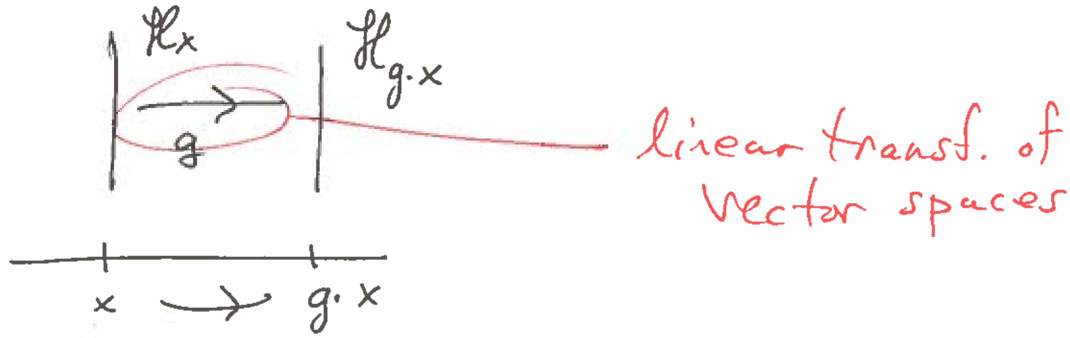
$$\begin{array}{ccc} x & \xrightarrow{\;g_1\;} & g_1\cdot x \\ & {\scriptstyle g_2 g_1}\searrow \quad \swarrow{\scriptstyle g_2} & \\ & g_2(g_1\cdot x) & \end{array}$$

Suppose we have a vector bundle over a G-space. $\pi: \mathcal{H} \to X$

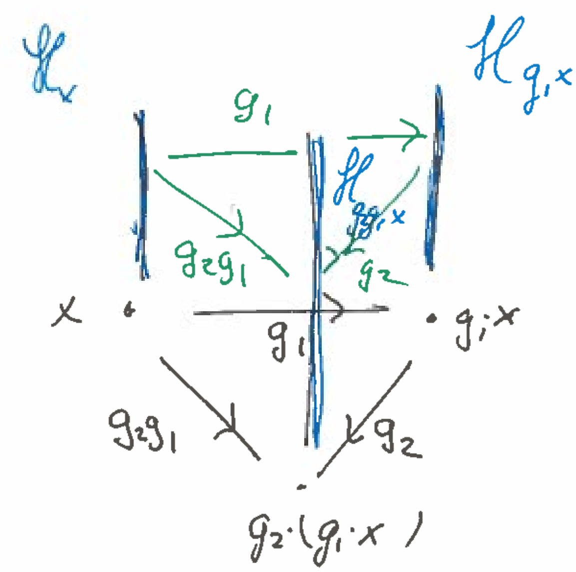
A *lifting* of the trans $x \xrightarrow{g} g\cdot x$ to $\mathcal{H}$ is a transformation on the total space so that

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{g} & \mathcal{H} \\ \pi\downarrow & & \downarrow\pi \\ X & \xrightarrow{g} & X \end{array}$$

and g is linear on the fibers



Suppose we can lift every $g \in G$ and the group elements compose:



Then $\mathcal{H}$ is a G-equivariant vector bundle.

Example Suppose $X = \{x_0\}$ is a single point. The G-action on a point must be trivial

$\mathcal{H}_{x_0} \circlearrowright g$: a linear tmn

$x_0 \circlearrowleft g$

Then a G-equiv. bundle (over a point) is the same thing as a rep^{n} of G.

Rmk (in general)

So: For a G-equiv. bundle, the fiber above any point $x_0 \in X$ is a rep. of $\mathrm{Stab}(x_0) \subset G$.

Now, again let us consider a v.b. / single point x_0

But suppose $\mathcal{H}_{x_0}$ is a projective rep of G.

So:

$$1 \longrightarrow A \longrightarrow \tilde{G} \xrightarrow{\pi_{\tilde{G}}} G \longrightarrow 1$$

c.e. by abelian group A.

Now we have a $\tilde{G}$ equivariant bundle covering the G-action on the base.

More generally:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\tilde{g}} & \mathcal{H} \\ \downarrow & & \downarrow \\ X & \xrightarrow{g = \pi_{\tilde{G}}(\tilde{g})} & X \end{array}$$

Now suppose we have:

1. V.b. $\pi : \mathcal{H} \longrightarrow X$
2. X is a G-space

Ask: Can we make $\mathcal{H}$ a G-equiv. bundle?

In general, <u>no</u>!

<u>Example</u> $\mathcal{H} = S^2 \times \mathbb{C}^2$

$SO(3)$ acts on S^2.
Can we make it act nontrivially on $\mathbb{C}^2$?
<u>No</u>!
But we can make the c.e. $SU(2)$ act.

Even more, let $P(\hat{x}) = \frac{1}{2}(1 + \hat{x}\cdot\vec{\sigma})$

$\mathcal{L} \subset S^2 \times \mathbb{C}^2$ line bundle defined by $P(\hat{x})$

$$u P(\hat{x}) u^{-1} = P\left(R(u)\cdot\hat{x}\right)$$

So we can make the magnetic monopole line bundle an $SU(2)$ equiv. bundle covering the $SO(3)$ action on the base.

But things can be more complicated:

$$\pi : \mathcal{H} \to X$$

X is a G-space.

At points $x_0 \in X$ with $\mathrm{Stab}(x_0) \neq \{1\}$

A central extension acts.

But no single $\tilde{G}$ exists making $\mathcal{H}$ a $\tilde{G}$-equiv. bundle.

This really happens! It is common in solid state physics for the bundle of electron Bloch wavefunctions over the Brillouin torus. What we have is a

"twisted equivariant G-bundle"

See D. Freed + G. Moore "Twisted Equivariant Matter"

(3) Toroidal CFT's

For simplicity: $\varphi: \Sigma_2 \longrightarrow T^d$

to write σ-model action need flat $G + B$-field:

Moduli space of σ-model data $= \mathcal{E} = \left\{ E = G + B \;\middle|\; \begin{array}{l} G = G^{tr} > 0 \\ B = -B^{tr} \end{array} \right\}$

$$\subset M_d(\mathbb{R})$$

Can show

$$\mathcal{E} \approx O(d,d;\mathbb{R}) / O(d) \times O(d)$$

For each $E \in \mathcal{E}$ can construct CFT so we have a bundle of CFT's

$$\begin{array}{ccc} \mathcal{H}_E & \hookrightarrow & \mathcal{H} \\ \downarrow & & \downarrow \\ E & \in & \mathcal{E} \end{array}$$

