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THE ANALYTIC GEOMETRY OF TWO-DIMENSIONAL CONFORMAL FIELD THEORY*

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Two-dimensional conformal field theory is formulated as analytic geometry on the universal moduli space of Riemann surfaces.

1. Introduction

Some years ago, Polyakov [1] proposed constructing all conformally invariant quantum field theories by using the constraint of conformal invariance to make concrete the fundamental principles of quantum field theory. This is the conformal bootstrap program. Conformal field theories describe the universality classes of critical phenomena, or equivalently the short distance limits of quantum field theories, so the conformal bootstrap program is an attempt to find all possible critical phenomena, and all possible quantum field theories, and to describe explicitly their short-distance behavior.

The subject of two-dimensional conformal field theory originated simultaneously in the theory of critical phenomena [21] and in string theory [3]. In recent years there has been progress in the two-dimensional conformal bootstrap program, based on investigation of the two-dimensional conformal anomaly and the Virasoro algebra [4–17]. There has also been progress in the two-dimensional super-conformal bootstrap program, leading to the discovery of supersymmetric critical phenomena [8, 18–22].

Two-dimensional conformal field theory also has several applications in mathematics. Modifications of the Ricci-flat Calabi-Yau spaces [23], and certain generalizations [24] are thought to provide examples of two-dimensional superconformal

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field theories [25,26]. Superconformal field theory describes the Calabi-Yau spaces in much the sense that supersymmetric quantum mechanics on a general manifold, the spectrum of the Dirac operator, probes riemannian geometry. But superconformal field theory should be a more powerful mathematical tool, since much more data about the geometry is encoded in a much richer mathematical structure. It might even be supposed that the Calabi-Yau space can be recovered from the superconformal field theory in the classical limit. In this sense, the present work is a step towards constructing the Calabi-Yau spaces by the conformal bootstrap. Other applications of two-dimensional conformal field theory have been found in the representation theory of affine algebras [27] and sporadic groups [28]. For example, the vertex operator construction of the monstrous moonshine module [28] is a two-dimensional conformal field theory in which the monster acts as symmetry group.

In this paper we formulate two-dimensional conformal field theory as analytic geometry on the universal moduli space of Riemann surfaces. The two-dimensional conformal bootstrap is thus translated into pure mathematics, as an analytic, and even eventually algebraic, bootstrap program. In a second paper [29] we apply these ideas to quantum string theory.

We take as fundamental problem the construction of the partition function of the two-dimensional conformal field theory on all compact Riemann surfaces without boundary. The space of Riemann surfaces is called the moduli space. The moduli are the parameters which describe the deformations of the conformal structure of the surface. They probe the local conformal properties of the field theory. The correlation functions of the surface stress-energy tensor are calculated by differentiating the partition function with respect to the moduli. In the limit of large genus any local conformal transformation on the surface is approximated by a variation of the moduli.

The correlation functions of all the local quantum fields are recovered from the partition function when tubes or *channels* in the surface are constricted down to nodes. The crossing symmetry of correlation functions follows from modular invariance of the partition function. In particular, when enough nodes are formed to make correlation functions on the two sphere, modular invariance implies SL_2 invariance, so that the reconstructed two-dimensional quantum field theory is conformally invariant. Real analyticity of the correlation functions, which is two-dimensional locality of the quantum field theory, follows from real analyticity of the partition function on moduli space.

The partition function must satisfy a fundamental factorization condition to permit a consistent reconstruction of correlation functions from the redundant data provided by the partition function. The factorization condition is just that the partition function of a surface with nodes is the product of the partition functions of the disconnected surfaces which remain when the node is removed.

We are thus motivated to define the *universal* moduli space of Riemann surfaces. The universal moduli space contains all compact, not necessarily connected, Riemann surfaces with nodes. It is made into a connected analytic space by allowing analytic deformations in which nodes form and are removed.

Our approach is to make an abstract mathematical formulation of conformal field theory in terms of analytic geometry on moduli space. We introduce a holomorphic vector bundle on moduli space and a projectively flat hermitian connection. The partition function is, essentially, the norm squared, in the hermitian metric, of a holomorphic section of the vector bundle. This definition ensures modular invariance of the partition function. The projective flatness combines with the analyticity of the section to produce the required analyticity of the partition function. We call this abstract version of a conformal field theory a *gauge system*.

This process of mathematical abstraction has several goals. We are motivated partly by the need for an abstract description of the quantum states of string; these concerns are described in [29]. In conformal field theory proper, we see the analytic bootstrap program as providing a mathematical language in which it might be possible to explicitly classify all possible two-dimensional conformal field theories, i.e., all possible two-dimensional critical phenomena, and in which it might be possible to effectively calculate their properties.

Aspects of the general philosophy underlying this work were previously discussed in the context of string theory [30]. The immediate precursor of the present work was the discovery by Cardy [15] that modular invariance is a powerful constraint on the genus 1 partition function of conformal field theories in the $c<1$ discrete series [8]. Modular invariance was already known as a crucial constraint in string theory [31–33]. While engaged in the present work, we were encouraged by the paper of Belavin and Knizhnik [34] on the partition of strings in flat spacetime, which also focuses on the complex analytic structure of moduli space, and by other recent studies of determinants of elliptic operators on surfaces and their factorization properties, on the moduli spaces of Riemann surfaces [34]. A number of ideas closely connected to aspects of this work have also been discussed by Martinec [35].

The organization of the paper is as follows. In sect. 2 we summarize some basic facts about the moduli space of Riemann surfaces. Sect. 3 describes the partition function of a conformal field theory in terms of a certain projective holomorphic line bundle on moduli space. In sect. 4 we describe the factorization condition which must be satisfied by the partition function. Sect. 5 discusses the partition function in genus 1, in particular the Ising model partition function, which gives an explicit model for the general construction. Sect. 6 describes the fundamental objects of the *gauge system*. In sect. 7 we formulate the factorization condition in the gauge system, which ensures that every gauge system is equivalent to a conformal field theory. In sect. 8 we define the universal moduli space of Riemann surfaces, and interpret the factorization condition as the condition that the gauge system be defined on the universal moduli space. Sect. 9 contains concluding remarks and some discussion of directions for future investigation.

2. Moduli space

We start by reviewing some basic facts about moduli space [36, 37]. The moduli space $\mathcal{M}_g$ is the space of conformal equivalence classes of compact, connected, smooth Riemann surfaces without boundary, of genus g. The genus, or number of handles, classifies the surface topologically. $\mathcal{M}_g$ is an analytic V-*manifold* or *orbifold*. This means that $\mathcal{M}_g$ is a smooth analytic manifold except for nongeneric branch points, called orbifold points, where $\mathcal{M}_g$ looks locally like a complex vector space modulo a finite group action. M_0 is the moduli space of 2-spheres, which is just a single point, since there is a unique complex structure on the sphere. $\mathcal{M}_1$, the moduli space of tori, is a branched covering of the complex plane. For $g > 1$, the complex dimension of $\mathcal{M}_g$ is $3g - 3$.

The universal analytic covering space of $\mathcal{M}_g$ is the Teichmüller space $\mathcal{T}_g$. $\mathcal{T}_g$ is a topologically trivial complex analytic manifold. $\mathcal{M}_g$ is the quotient of $\mathcal{T}_g$ by the action of a discrete group Γ_g called the mapping class group, or modular group: $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$.

In genus 1, the Teichmüller space $\mathcal{T}_1$ is the upper half plane, $\operatorname{Im}\tau > 0$. The torus parametrized by τ is the quotient of the complex plane by the lattice generated by 1 and τ. A complex coordinate on the torus τ is the coordinate w on the complex plane, modulo the equivalence relation $w \sim w + m\tau + n$, $m, n \in \mathbb{Z}$. The modular group Γ_1 consists of the ordinary modular transformations $\tau \to (a\tau + b)/(c\tau + d)$, for $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$.

The moduli space $\mathcal{M}_g$ can be regarded as the space of riemannian metrics on a surface with g handles, modulo equivalence under diffeomorphisms, or reparametrizations, of the surface, and also under local conformal rescalings, or Weyl transformations. The Teichmüller space $\mathcal{T}_g$ can be regarded as the space of metrics on a surface of genus g with constant curvature and unit volume, modulo diffeomorphisms of the surface which can be deformed to the identity. The mapping class group Γ_g is the group of all diffeomorphisms of the surface, modulo the connected component of the identity. The subgroup of Γ_g which leaves a point $t \in \mathcal{T}_g$ fixed, the little group at t, is the isometry group of the constant curvature metric corresponding to t. The little group is the automorphism group of the Riemann surface in $\mathcal{M}_g$ represented by t. The surfaces with nontrivial automorphism groups are the branch points or orbifold points of $\mathcal{M}_g$.

A Riemann surface m in $\mathcal{M}_g$ corresponds to a conformal class of metrics which can be expressed locally in the form $\mathrm{d}s^2 = \mathrm{e}^f|\mathrm{d}z|^2$, where z is a local complex coordinate on m, $f(\bar{z}, z)$ is an arbitrary, locally defined function. Any other Riemann surface in $\mathcal{M}_g$ can be expressed as a conformal class of the form $\mathrm{d}s^2 = \mathrm{e}^f|\mathrm{d}z + \mu\,\mathrm{d}\bar{z}|^2$ where $\mu(\bar{z}, z)(\mathrm{d}\bar{z}/\mathrm{d}z)$ is a globally defined tensor field on the surface m. Tensor fields of the form $\mu(\bar{z}, z)(\mathrm{d}\bar{z}/\mathrm{d}z)$ are called Beltrami differentials. The infinitesimal analytic variations of m, forming the holomorphic tangent space $\mathrm{T}_m\mathscr{M}_g^{1,0}$, are the Beltrami differentials μ modulo the subspace of Beltrami differentials of the form $\mu = \bar{\partial} v$ for some vector field $v(\bar{z}, z)(\mathrm{d}z)^{-1}$ globally defined on the surface. The Beltrami differentials $\mu = \bar{\partial} v$ represent infinitesimal reparametrizations, and not true variations of the conformal structure. The dual space, the holomorphic cotangent space $\mathrm{T}_m^*\mathscr{M}_g^{1,0}$, is the space of holomorphic quadratic differentials $\nu(z)(\mathrm{d}z)^2$ on the surface m. The pairing between a 1-form $\nu(z)(\mathrm{d}z)^2$ and a tangent vector represented by Beltrami differential $\mu(\bar{z}, z)(\mathrm{d}\bar{z}/\mathrm{d}z)$ is

$$(\nu, \mu) = \frac{i}{2\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\nu(z)\mu(\bar{z}, z) = \frac{1}{\pi}\int \mathrm{d}(\mathrm{Re}\,z)\,\mathrm{d}(\mathrm{Im}\,z)\,\nu(z)\mu(\bar{z}, z). \quad (1)$$