When quantizing we find a space of zeromodes

$$\{(p_L; p_R)\} =: \Gamma \hookrightarrow \mathbb{R}^{d,d}$$

Embedding of ! even unimodular lattice $\mathrm{II}^{d,d} \hookrightarrow \mathbb{R}^{d,d}$

Conversely, given such an embedding

$$II^{d,d} \hookrightarrow \Gamma \subset \mathbb{R}^{d,d}$$

we can construct a CFT $\mathcal{C}_\Gamma$:

$$\begin{array}{ccc} & \Gamma \otimes \mathbb{R} & \\ {\scriptstyle \pi_L} \swarrow & & \searrow {\scriptstyle \pi_R} \\ V_R & & V_R \end{array}$$

$$\mathcal{H}_\Gamma := S^\bullet\left(\bigoplus_{n>0} q^n V_R \otimes \mathbb{C}\right) \otimes S^\bullet\left(\bigoplus_{n>0} q^n V_R \otimes \mathbb{C}\right) \otimes \mathbb{C}(\hat{\Gamma})$$

~~where $1 \to A \to \hat{\Gamma} \to \Gamma \to 1$ is a c.e. of Γ (defined by cocycles)~~

$$H = \frac{1}{2}P_L^2 + \frac{1}{2}P_R^2 + H^{osc}$$

$\mathbb{L}$ = moduli space of embeddings $II^{d,d} \hookrightarrow \mathbb{R}^{d,d}$

$$\cong \underbrace{O(d,d,\mathbb{Z})}_{:= \mathcal{J}} \backslash O(d,d;\mathbb{R})$$

so the situation is

VTX OP SIDE

$$\begin{array}{ccc} \mathcal{H}_\Gamma & & \mathcal{H} \\ \downarrow & & \downarrow \\ \Gamma & \in & \mathbb{L} \\ & & \cong \\ & & O(d,d;\mathbb{Z})\backslash O(d,d;\mathbb{R}) \end{array}$$

$O(d,d;\mathbb{Z})$ = T-duality group

σ-Model side

$$\begin{array}{ccc} \mathcal{H} & & \mathcal{H}_E \\ \downarrow & & \downarrow \\ \mathcal{N} & \ni & E \\ \cong & & \\ O(d,d;\mathbb{R})/O(d)\times O(d) & & \end{array}$$

$O(d)\times O(d)$ aut of CFT $\mathcal{H}_\Gamma$

$\mathcal{T}$ acts on σ-model data $E \to (aE+b)(cE+d)^{tr,-1}$

$$\mathcal{W} = O(d,d;\mathbb{Z})\backslash O(d,d;\mathbb{R})/O(d)\times O(d)$$
$$= \text{moduli space of CFTs}$$

Main question: Is $\mathcal{H} \to \mathcal{N}$ $\mathcal{T}$-equivariant?

Already we know the answer is "no" from Gaussian model.

But in that case one could make it $\mathbb{Z}_4$-equiv.

To get at this question we look at the points in $\mathcal{N}$ with nontrivial stabilizer. These correspond to orbifold points in $\mathcal{W}$ and are more easily thought of in terms of Γ

$$C(\Gamma) := \mathrm{Aut}(\Gamma) \cap \left(O(d)_L \times O(d)_R \right)$$

Always contains $p \to -p$. But this does not act effectively on $\mathbb{L}$ or $\mathbb{H}$ so we should quotient out

$$\bar{C}(\Gamma) = C(\Gamma) / \langle \text{trivial involution} \rangle$$

$$\mathcal{W}_{ESP} := \{ [\Gamma] \mid \bar{C}(\Gamma) \text{ is nontrivial} \}$$

can lift that locus to $\mathbb{L}$ or $\mathbb{H}$.

This is a complicated singular sublocus. To get some idea of what $\mathcal{H}$ looks like over this space we begin with some very special points w/ nonabelian symmetry.

4 Weyl Groups

Suppose $\mathfrak{g}$ = semisimple, simply laced, rank d

$= \oplus A, D, E$

$$\Gamma(\mathfrak{g}) := \{ (p_L ; p_R) \in \Lambda_{wt}(\mathfrak{g}) \times \Lambda_{wt}(\mathfrak{g}) \mid p_L - p_R \in \Lambda_{rt}(\mathfrak{g}) \}$$

- $C(\Gamma(\mathfrak{g})) = W(\mathfrak{g})_L \times W(\mathfrak{g})_R$

- $\mathcal{H}_{\Gamma(\mathfrak{g})}$ = WZW model for G = simply connected cover

$$\underset{G \text{ simple}}{=} \bigoplus_{\theta\cdot\lambda \leq 1} V_\lambda \otimes \overline{V}_\lambda$$

V_λ = integ. hwt rep of $\widehat{LG}^{(1)}$.

Does $C(\Gamma(\mathfrak{g}))$ act on $\mathcal{H}_{\Gamma(\mathfrak{g})}$?

In general – <u>NO</u>!

Need to correct a misconception that is common in the string theory literature.

> $W(\mathfrak{g})$ is <u>NOT</u> a subgroup of G!!

<u>Rather:</u> $T \subset G$ Maximal torus

$$T \triangleleft N(T) := \{ g \in G \mid gTg^{-1} = T \} \subset G$$

($\triangleleft$ ↑ normal subgroup)

$$W(\mathfrak{g}) := N(T)/T$$

e.g. $G = SU(2)$ $\quad T = \left\{ \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \;\middle|\; |x| = 1 \right\}$

$$N(T) = T \amalg \left\{ \begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix} \;\middle|\; |x| = 1 \right\}$$

Conj. by any element in nontrivial component induces the Weyl group action on T.

But $\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}^2 = -\mathbb{1}$ <u>all of order 4</u>.

General story is subtle and goes back to J. Tits from mid-1960's.

You can do the case of $SU(N)$ yourself:

$$W(su(N)) \cong S_N$$
$$= \langle \sigma_i \rangle$$

(σ_i ↑ reflections in sple roots)

The general lift of σ_i into $N(T)$ has the form

$$g_i = \begin{pmatrix} z_1^{(i)} & & & \\ & \ddots & & \\ & & \begin{matrix} & i & i+1 \\ i & 0 & x_i \\ i+1 & y_i & 0 \end{matrix} & \\ & & & \ddots \quad z_N^{(i)} \end{pmatrix}$$

$x_i, y_i, z_k^{(i)}$ phases and $x_i y_i \prod z_k^{(i)} = -1$

Conj. by g_i induces Weyl group action.

$\langle g_i \rangle = W(\vec{x}, \vec{y}, \vec{z})$: Finite group if all are roots of 1

Can we find one isomorphic to S_N?

N even : NO
N odd : YES

e.g.

$$g_1^W = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix} \qquad g_2^W = \begin{pmatrix} -1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

But this is special - in general there is a <u>canonical</u> <u>lift</u>, called the Tits lift $g_i \longrightarrow \sigma_i$

$$g_i = \exp \frac{\pi}{2}(e_i - f_i) \qquad \text{order } 4$$

↑ ↗ Serre gen's

$$1 \longrightarrow T_2 \longrightarrow \tilde{W}^T \longrightarrow W \longrightarrow 1$$

T_2 = points in T of order 2.

e.g. for $SU(3)$

$$g_1^T = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \qquad g_2^T = \begin{pmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$$

Conclusion: On various sub-loci of ~~B~~ the symmetry $C(\Gamma)$ does <u>NOT</u> lift to an action on $\mathcal{H}_\Gamma$ but rather some extension – in general <u>NOT</u> a central extension – will lift.

(5) Doomed To Fail

W_{ESP} has many other points other than the A-D-E loci

<u>Example:</u> (Related to how Jeff and I got into these considerations.)

Let $G \subset Co_0$ be any subgroup of the Conway group fixing a $(24-d)$-dim'l lattice.

Then, $\exists \Gamma$ w/ $C(\Gamma) = G_L \times G_R$.

We don't have the crutch of nonabelian symmetry but we can use modular covariance to decide when the formula:

$$g|p\rangle \overset{?}{=} |g\cdot p\rangle \quad \forall p \in \Gamma \qquad (*)$$

is doomed to fail. (Given action of g on Γ its action on the oscillators is completely determined since we know the action on $\Gamma\otimes\mathbb{R}$)

Given (*) we have

$$g \underset{1}{\square} = B_+^{n_+} B_-^{n_-} \bar{B}_+^{\tilde{n}_+} \bar{B}_-^{\tilde{n}_-} \;(\#)_{\Gamma g}$$

Find (1) If $\ell := \text{order}(g)$ is odd there is no obstruction to modular covariance

(2) If ℓ is <u>even</u> and $\exists p \in \Gamma$ with

$$p \cdot g^{\ell/2} \cdot p = 1 \mod 2 \quad (**)$$

then modular covariance fails.

(3) If $\forall p \in \Gamma$ $p \cdot g^{\ell/2} \cdot p = 0 \mod 2$ modular covariance is o.k.

(4) When the criterion (**) holds if we assume there is an action $\tilde{g}$ on $\mathcal{C}_\Gamma$ with $\tilde{g}^{2\ell} = 1$ then modular covariance holds.

We can rephrase the "doomed to fail" criterion (**):

It is possible to show $\exists\, W_g \in \Gamma^g \subset \Gamma$ with

$$P \cdot g^{\ell/2} \cdot P = W_g \cdot P \mod 2.$$

Of course W_g is only defined up to translation by elements of 2Γ. We can call W_g a "twisted characteristic vector".

(**) $\Longleftrightarrow$ $W_g \not\equiv 0$.

We will come back to (**).

6. Cocycles

In the vertex operator construction of toroidal CFT's we must include "cocycles" c_p these are operators on $\mathcal{C}(\Gamma)$ so we can write

$$V_p = c_p : e^{ip\cdot X} :$$

$$c_{p^1} c_{p^2} = \varepsilon(p^1, p^2)\, c_{p^1+p^2}$$

ε valued in some subgroup $A \subset U(1)$.

OPE: $$V_{p^1} V_{p^2} \sim \varepsilon(p^1,p^2)\, z_{12}^{p_L^1\cdot p_L^2}\, \bar{z}_{12}^{p_R^1\cdot p_R^2}\, V_{p^1+p^2} + \cdots$$

Assoc.: $\Rightarrow$ ε is a cocycle

$\Rightarrow$ What actually acts on $\mathcal{C}(\Gamma)$ is a c.e.

$$1 \to A \to \hat{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

Locality: $$\frac{\varepsilon(p^1,p^2)}{\varepsilon(p^2,p^1)} = e^{i\pi p^1\cdot p^2}$$

$\exists$ Many mistakes in the literature for explicit formulae on cocycles.

e.g. often - even for the Gaussian model

A is taken to be $\{\pm 1\}$ and, e.g.

$$p = ne + wf$$

$$\varepsilon(p^1, p^2) = \exp(i\pi n_1 w_2)$$

satisfies locality condition but is not compatible with $SU(2)$ symmetry.

Correct formula

$$\varepsilon(p^1, p^2) = e^{-\frac{i\pi}{2}(n_1 w_2 - n_2 w_1) - i\pi w_1 w_2}$$

In general $Aut(\Gamma)$ gets extended to $Aut(\hat{\Gamma})$

$$g \cdot (a, p) = (a\,\xi_g(p), g \cdot p)$$

Where

$$\frac{\xi_g(p^1 + p^2)}{\xi_g(p^1)\,\xi_g(p^2)} = \frac{\varepsilon(gp^1, gp^2)}{\varepsilon(p^1, p^2)}$$

If g is an involution the lift to $Aut(\hat{\Gamma})$ is <u>not</u> an involution if

$$\xi_g(p)\,\xi_g(gp) \neq 1.$$

This is what happens in the Gaussian model.

7 Truth & Consequences

7.1 Asymmetric Orbifolds

- Orbifold groups are generally thought of as subgroups of $C(\Gamma)$. This is a basic misconception. You can only orbifold by a subgroup of the Aut's of the CFT.

- We don't know the full set of consistency conditions for toroidal orbifolds!

  * Level matching: Clearly necessary. Closeley related to modular covariance.

  * Freed-Vafa: Higher genus symmetric orbifolds. Potentially new conditions but all cases they could compute reduced to level matching

  * Narain-Sarmadi-Vafa

Stated the following: If $g \in C(\Gamma)$ is in your orbifold group and $g^{\ell} = 1$ has even order ℓ then it is necessary for consistency that

$$\forall p \in \Gamma \qquad p \cdot g^{\ell/2} \cdot p = 0 \mod 2$$

i.e. $W_g \simeq 0$.

We have shown this is not right.

Example $\mathcal{C} = \mathcal{C}_1 = WZW(SU(2))_1$

$\mathcal{C}^N /$ Diagonal T-duality

* Satisfies level matching for $N = 0(4)$
* Torus partition function has pos. integer $q, \bar{q}$ expansion. (Stronger than mod. invce)

More generally, for nontrivial involutions $g \in O(\Gamma)$ that violate (**) the $\mathbb{Z}_4$ orbifold by $\tilde{g}$ has a perfectly good 1-loop p.f. provided

$$W_g^2 = 0 \mod 4$$

This appears to be a new consistency condition.