This normalization of the pairing will eliminate factors of π later on. The condition that the pairing be well defined on the equivalence classes $\mu \sim \mu + \bar{\partial} v$ of Beltrami differentials gives, after integration by parts, the condition of analyticity on $\nu(z)$.

We will also need some basic facts about Riemann surfaces with nodes [37]. A node in some tube or channel of a smooth surface is formed by pinching a circumferential circle around the tube down to a point. When we add to the moduli space $\mathscr{M}_g$ the surfaces with nodes, we get $\overline{\mathscr{M}}_g$, the moduli space of *stable* Riemann surfaces of genus g. $\overline{\mathscr{M}}_g$ is a *compact* orbifold.

On a surface with node(s), a neighborhood of each node can be described by two coordinate disks $\{z_i\colon |z_i| < 1\}$, $i = 1,2$. The two disks are attached together at their origins $z_1, z_2 = 0$ to form the node. The opening of the node is parametrized by a single complex coordinate q on moduli space. Remove the sub-disks $|z_i| < |q|^{1/2}$ and attach the resulting pair of annuli at their inner boundaries $|z_i| = |q|^{1/2}$ by identifying z_2 with q/z_1. This coordinate neighborhood on the surface is mapped to a single annulus $|q|^{1/2} < |z| < |q|^{-1/2}$, by

$$z = \begin{cases} q^{1/2}/z_2 & \text{if } |q|^{1/2} < |z| \leq 1 \\ q^{-1/2}z_1 & \text{if } 1 \leq |z| < |q|^{-1/2}. \end{cases} \quad (2)$$

As $q \to 0$, closing the node, z goes to a coordinate on the 2-sphere punctured at the origin and at ∞. A further transformation $w = (2\pi i)^{-1}\ln z$ pictures the opened node as a long cylinder or tube. Writing $q = \mathrm{e}^{2\pi i\tau}$, the length of the tube is $\mathrm{Im}\,\tau$, and the twist in the tube is $\mathrm{Re}\,\tau$. The closed node corresponds to a tube of infinite length. $\mathscr{M}_g$ is compactified to $\overline{\mathscr{M}}_g$ by adding the points $q = 0$ to the neighborhood $|q| > 0$ in $\mathscr{M}_g$.

The surfaces with nodes form a subvariety $\mathscr{D}_g = \overline{\mathscr{M}}_g - \mathscr{M}_g$ in $\overline{\mathscr{M}}_g$ called the compactification divisor. The compactification divisor $\mathscr{D}_g$ decomposes into a union of irreducible components $\mathscr{D}_{g,k}$, $k = 0, 1, \ldots, \lfloor\frac{1}{2}g\rfloor$, which are distinguished by the topological effect of removing a node. $\mathscr{D}_{g,0}$ consists of the surfaces which become, on removal of some node, a connected surface of genus $g-1$ with two punctures. Such a node lies within a handle in the surface. $\mathscr{D}_{g,k}$, $k>0$, consists of the surfaces which become, on removal of some node, two disconnected surfaces, one of genus k and one of genus $g-k$, each with one puncture. Such a node lies in a tube which is the only channel connecting the two components.

The generic surface $m_{\mathscr{D}} \in \mathscr{D}_g$ has exactly one node, and lies in only one of the irreducible components $\mathscr{D}_{g,k}$. The surfaces with multiple nodes lie in the intersections of the $\mathscr{D}_{g,k}$. All of the intersections are transversal. That is, the closing of each node is described by an independent coordinate q. The multiple intersections are fiber bundles over moduli spaces of Riemann surfaces with multiple disconnected components.

Each irreducible component of $\mathscr{D}_g$ is a fiber bundle whose fibers are the locations (x_1, x_2) of the punctures:

$$
\begin{array}{ccc}
\mathscr{D}_{g,0} \rightarrow \overline{\mathscr{M}}_g & \qquad & \mathscr{D}_{g,k} \rightarrow \overline{\mathscr{M}}_g \\
\downarrow & & \downarrow \\
\overline{\mathscr{M}}_{g-1} & & \overline{\mathscr{M}}_k \times \overline{\mathscr{M}}_{g-k}
\end{array}
\tag{3}
$$

For a point $m_{\mathscr{D}} \in \mathscr{D}_{g,0}$ we write $m_{\mathscr{D}} = (m^{(g-1)}, x_1, x_2)$ where $m^{(g-1)}$ is a surface in $\overline{\mathscr{M}}_{g-1}$ and (x_1, x_2) is an unordered pair of points in $m^{(g-1)}$. We write a point $m_{\mathscr{D}} \in \mathscr{D}_{g,k}$, $k>0$, as $m_{\mathscr{D}} = (m_1^{(k)}, m_2^{(g-k)}, x_1, x_2)$ where $m_1^{(k)}$ is a surface in $\overline{M}_k$, $m_2^{(g-k)}$ is in $\overline{M}_{g-k}$ and x_i is a point in m_i. When $g=2k$, we need to divide by the equivalence $(m_1^{(k)}, m_2^{(k)}, x_1, x_2) \sim (m_2^{(k)}, m_1^{(k)}, x_2, x_1)$. The base of the fiber bundle $\mathscr{D}_{g,k}$, $k>0$, can be regarded as the moduli space of Riemann surfaces with two disconnected components, of genus k and $g-k$.