So, what *are* the consistency conditions for orbifolds?

Not known. Preliminary discussions w/ D. Gaiotto and N. Seiberg suggest that modern (topological) ideas of TFT (as used now in phases of matter) might be a good tool to answer the question: Need a general set of consistency conditions for G-equivariant unitary modular tensor categories – These have been studied.

7.2 Symmetry Surfing

GHV Theorem on symmetries of K3 sigma models is about classifying $C(\Gamma)$ (Used heterotic/typeII duality). Leaves open the possibility that actually it is extensions of $C(\Gamma)$ that act.

7.3 T-Duality As A Gauge Symmetry

Famous statement of Dine & Seiberg.
"Proved" in review of Giveon Porrati Rabinovici

- $C(\Gamma(\mathfrak{g}))$ + simple generator generate $O(d,d;\mathbb{Z}) = \mathcal{T}$

- Then $W(\mathfrak{g})_L \times W(\mathfrak{g})_R \subset G_L \times G_R$

but $G_L \times G_R$ is a gauge group.

$\therefore$ entire group $\mathcal{T}$ is a gauge symmetry

ASK: What's wrong with this picture?

Yes! $W(\mathfrak{g}) \not\subset G$ and sometimes no lift exists. $\Longrightarrow$ Two options:

Ⓐ $\exists\ \tilde{\mathcal{T}} \to \mathcal{T}$ so

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\tilde{\mathcal{T}}} & \mathcal{H} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{H} & \longrightarrow & \mathbb{H} \end{array}$$

then perhaps $\tilde{\mathcal{T}}$ is a gauge symmetry

Ⓑ No such $\tilde{\mathcal{T}}$ exists.

Don't know which is right, but given the experience with band structure I strongly suspect Ⓑ is the correct option ∎

12:27:30
12:47 end (1)
12:57 end (2)
13:08 end (3)

13:11 Start (4)
13:25 end (4)
13:30 end (5)
13:38 end (6)
13:51 end.

Über T-Dualische Transformationen: Giving Orbifold Groups A Lyft

All on side board

Aspen – June 15 – 2017

Work in progress w/ J. Harvey

Outline

1. $r \to 1/r$
2. Lifting Symmetries & Equivariant Vector Bundles
3. Toroidal CFT's
4. Weyl Groups
5. ~~General Nontrivial Involutions~~ Doomed To Fail
6. Cocycles
7. Truth & Consequences
   - Orbifolds
   - Symmetry Surfing
   - Is T-Duality A Gauge Symmetry of String Theory

Preamble:

This talk is probably relevant to the Moonshine workshop. It grew out of a project w/ Jeff Harvey where we were considering toroidal orbifolds of heterotic string theory which - by construction - have large discrete symmetries associated with subgroups of the Conway group. Then the consequences for type II on CY manifolds is potentially interesting. For example these groups would be symmetries of algebras of BPS states.

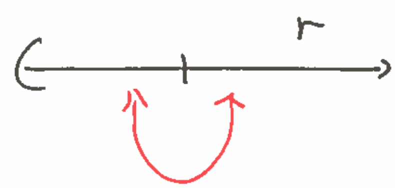
But in the course of our work we realized we had some basic misconceptions about symmetries even in toroidal string theory. So we'll be going back to some old and well-worn stuff — a real blast from the past — but with some new twists - at least they were new to us.

(1.) $r \longrightarrow 1/r$

The main point is very easily demonstrated just w/ the 2D CFT of a periodic scalar field of radius r : $\mathcal{C}_r$

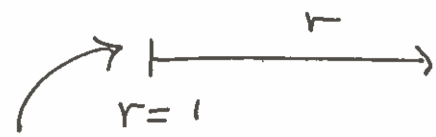
Action $\sim \quad r^2 \int_{\Sigma_2} (\partial\varphi)^2 \qquad \varphi \sim \varphi + 2\pi$

Moduli space of σ-model data: $\mathbb{R}_+$



Everybody knows $r \longrightarrow 1/r$

Moduli space of CFT's:



$\mathbb{Z}_2$-orbifold point.

<u>Commonly said</u> At $r=1$ there is enhanced $SU(2)$ symmetry and $\mathbb{Z}_2$ of $r \to 1/r$ is the Weyl group action, so T-duality is order 2!

Take a poll: How many people think you can orbifold the Gaussian model by an order 2 T-duality transformation?

Let's be a little more careful:

In the CFT we can split φ into Leftmovers + Right movers:

$$\varphi = \varphi_L + \varphi_R$$

Two symmetries:

$$\sigma_L : (\varphi_L, \varphi_R) \longrightarrow (-\varphi_L, \varphi_R)$$
$$\sigma_R : (\varphi_L, \varphi_R) \longrightarrow (\varphi_L, -\varphi_R)$$

Handling the zeromodes:

$$\{(p_L, p_R)\} = \Gamma_r = \mathbb{Z} e_r \oplus \mathbb{Z} f_r \subset \mathbb{R}^{1,1}$$

$$e_r = \frac{1}{\sqrt{2}}\left(\frac{1}{r}, \frac{1}{r}\right) \qquad f_r = \frac{1}{\sqrt{2}}(r, -r)$$

$$\sigma_L \cdot \Gamma_r = \sigma_R \cdot \Gamma_r = \Gamma_{1/r}$$

At $r = 1$ it would seem we have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of $\mathcal{C}_1$.

~~As we noted~~

Now $\mathcal{C}_1$ is the SU(2) WZW model of level 1 and has

$$\widetilde{LSU(2)}_L \times \widetilde{LSU(2)}_R$$

dynamical symmetry.

Left-moving currents:

$J^3 = \frac{1}{\sqrt{2}} \partial \varphi_L \qquad J^\pm = e^{\pm i \sqrt{2} \varphi_L}$

Action of σ_L:

σ_L: $J^3 \to -J^3$, $J^\pm \to J^\mp$

But $J^\pm = J^1 \pm i J^2$,

J^a in the $\underline{3}$ of $SO(3)$ and this is a 180° degree rotation in $SO(3)$

But the Hilbert space of states $\mathcal{H}_{r=1}$ is Not a rep of $SO(3)_L$.

e.g. $V_{\pm\pm} = e^{\frac{i}{\sqrt{2}}(\pm \varphi_L \pm \varphi_R)}$ in $(2;2)$ of $SU(2)_L \times SU(2)_R$

So we must lift the 180° rot. to $SU(2)_L$ Where it becomes order 4, not order 2.