The generic surface in $\mathscr{D}_{g,k}$, $k \neq 1$ has no automorphisms and so is a smooth point of $\overline{\mathscr{M}}_g$. The divisor $\mathscr{D}_{g,1}$ is slightly special, since every surface in $\mathscr{D}_{g,1}$ has a node which pinches off a torus, and every torus has the automorphism $(w + m\tau + n) \rightarrow -(w + m\tau + n)$. Thus $\mathscr{D}_{g,1}$ consists entirely of orbifold points. As a result, the coordinate transversal to $\mathscr{D}_{g,1}$ is q^2 in place of q.

For surfaces with nodes, the automorphism group is not the same as the little group. The automorphism group is always finite. Generically it is the trivial group, or, for $\mathscr{D}_{g,1}$, the group $\mathbb{Z}_2$. But the little group is always infinite, since it contains the Dehn twists $q \rightarrow e^{2\pi i n} q$, $n \in \mathbb{Z}$, around each node.

3. The partition function

The partition function of a two-dimensional quantum field theory on a compact surface is a function $Z[g]$ of the surface metric g. $Z[g]$ is invariant under the group of diffeomorphisms of the surface acting on metrics. When the quantum field theory is conformally invariant, the partition function transforms covariantly under local rescalings of the surface metric [4]:

$$Z\left[e^f g\right] = Z[g]e^{cS_L(f,g)}, \tag{4}$$

where $S_L(f, g)$ is the Liouville action of the function f in the metric g. The Liouville action is defined by integrating the trace anomaly for infinitesimal local rescalings:

$$\left.\frac{\partial}{\partial f}\right|_{f=0} S_L(f, g) = \frac{1}{48\pi}\sqrt{g}\,R, \tag{5}$$

where $\sqrt{g}\,R$ is the scalar curvature density of the metric g. The real number c is both the coefficient of the surface trace anomaly and the central charge of the Virasoro algebra [5]. By eq. (4), the partition function depends in an essential way only on the conformal class of the surface; its dependence on the local scale of the metric is prescribed. To be more precise, the partition function can be regarded as a section $Z(\overline{m}, m)$ of a real line bundle over $\mathscr{M}_g$. $Z(\overline{m}, m)$ becomes a *function* of m only after a specific metric is chosen on the surface m.

The partition function depends implicitly on a choice of renormalization scheme. We assume henceforth that some choice of renormalization scheme has been made. In sect. 6 below, we consider how the dependence on the renormalization scheme is described in the language of analytic geometry.

We now give a description of the partition function in terms of the complex analytic geometry of moduli space. The strategy is to interpret the expectation value of the locally analytic stress-energy tensor $T(z)$ of the conformal field theory [5] as a hermitian connection in a holomorphic line bundle over moduli space. The partition function is then obtained by integrating the stress-energy tensor with respect to the moduli [5]. In a coordinate neighborhood of a surface m, with local complex coordinate z, the expectation value of the analytic stress-energy tensor is a locally defined quadratic differential $T(\overline{m}, m, z)(\mathrm{d}z)^2$. It is calculated by varying the partition function with respect to the surface geometry:

$$\frac{i}{2\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,T(\overline{m}, m, z)\mu(\bar{z}, z) = -Z^{-1}(\delta_\mu Z). \tag{6}$$

$\delta_\mu Z$ is the variation of Z under the infinitesimal variation of the surface metric $|\mathrm{d}z|^2 \to |\mathrm{d}z + \mu\,\mathrm{d}\bar{z}|^2$ for μ a Beltrami differential compactly supported near z. The variation is taken around *any* metric in the conformal class of m which is precisely equal to $|\mathrm{d}z|^2$ in the neighborhood of z. By the diffeomorphism invariance of $Z[g]$, $\delta_\mu Z = \delta_{\mu+\bar{\partial}v} Z$, so $T(z)$ is locally analytic in z. By the locality of the trace anomaly, eq. (5), $T(z)(\mathrm{d}z)^2$ is independent of the choice of metric away from z, and thus is well defined on moduli space.

$T(\overline{m}, m, z)(\mathrm{d}z)^2$ is *not* a globally defined quadratic differential on the surface m. Under a change of coordinate $z = z(w)$, the stress-energy tensor transforms anomalously [6]:

$$T(w)(\mathrm{d}w)^2 = T(z)(\mathrm{d}z)^2 + \tfrac{1}{12}cS(z:w)(\mathrm{d}w)^2,$$

$$S(z:w) = z'''/z' - \tfrac{3}{2}(z''/z')^2, \tag{7}$$

where $S(z:w)$ is the schwarzian derivative.