T-Duality of $\mathcal{C}_1$ is order 4!

Credits:

1. Something similar is in Nahm-Wendland: Mirror symmetry for $T^4/\mathbb{Z}_2$ is order 4. Not quite: same: They have an order 4 turn on the σ-model data. We have an order 2 turn on σ-model data but the lift to the CFT is order 4.

2. Above discussion comes from conversations w/ N. Seiberg. We had stumbled on this another way — I'll now explain

Our original route to this observation was via <u>modular</u> <u>covariance</u> – let us define this term

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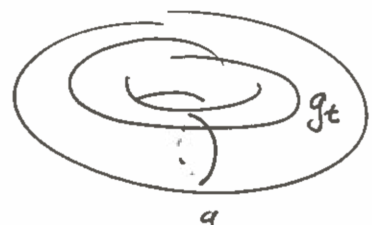
Quite generally, suppose G acts (as a group of aut's) on a CFT $\mathcal{C}$.

Then we can consider the torus amplitude with twisted b.c.'s



$$Z(g_t, g_s; \tau) := \quad := \quad g_t \square_{g_s} \qquad \text{if } [g_s, g_t] = 1$$

In the Hamiltonian viewpoint there is a twisted statespace $\mathcal{H}^{g_s}$ and g_t acts on it and

$$Z(g_t, g_s; \tau) = \mathrm{Tr}_{\mathcal{H}^{g_s}}\, g_t\, q^{H}\, \bar{q}^{\tilde{H}} \qquad H = L_0 - c/24$$

<u>Def</u>: Modular Covariance says

$\forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

$$Z(g_t, g_s; \gamma\cdot\tau) = e^{i\phi(\gamma)}\, Z(\underbrace{g_s^{-b} g_t^{d}\, ; g_s^{a} g_t^{-c}}_{\text{action of large diffeo on b.c.'s}} ; \tau)$$

Returning to our Gaussian model $\mathcal{C}_1$

Suppose (counterfactually) σ_L acts on $\mathcal{C}_1$ with order 2.

Then we can compute $\sigma_L \underset{1}{\square}$ from trace:

$$\sigma_L \,|\, \{n_i\}, \{\tilde{n}_i\}; p\rangle \overset{?}{=} (-1)^N \,|\, \{n_i\}, \{\tilde{n}_i\}; \sigma_L \cdot p\rangle$$

N = # left-moving oscillators

This is the formula assumed in the literature!

$$\sigma_L \underset{1}{\square} = B_- \overline{B}_+ \overline{\vartheta_3(2\tau)}$$

$$B_\pm = \frac{1}{q^{1/24} \prod (1 \mp q^n)}, \quad \overline{\vartheta_3(2\tau)} = (\#)_{\Gamma_{8L}}$$

Now act w/ modular transformations:

$$\sigma_L \underset{1}{\square} \xrightarrow{S} 1 \underset{\sigma_L}{\square} \xrightarrow{T} \sigma_L \underset{\sigma_L}{\square} \xrightarrow{T} \sigma_L^2 \underset{\sigma_L}{\square} \xrightarrow{S} \sigma_L \underset{\sigma_L^2}{\square}$$

Compute:

$$\sigma_L \underset{\sigma_L^2}{\square} = e^{-i\pi/4} B_- \overline{B}_+ \; \overline{\vartheta_2(2\tau)}$$

~~$\vartheta_4(2\tau)$~~

IS NOT proportional to $\sigma_L \underset{1}{\square}$!!

Assuming σ_L has order 4 lift $\tilde{\sigma}_L$ –

Can construct modular covariant $h \underset{g}{\square}$.

There is a 3^{rd} way to understand this phenomenon in terms of cocycles for vertex operators – we'll come back to that later. That method shows the correct action:

$$\tilde{\sigma}_L \,|\{n_i\},\{\tilde{n}_i\};p\rangle = e^{-\frac{i\pi}{2}(n+w)^2}(-1)^N\,|\{n_i\},\{\tilde{n}_i\};\sigma_L\cdot p\rangle$$

(2) Equivariant Bundles & Liftings

We want to generalize the above, and put it into proper geometrical context.

Review some standard geometrical notions

X is a (left) G-space if we have transformations

$$x \xrightarrow{\;g\;} g\cdot x$$

s.t.

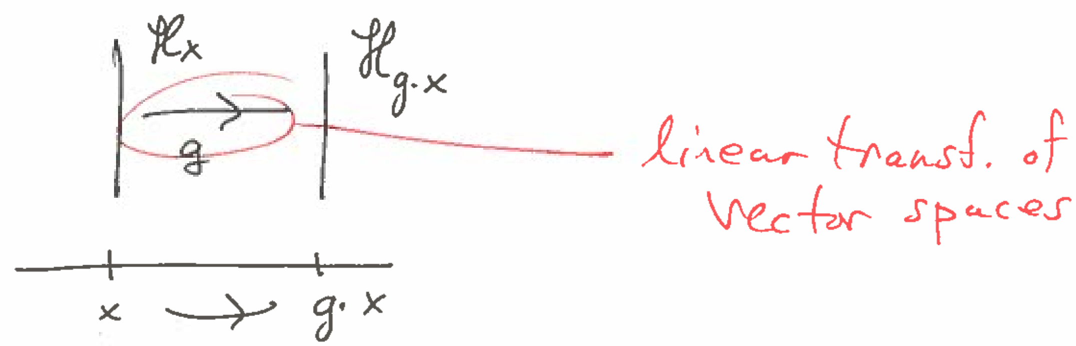
$$\begin{array}{ccc} x & \xrightarrow{\;g_1\;} & g_1\cdot x \\ & {\scriptstyle g_2g_1}\searrow \quad \swarrow{\scriptstyle g_2} & \\ & g_2\cdot(g_1\cdot x) & \end{array}$$

Suppose we have a vector bundle over a G-space. $\pi: \mathcal{H} \longrightarrow X$

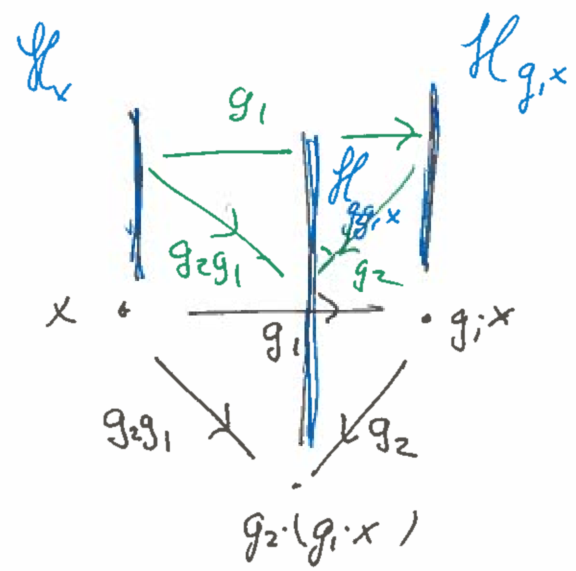
A <u>lifting</u> of the trun $x \xrightarrow{g} g\cdot x$ to $\mathcal{H}$ is a transformation on the total space so that

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{g} & \mathcal{H} \\ \pi\downarrow & & \downarrow\pi \\ X & \xrightarrow{g} & X \end{array}$$

and g is linear on the fibers



Suppose we can lift every $g \in G$ and ~~the~~ the group elements compose:



Then $\mathcal{H}$ is a G-equivariant vector bundle.

<u>Example</u> Suppose $X = \{x_0\}$ is a single point. The G-action on a point must be trivial.

$\mathcal{H}_{x_0} \circlearrowright g$: a linear tmn

$\cdot x_0 \circlearrowleft g$

Then a G-equiv. bundle (over a point) is the same thing as a rep^{n} of G.

Rmk (in general)

<u>So</u>: For a G-equiv. bundle, the fiber above any point $x_0 \in X$ is a rep: of $\mathrm{Stab}(x_0) \subset G$.

Now, again let us consider a v.b. / single point x_0

But suppose $\mathcal{H}_{x_0}$ is a <u>projective</u> <u>rep</u> of G.

So:

$$1 \longrightarrow A \longrightarrow \tilde{G} \xrightarrow{\pi_{\tilde{G}}} G \longrightarrow 1$$

c.e. by abelian group A.

Now we have a $\tilde{G}$ equivariant bundle covering the G-action on the base.

More generally:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\tilde{g}} & \mathcal{H} \\ \downarrow & & \downarrow \\ X & \xrightarrow{g = \pi_{\tilde{G}}(\tilde{g})} & X \end{array}$$

Now suppose we have:

1. v.b. $\pi : \mathcal{H} \longrightarrow X$
2. X is a G-space

Ask: Can we make $\mathcal{H}$ a G-equiv. bundle?

In general, <u>no</u>!

<u>Example</u> $\mathcal{H} = S^2 \times \mathbb{C}^2$

$SO(3)$ acts on S^2.

Can we make it act nontrivially on $\mathbb{C}^2$?

<u>No</u>!

But we can make the c.e. $SU(2)$ act.

Even more, let $P(\hat{x}) = \frac{1}{2}(1 + \hat{x} \cdot \vec{\sigma})$

$\mathcal{L} \subset S^2 \times \mathbb{C}^2$ line bundle defined by $P(\hat{x})$

$$u P(\hat{x}) u^{-1} = P\big(R(u) \cdot \hat{x}\big)$$

So we can make the magnetic monopole line bundle an $SU(2)$ equiv. bundle covering the $SO(3)$ action on the base.

But ~~[illegible]~~ things can be more complicated:

$$\pi: \mathcal{H} \to X$$

X is a G-space.

At points $x_0 \in X$ with $\mathrm{Stab}(x_0) \neq \{1\}$

A central extension acts.

But no single $\tilde{G}$ exists making $\mathcal{H}$ a $\tilde{G}$-equiv. bundle.

This really happens! It is common in solid state physics for the bundle of electron Bloch wavefunctions over the Brillouin torus. What we have is a

"twisted equivariant G-bundle"

See D. Freed + G. Moore "Twisted Equivariant Matter"

(3) Toroidal CFT's

For simplicity: $\varphi : \Sigma_2 \longrightarrow T^d$

to write σ-model action need flat $G + B$-field:

Moduli space of σ-model data $= \mathcal{H} = \left\{ E = G + B \;\middle|\; \begin{array}{l} G = G^{tr} > 0 \\ B = -B^{tr} \end{array} \right\}$

$$\subset M_d(\mathbb{R})$$

Can show

$$\mathcal{H} \approx O(d,d;\mathbb{R}) / O(d) \times O(d)$$

For each $E \in \mathcal{H}$ can construct CFT so we have a bundle of CFT's

$$\begin{array}{ccc} \mathcal{H}_E & \hookrightarrow & \mathcal{H} \\ \downarrow & & \downarrow \\ E & \in & \mathcal{H} \end{array}$$

When quantizing we find a space of zeromodes

$$\{ (p_L ; p_R) \} =: \Gamma \hookrightarrow \mathbb{R}^{d,d}$$

Embedding of ! even unimodular lattice $\quad \mathrm{II}^{d,d} \hookrightarrow \mathbb{R}^{d,d}$

Conversely, given such an embedding

$$II^{d,d} \hookrightarrow \Gamma \subset \mathbb{R}^{d,d}$$

we can construct a CFT $\mathcal{C}_\Gamma$:

$$\begin{array}{ccc} & \Gamma \otimes \mathbb{R} & \\ \pi_L \swarrow & & \searrow \pi_R \\ V_R & & V_R \end{array}$$

$$\mathcal{H}_\Gamma := S^\bullet\Big(\bigoplus_{n>0} q^n V_L \otimes \mathbb{C}\Big) \otimes S^\bullet\Big(\bigoplus_{n>0} q^n V_R \otimes \mathbb{C}\Big) \otimes \mathbb{C}(\Gamma)$$

~~where $1 \to A \to \hat{\Gamma} \to \Gamma \to 1$ is a c.e. of Γ (defined by cocycles)~~

$$H = \tfrac{1}{2} P_L^2 + \tfrac{1}{2} P_R^2 + H^{osc}$$

$\mathbb{L}$ = moduli space of embeddings $II^{d,d} \hookrightarrow \mathbb{R}^{d,d}$

$$\cong \underbrace{O(d,d,\mathbb{Z})}_{:= \mathcal{T}} \backslash O(d,d;\mathbb{R})$$

so the situation is

VTX OP SIDE

$$\mathcal{H}_\Gamma \to \Gamma \in \mathbb{L} \leftarrow \mathcal{H}$$

$$\mathbb{L} \cong \underbrace{O(d,d;\mathbb{Z})}_{\text{T-duality group}} \backslash O(d,d;\mathbb{R})$$

σ-Model side

$$\mathcal{H} \to \mathfrak{E} \ni E \leftarrow \mathcal{H}_E$$

$$\mathfrak{E} \cong O(d,d;\mathbb{R}) / O(d) \times O(d)$$

$O(d)\times O(d)$ aut of CFT $\mathcal{H}_\Gamma$

$\mathcal{T}$ acts on σ-model data $E \to (aE+b)(cE+d)^{\text{tr},-1}$

$$\mathcal{N} = O(d,d;\mathbb{Z}) \backslash O(d,d;\mathbb{R}) / O(d) \times O(d)$$
$$= \text{moduli space of CFTs}$$

Main question: Is $\mathcal{H} \to \mathfrak{E}$ $\mathcal{T}$-equivariant?