Define a *c-connection* [38] $A(\overline{m}, m)$ on $\overline{\mathcal{M}}_g$ to be a collection of quadratic differentials $A(\overline{m}, m, z)(\mathrm{d}z)^2$ defined on coordinate neighborhoods in the surface m, which transform under changes of coordinate exactly like $T(z)$ in eq. (7). We will show that the c-connections define a holomorphic *projective line bundle* E_c on $\overline{\mathcal{M}}_g$, such that the c-connections on $\overline{\mathcal{M}}_g$ are exactly the connections in E_c. The interpretation of c-connections as connections in a line bundle is consistent with the fact that the difference of two c-connections is a globally defined quadratic differential on the surface, and thus a 1-form on moduli space.

A projective line bundle E_c, like an ordinary line bundle, is defined by holomorphic transition functions $g_{\alpha\beta}(m)$ on intersections $\mathrm{U}_\alpha \cap \mathrm{U}_\beta$ of neighborhoods in a covering of $\overline{\mathcal{M}}_g$. A local holomorphic section $e(m)$ of E_c is defined by holomorphic functions $f_\alpha(m)$ which satisfy $f_\alpha = g_{\alpha\beta} f_\beta$ on intersections $\mathrm{U}_\alpha \cap \mathrm{U}_\beta$. Equivalently, the transition functions can be regarded as the ratios $g_{\alpha\beta}(m) = e_\alpha(m)^{-1} e_\beta(m)$, where e_α and e_β are local sections of E_c over U_α and U_β respectively.

The transition functions of a projective line bundle satisfy weaker compatibility conditions than those of an ordinary line bundle. On each triple intersection, $\mathrm{U}_\alpha \cap \mathrm{U}_\beta \cap \mathrm{U}_\gamma$, the compatibility condition is

$$g_{\alpha\gamma}(m) = g_{\alpha\beta}(m) g_{\beta\gamma}(m) \sigma_{\alpha\beta\gamma} \tag{8}$$

for some nonzero constants $\sigma_{\alpha\beta\gamma}$. Clearly, if the constant $\sigma_{\alpha\beta\gamma} \neq 1$ then there can be no local sections over the triple intersection. The natural extension of the notion of section is that of *projective* section, which is only defined up to multiplication by a nonzero complex constant.

The constants $\sigma_{\alpha\beta\gamma}$ determine a cohomology class $[\sigma] \in \mathrm{H}^2(\overline{\mathcal{M}}_g, \mathbb{C}^*)$, where $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers. The cohomology class $[\sigma]$ is the obstruction to representing the projective line bundle as an ordinary holomorphic line bundle. If we define $f_{\alpha\beta} = (1/2\pi i)\ln g_{\alpha\beta}$, choosing an arbitrary branch of the logarithm, then $\omega_{\alpha\beta\gamma} = f_{\alpha\gamma} - f_{\alpha\beta} - f_{\beta\gamma}$ is a 2-cocycle in $\overline{\mathcal{M}}_g$ with complex coefficients representing the Chern class $c_1(\mathrm{E}_c) = [\omega] \in \mathrm{H}^2(\overline{\mathcal{M}}_g, \mathrm{C})$. The obstruction class is $[\sigma] = \mathrm{e}^{2\pi i[\omega]}$, so E_c can be represented as a line bundle if and only if $c_1(\mathrm{E}_c) \in \mathrm{H}^2(\overline{\mathcal{M}}_g, \mathbb{Z})$. As a projective holomorphic line bundle on $\overline{\mathcal{M}}_g$, E_c is characterized by its Chern class.

In order to construct a projective line bundle E_c from the c-connections, we only need to describe the curvature map which takes a locally defined c-connection $A(\bar{m}, m)$ on $\bar{\mathcal{M}}_g$ to its curvature 2-form $F_A(\bar{m}, m)$ on $\mathcal{M}_g$. The properties which the curvature map must have are: (i) it should depend only on the first derivative of A with respect to m, and (ii) it should satisfy $F_{A+\nu} = F_A + \mathrm{d}\nu$, for any locally defined 1-form $\nu(\bar{m}, m)$.

Before describing the curvature map, we explain how it is used to construct E_c. Choose, on each neighborhood U_α on $\bar{\mathcal{M}}_g$, a holomorphic c-connection $A_\alpha(m)$ satisfying $F_A = 0$. Associate with $A_\alpha(m)$, by definition, a nowhere zero local holomorphic section $e_\alpha(m)$ of E_c over U_α, such that $A_\alpha(m) = -e_\alpha(m)^{-1}\,\partial e_\alpha(m)$. The transition functions of E_c are defined by $g_{\alpha\beta} = e_\alpha^{-1} e_\beta$:

$$\partial f_{\alpha\beta} = \frac{1}{2\pi i}\left(A_\alpha - A_\beta\right), \qquad g_{\alpha\beta} = \mathrm{e}^{2\pi i f_{\alpha\beta}}, \tag{9}$$

making an arbitrary choice of integration constant for each $f_{\alpha\beta}$. Clearly the $g_{\alpha\beta}$ given by eq. (9) satisfy the compatibility conditions (8), and so define a projective line bundle E_c.