Already we know the answer is "no" from Gaussian model.

But in that case one could make it $\mathbb{Z}_4$-equiv.

To get at this question we look at the points in $\mathfrak{E}$ with nontrivial stabilizer. These correspond to orbifold points in $\mathcal{N}$ and are more easily thought of in terms of Γ

$$C(\Gamma) := \mathrm{Aut}(\Gamma) \cap \left(O(d)_L \times O(d)_R \right)$$

Always contains $p \to -p$. But this does not act effectively on $\mathbb{L}$ or $\mathbb{H}$ so we should quotient out

$$\bar{C}(\Gamma) = C(\Gamma) / \langle \text{trivial involution} \rangle$$

$$\mathcal{N}_{ESP} := \{ [\Gamma] \mid \bar{C}(\Gamma) \text{ is nontrivial} \}$$

can lift that locus to $\mathbb{L}$ or $\mathbb{H}$.

This is a complicated singular sublocus. To get some idea of what $\mathcal{H}$ looks like over this space we begin with some very special points w/ nonabelian symmetry.

4 Weyl Groups

Suppose $\mathfrak{g}$ = semisimple, simply laced, rank d

$$= \oplus A, D, E$$

$$\Gamma(\mathfrak{g}) := \{ (p_L; p_R) \in \Lambda_{wt}(\mathfrak{g}) \times \Lambda_{wt}(\mathfrak{g}) \mid p_L - p_R \in \Lambda_{rt}(\mathfrak{g}) \}$$

- $C(\Gamma(\mathfrak{g})) = W(\mathfrak{g})_L \times W(\mathfrak{g})_R$
- $\mathcal{H}_{\Gamma(\mathfrak{g})}$ = WZW model for G = simply connected cover

$$\underset{G \text{ simple}}{=} \bigoplus_{\theta\cdot\lambda \le 1} V_\lambda \otimes \overline{V}_\lambda$$

V_λ = integ. hwt rep of $\widehat{LG}^{(1)}$.

Does $C(\Gamma(\mathfrak{g}))$ act on $\mathcal{H}_{\Gamma(\mathfrak{g})}$?

In general – NO!

Need to correct a misconception that is common in the string theory literature.

$W(\mathfrak{g})$ is NOT a subgroup of G !!

<u>Rather:</u> $T \subset G$ Maximal torus

$$T \triangleleft N(T) := \{ g \in G \mid gTg^{-1} = T \} \subset G$$

($\uparrow$ normal subgroup)

$$W(\mathfrak{g}) := N(T)/T$$

e.g. $G = SU(2)$ $\quad T = \left\{ \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \;\middle|\; |x| = 1 \right\}$

$$N(T) = T \amalg \left\{ \begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix} \;\middle|\; |x| = 1 \right\}$$

Conj. by any element in nontrivial component induces the Weyl group action on T.

But $\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}^2 = -\mathbb{1}$ <u>all of order 4</u>.

General story is subtle and goes back to J. Tits from mid-1960's.

You can do the case of $SU(N)$ yourself:

$$W(su(N)) \cong S_N = \langle \sigma_i \rangle$$

($\uparrow$ reflections in sple roots)

The general lift of σ_i into $N(T)$ has the form

$$g_i = \begin{pmatrix} z_1^{(i)} & & & & \\ & \ddots & & & \\ & & \begin{matrix} & i & i+1 \\ i & 0 & x_i \\ i+1 & y_i & 0 \end{matrix} & & \\ & & & \ddots & \\ & & & & z_N^{(i)} \end{pmatrix}$$

$x_i, y_i, z_k^{(i)}$ phases and $x_i y_i \prod z_k^{(i)} = -1$

Conj. by g_i induces Weyl group action.

$\langle g_i \rangle = W(\vec{x}, \vec{y}, \vec{z})$: Finite group if all are roots of 1

Can we find one isomorphic to S_N ?

N even : NO
N odd : YES

e.g.

$$g_1^W = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix} \qquad g_2^W = \begin{pmatrix} -1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

But this is special - in general there is a <u>canonical</u> <u>lift</u>, called the Tits lift $g_i \longrightarrow \sigma_i$

$$g_i = \exp \frac{\pi}{2} (e_i - f_i) \qquad \text{order } 4$$

↑ ↗ Serre gen's

$$1 \longrightarrow T_2 \longrightarrow \tilde{W}^T \longrightarrow W \longrightarrow 1$$

T_2 = points in T of order 2.

e.g. for $SU(3)$

$$g_1^T = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \qquad g_2^T = \begin{pmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$$

Conclusion: On various sub-loci of the symmetry $C(\Gamma)$ does NOT lift to an action on $\mathcal{H}_\Gamma$ but rather some extension – in general NOT a central extension – will lift.

(5) Doomed To Fail

$\mathcal{W}_{ESP}$ has many other points other than the A-D-E loci

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<u>Example:</u> (Related to how Jeff and I got into these considerations.)

Let $G \subset Co_0$ be any subgroup of the Conway group fixing a $(24-d)$-dim'l lattice.

Then, $\exists\, \Gamma$ w/ $C(\Gamma) = G_L \times G_R$.

We don't have the crutch of nonabelian symmetry but we can use modular covariance to decide when the formula:

$$g|p\rangle \overset{?}{=} |g\cdot p\rangle \quad \forall p \in \Gamma \qquad (*)$$

is doomed to fail. (Given action of g on Γ its action on the oscillators is completely determined since we know the action on $\Gamma \otimes \mathbb{R}$)

Given (*) we have

$$g \underset{1}{\square} = B_+^{n_+} B_-^{n_-} \bar{B}_+^{\tilde{n}_+} \bar{B}_-^{\tilde{n}_-} \ (\#)_{\Gamma g}$$

Find (1) If $\ell := \text{order}(g)$ is odd there is no obstruction to modular covariance

(2) If ℓ is <u>even</u> and $\exists\, p \in \Gamma$ with

$$p \cdot g^{\ell/2} \cdot p = 1 \mod 2 \qquad (**)$$

then modular covariance fails.