The curvature F_A of a c-connection A is calculated by differentiating A with respect to $\bar{m}$ and m, and then antisymmetrizing. The $(1,1)$-component of the curvature tensor, $F_A^{1,1} = \partial A/\partial\bar{m}$, is a well-defined $(1,1)$-form on $\bar{\mathcal{M}}_g$ because $\partial S(z:w)/\partial\bar{m} = 0$. The calculation of $F_A^{2,0}$, and by analogy $F_A^{0,2}$, is more complicated. We start by using the techniques of ref. [5] to represent $\partial A/\partial m$ as a globally defined meromorphic kernel on the surface. Let μ be a Beltrami differential representing the infinitesimal variation $m_\mu = m + \delta m$. Let $z_\mu = z + \delta z(\bar{z}, z)$ be a corresponding variation of a complex coordinate z on the surface, i.e. $(\partial/\partial\bar{z})\delta z = \mu(\bar{z}, z)$. For simplicity we discuss the curvature of a c-connection $A(m, z)(\mathrm{d}z)^2$ which is analytic in m. Write the c-connection A, defined near m, in the form $A(m_\mu, z_\mu)(\mathrm{d}z_\mu)^2$, and the variation of A as

$$A(m_\mu, z_\mu) = A(m, z) + \delta m\frac{\partial A}{\partial m}(m, z) + \delta z\frac{\partial A}{\partial z}(m, z). \tag{10}$$

Define the operator K from Beltrami differentials μ to quadratic differentials $K_\mu(\bar{z}, z)(\mathrm{d}z)^2$ by

$$K\mu(\bar{z}, z) = \delta m\frac{\partial A}{\partial m}(m, z) + \delta z\frac{\partial A}{\partial z}(m, z) + 2A(m, z)\frac{\partial}{\partial z}\delta z + \tfrac{1}{12}c\frac{\partial^3}{\partial z^3}\delta z. \tag{11}$$

It is a straightforward calculation, using the transformation law (7), to show that $K\mu$ is a globally defined quadratic differential on the surface m. The key step is to find that $\partial S(z:w)/\partial m = f(z) - f(w)$ for a local function $f(z)$ on the surface. It is also straightforward to calculate

$$\bar{\partial}(K\mu) = (\partial A)\mu + 2A\,\partial\mu + \tfrac{1}{12}c\,\partial^3\mu\,. \tag{12}$$

The fact that $\bar{\partial}(K\mu)$ is local in μ implies that K can be represented by a globally defined meromorphic kernel $K(z,w)(\mathrm{d}z)^2(\mathrm{d}w)^2$ on the surface, with singularities only on the diagonal $z=w$. This kernel is *not* well defined as a bilinear form on the tangent space $\mathrm{T}_m\overline{\mathcal{M}}^{1,0}$ because of its poles. The singularities are determined by eq. (12):

$$K(z,w) \sim \tfrac{1}{2}c(z-w)^{-4} + (z-w)^{-2}(A(z)+A(w))\,. \tag{13}$$

Because of the $z \leftrightarrow w$ symmetry in the singular part of $K(z,w)$, the curvature form defined by

$$F_A^{2,0}(m,z,w)(\mathrm{d}z)^2(\mathrm{d}w)^2 = K(z,w)(\mathrm{d}z)^2(\mathrm{d}w)^2 - (z \leftrightarrow w) \tag{14}$$

is a well-defined 2-form on moduli space at m. It is straightforward to show that $F_{A+\nu} = F_A + \mathrm{d}\nu$, for any locally defined 1-form $\nu(\bar{m},m)$. Thus $A \mapsto F_A$ is a curvature map, and E_c is a well-defined projective line bundle.

Now return to conformal field theory. The expectation value of the stress-energy tensor, $T(\bar{m},m,z)(\mathrm{d}z)^2$, is a c-connection and thus a connection in E_c. We stress that $T(\bar{m},m)$ is smooth on all of $\overline{\mathcal{M}}_g$, including the compactification divisor. To see that $T(\bar{m},m)$ is regular at $\mathcal{D}_g$, use the coordinate q for the closing of a node and the coordinate z given in eq. (2) for a neighborhood of the almost closed node. Recall that z goes to a coordinate on the punctured plane in the limit $q \to 0$. A quantum field on the z-neighborhood, in particular the stress-energy tensor $T(z)$, goes to its value at the node, because at $q=0$ the whole punctured plane becomes identified with the node. On the other hand, as we will see in detail in the next section, the expectation value of a field in the z-neighborhood goes to its ground state expectation on the plane. The ground state expectation value of the stress-energy is zero by $\mathrm{SL}_2(\mathbb{C})$ invariance on the two sphere. Therefore $\lim_{q\to 0}T(z)=0$. This is exactly the condition for regularity of a c-connection at the compactification divisor.

The physical connection $T(\bar{m},m)$ is neither holomorphic in m, nor flat. It is hermitian, i.e. its curvature form F_T is a $(1,1)$-form on $\mathcal{M}_g$. The curvature form F_T is the connected two-point correlation function of the stress-energy tensor. We temporarily abuse notation by writing $T(z)$ for the quantum field rather than its expectation value. The two-point function $\langle T(z)T(w)\rangle$ is symmetric in z and w, so all the curvature of T is given by the $(1,1)$-form on $\overline{\mathcal{M}}_g$

$$F_T(\bar{z},w) = -\langle \bar{T}(\bar{z})T(w)\rangle + \langle \bar{T}(\bar{z})\rangle\langle T(w)\rangle\,. \tag{15}$$

The existence of a hermitian connection in E_c implies that the transition functions $f_{\alpha\beta}(m)$ defined in eq. (9) can be modified so that the cocycle $\omega_{\alpha\beta\gamma}$ becomes real, and the cocycle $\sigma_{\alpha\beta\gamma}$ takes values in U(1). E_c is then a projective *unitary* line bundle over $\overline{\mathcal{M}}$. The Chern class is real, $c_1(\mathrm{E}_c) \in \mathrm{H}^2(\overline{\mathcal{M}}_g, \mathbb{R})$; and the obstruction class is unitary, $e^{2\pi i c_1(\mathrm{E}_c)} \in \mathrm{H}^2(\overline{\mathcal{M}}_g, \mathrm{U}(1))$. The Chern class can be calculated directly from the definition of E_c, but we will instead develop the requisite information in the context of conformal field theory, in the next two sections. The result will be $c_1(\mathrm{E}_c) = \frac{1}{2}c_1(\lambda_{\mathrm{H}})$, where (λ_{H}) is the Hodge line bundle on $\overline{\mathcal{M}}_g$ [40, 37]. The Hodge line bundle is just the determinant line bundle of the $\bar{\partial}$ operator on functions. More explicitly, it is the determinant bundle of the rank-g vector bundle over moduli space whose fiber at m is the g-dimensional vector space of holomorphic 1-forms on the surface m. It makes sense to take arbitrary powers of projective line bundles, so we can write $\mathrm{E}_c = (\lambda_{\mathrm{H}})^{c/2}$. On page 51 of ref. [40] it is shown that $2c_1(\lambda_{\mathrm{H}})$ is a generator of $\mathrm{H}^2(\overline{\mathcal{M}}_g, \mathbb{Z})$. Therefore E_c can be represented as a line bundle if and only if $\frac{1}{4}c \in \mathbb{Z}$.

The Chern class $c_1(\mathrm{E}_c)$ is related by Poincaré duality to a divisor class in $\mathrm{H}_{6g-8}(\overline{\mathcal{M}}_g, \mathbb{R})$. By the result of ref. [37] on $c_1(\lambda_{\mathrm{H}})$,

$$c_1(\mathrm{E}_c) = \tfrac{1}{24}c\left(\left[\omega_{\mathrm{wp}}/2\pi^2\right] + \delta\right),$$

$$\delta = \mathscr{D}_{g,0} + \tfrac{1}{2}\mathscr{D}_{g,1} + \mathscr{D}_{g,2} + \cdots + \mathscr{D}_{g,\lfloor g/2 \rfloor}, \tag{16}$$

where $[\omega_{\mathrm{wp}}]$ is Poincaré dual to the cohomology class of the Weil-Petersson Kähler form. The coefficient $\frac{1}{2}$ for $\mathscr{D}_{g,1}$ is due to the generic $\mathbb{Z}_2$ automorphism group of surfaces in $\mathscr{D}_{g,1}$, compensating for the fact that q^2 is the traversal coordinate for $\mathscr{D}_{g,1}$. Eq. (16) holds for genus $g > 1$. For genus 1, there is only one independent cohomology class, and $c_1(\mathrm{E}_c) = \frac{1}{24}c\mathscr{D}_{1,0}$, where $\mathscr{D}_{1,0}$ is the single point $q = 0$ in $\overline{\mathcal{M}}_1$.

The partition function of the conformal field theory is recovered by integrating the stress-energy tensor [5]. That is, define $Z(\overline{m}, m)$ to be the hermitian section of $\overline{\mathrm{E}}_c \otimes \mathrm{E}_c$ compatible with the connection T:

$$T = -Z^{-1}\partial Z, \qquad \overline{T} = -Z^{-1}\bar{\partial} Z. \tag{17}$$

Equivalently, $Z(\overline{m}, m)$ can be regarded as a hermitian metric in E_{-c} with connection $-T$. Eq. (17) determines $Z(\overline{m}, m)$ up to a multiplicative constant on each moduli space $\overline{\mathcal{M}}_g$. In the next section these constants are determined by the factorization condition. We stress that the regularity of T implies that the partition function $Z(\overline{m}, m)$ is a regular section of $\overline{\mathrm{E}}_c \otimes \mathrm{E}_c$ on all of $\overline{\mathcal{M}}_g$, including the compactification divisor.