(3) If $\forall\, p \in \Gamma$ $\ p \cdot g^{\ell/2} \cdot p = 0 \mod 2$ modular covariance is o.k.

(4) When the criterion (**) holds if we assume there is an action $\tilde{g}$ on $\mathcal{C}_\Gamma$ with $\tilde{g}^{2\ell} = 1$ then modular covariance holds.

We can rephrase the "doomed to fail" criterion (**):

It is possible to show $\exists\, W_g \in \Gamma^g \subset \Gamma$ with

$$p \cdot g^{l/2} \cdot p = W_g \cdot p \mod 2.$$

Of course W_g is only defined up to translation by elements of 2Γ. We can call W_g a "twisted characteristic vector".

$$(**) \iff W_g \not\cong 0.$$

We will come back to (**).

(6.) Cocycles

In the vertex operator construction of toroidal CFT's we must include "cocycles" c_p these are operators on $\mathcal{H}(\Gamma)$ so we can write

$$V_p = c_p : e^{ip\cdot X} :$$

$$c_{p^1} c_{p^2} = \varepsilon(p^1, p^2)\, c_{p^1+p^2}$$

ε valued in some subgroup $A \subset U(1)$.

OPE: $$V_{p^1} V_{p^2} \sim \varepsilon(p^1,p^2)\, z_{12}^{p_L^1 \cdot p_L^2}\, \bar{z}_{12}^{p_R^1 \cdot p_R^2}\, V_{p^1+p^2} + \cdots$$

Assoc.: $\Rightarrow$ ε is a cocycle

$\Rightarrow$ What actually acts on $\mathcal{H}(\Gamma)$ is a c.e.

$$1 \to A \to \hat{\Gamma} \to \Gamma \to 1$$

Locality: $$\frac{\varepsilon(p^1,p^2)}{\varepsilon(p^2,p^1)} = e^{i\pi p^1\cdot p^2}$$

$\exists$ Many mistakes in the literature for explicit formulae on cocycles.

e.g. often - even for the Gaussian model

A is taken to be $\{\pm 1\}$ and, e.g.

$$p = ne + wf$$

$$\varepsilon(p^1, p^2) = \exp(i\pi n_1 w_2)$$

satisfies locality condition but is not compatible with $SU(2)$ symmetry.

Correct formula

$$\varepsilon(p^1, p^2) = e^{-\frac{i\pi}{2}(n_1 w_2 - n_2 w_1) - i\pi w_1 w_2}$$

In general $Aut(\Gamma)$ gets extended to $Aut(\hat{\Gamma})$

$$g \cdot (a, p) = (a\xi_g(p), g \cdot p)$$

where
$$\frac{\xi_g(p^1 + p^2)}{\xi_g(p^1)\xi_g(p^2)} = \frac{\varepsilon(gp^1, gp^2)}{\varepsilon(p^1, p^2)}$$

If g is an involution the lift to $Aut(\hat{\Gamma})$ is <u>not</u> an involution if

$$\xi_g(p)\,\xi_g(gp) \neq 1.$$

This is what happens in the Gaussian model.

7 Truth & Consequences

7.1 Asymmetric Orbifolds

- Orbifold groups are generally thought of as subgroups of $C(\Gamma)$. This is a basic misconception. You can only orbifold by a subgroup of the Aut's of the CFT.
- We don't know the full set of consistency conditions for toroidal orbifolds!

  * Level matching : Clearly necessary. Closeley related to modular covariance.
  * Freed-Vafa : Higher genus symmetric orbifolds. Potentially new conditions but all cases they could compute reduced to level matching
  * Narain-Sarmadi-Vafa

Stated the following : If $g \in C(\Gamma)$ is in your orbifold group and $g^{\ell}=1$ has even order ℓ then it is necessary for consistency that

$$\forall p \in \Gamma \quad p \cdot g^{\ell/2} \cdot p = 0 \mod 2$$

i.e. $W_g \approx 0$.

We have shown this is not right.

Example $\mathcal{C} = \mathcal{C}_1 = WZW(SU(2))_1$

$\mathcal{C}^N$ / Diagonal T-duality

* Satisfies level matching for $N = 0(4)$
* Torus partition function has pos. integer $q, \bar{q}$ expansion. (Stronger than mod. invce)

More generally, for nontrivial involutions $g \in \mathcal{O}(\Gamma)$ that violate (**) the $\mathbb{Z}_4$ orbifold by $\tilde{g}$ has a perfectly good 1-loop p.f. provided

$$W_g^2 = 0 \mod 4$$

This appears to be a new consistency condition.

So, what *are* the consistency conditions for orbifolds?

Not known. Preliminary discussions w/ D. Gaiotto and N. Seiberg suggest that modern ~~topological~~ ideas of TFT (as used now in phases of matter) might be a good tool to answer the question: Need a general set of consistency conditions for G-equivariant unitary modular tensor categories – These have been studied.

7.2 Symmetry Surfing

GHV Theorem on symmetries of K3 sigma models is about classifying $C(\Gamma)$ (used heterotic/type II duality). Leaves open the possibility that actually it is extensions of $C(\Gamma)$ that act.

7.3 T-Duality As A Gauge Symmetry

Famous statement of Dine & Seiberg.
"Proved" in review of Giveon Porrati Rabinovici

- $C(\Gamma(\mathfrak{g}))$ + simple generator generate $O(d,d;\mathbb{Z}) = \mathcal{T}$

- Then $W(\mathfrak{g})_L \times W(\mathfrak{g})_R \subset G_L \times G_R$

but $G_L \times G_R$ is a gauge group.

$\therefore$ entire group $\mathcal{T}$ is a gauge symmetry

ASK: What's wrong with this picture?

Yes! $W(\mathfrak{g}) \not\subset G$ and sometimes no lift exists. $\Longrightarrow$ Two options:

(A) $\exists\ \tilde{\mathcal{T}} \to \mathcal{T}$ so

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\tilde{\mathcal{T}}} & \mathcal{H} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{H}/ & \longrightarrow & \mathcal{H}/ \end{array}$$

then perhaps $\tilde{\mathcal{T}}$ is a gauge symmetry

(B) No such $\tilde{\mathcal{T}}$ exists.

Don't know which is right, but given the experience with band structure I strongly suspect (B) is the correct option ■

12:27:30
12:47 end (1)
12:57 end (2)
13:08 end (3)

13:11 Start (4)
13:25 end (4)
13:30 end (5)
13:38 end (6)
13:51 end.