The partition function, as a section of $\overline{\mathrm{E}}_c \otimes \mathrm{E}_c$, is converted back into a function $Z[g]$ of surface metrics, by associating to each surface metric g a hermitian metric h^g in E_c. Then $Z[g] = h^g Z(\overline{m}, m)$, where m is the conformal class of the metric g. The metric h^g is simplest to construct for the constant curvature metrics g_{cc}. In a coordinate neighborhood on the surface, the constant curvature metric is of the form $\mathrm{d}s^2 = \mathrm{e}^f|\mathrm{d}z|^2$ with $\partial^2\bar{\partial}f = 0$. Define a $(-c)$-connection by $A^{\mathrm{cc}}(z) = \frac{1}{12}c(\partial_z^2 f - \frac{1}{2}(\partial_z f)^2)$. $A^{\mathrm{cc}}(z)$ is locally analytic in z because the curvature is constant. Integrating A^{cc} gives a hermitian metric $h^{g_{\mathrm{cc}}}$ in E_c. The gradient of $Z[g_{\mathrm{cc}}]$ as a function on Teichmüller space is the 1-form $A^{\mathrm{cc}} + T$, so the constant curvature partition function is $Z[g_{\mathrm{cc}}] = h^{g_{\mathrm{cc}}} Z(\bar{m}, m)$. The general metric g can be written in the form $g = \mathrm{e}^f g_{\mathrm{cc}}$ for some function f and a unique constant curvature metric g_{cc}. If we associate with g the hermitian metric in E_c

$$h^g = \exp(cS_{\mathrm{L}}(f, g_{\mathrm{cc}}))h^{g_{\mathrm{cc}}}, \tag{18}$$

then, by eq. (4), the partition function, as a function of the metric, is given by

$$Z[g] = h^g Z(\bar{m}, m). \tag{19}$$

The projective line bundle E_c thus encodes all of the information about surface metric geometry which is needed to formulate conformal field theory.

As an aside, we remark that in the language of sheaf theory [39] the projective line bundles are described by $\mathrm{H}^1(\mathcal{O}^*/\mathbb{C}^*)$, where $\mathcal{O}^*$ is the sheaf of local-nowhere zero-holomorphic functions on $\overline{\mathcal{M}}_g$. The Chern class and obstruction class are given by long exact sequences associated with the exact diagram of sheaves:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{C} & \overset{\exp}{\rightarrow} & \mathbb{C}^* & \rightarrow & 0 \\
 & & \updownarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathcal{O} & \overset{\exp}{\rightarrow} & \mathcal{O}^* & \rightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}/\mathbb{C} & \leftrightarrow & \mathcal{O}^*/\mathbb{C}^* & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{20}$$

$\mathcal{O}$ is the sheaf of local holomorphic functions on $\overline{\mathcal{M}}_g$ and $\mathbb{Z}$, $\mathbb{C}$, and $\mathbb{C}^*$ are the constant sheaves. The long exact sequences associated with this diagram give an explicit description of the possible projective line bundles. A simple exercise in diagram chasing shows that when a projective line bundle can be represented as an ordinary line bundle, the representation is unique only up to tensoring by line bundles with vanishing first Chern class.

4. Factorization on the compactification divisor

At each point $m_{\mathscr{D}}$ in the compactification divisor $\mathscr{D}_g$, and for each node in the surface $m_{\mathscr{D}}$, the partition function has an expansion of the form

$$Z(\overline{m}, m) = \sum_{\varphi} q^{h_\varphi} \bar{q}^{\bar{h}_\varphi} Z_\varphi(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}}), \tag{21}$$

where q is a transversal coordinate parametrizing the closing of the node, and $m = (m_{\mathscr{D}}, q)$. This expansion in q is derived by parametrizing a neighborhood of the node as the annulus $|q|^{1/2} < |z| < |q|^{-1/2}$, as in eq. (2). The choice of this particular coordinate system singles out a class of c-connections A having $A(z) = 0$, which provides a local basis for E_c in which to carry out the expansion. Equivalently, we can calculate in the conformal field theory on the annular neighborhood, using the local metric $\mathrm{d}s^2 = |\mathrm{d}z|^2$.

The conformal field theory on the part of the surface outside the annular neighborhood provides particular external states on the boundary circles $|z| = |q|^{1/2}$ and $|z| = |q|^{-1/2}$. The partition function is the expectation value of $q^{L_0}\bar{q}^{\bar{L}_0}$ between these states, where L_0 and $\bar{L}_0$ are the Virasoro operators generating dilation in the complex plane. If the node separates the surface into two components, then the states on the two boundary circles are independent. If the node lies in a handle, then the states are coupled, forming a density matrix. In general,

$$Z(\overline{m}, m) = \operatorname{tr} q^{L_0} \bar{q}^{\bar{L}_0} \rho(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}}), \tag{22}$$

where the density matrix $\rho(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}})$ is independent of q, since the surface outside the annular neighborhood is held fixed while q varies.

The expansion (21) is produced by inserting a complete set of states $|\varphi\rangle\langle\varphi|$ in the channel containing the node, where the $|\varphi\rangle$ are the eigenstates of L_0 and $\bar{L}_0$ with eigenvalues h_φ, $\bar{h}_\varphi$:

$$Z(\overline{m}, m) = \sum_{\varphi} q^{h_\varphi} \bar{q}^{\bar{h}_\varphi} \langle\varphi|\rho(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}})|\varphi\rangle . \tag{23}$$

In a strict conformal field theory, the weights h_φ, $\bar{h}_\varphi$ are nonnegative. The partition function is single-valued, so $h_\varphi - \bar{h}_\varphi$ must be an integer. The leading term $Z_0(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}})$ is the contribution of the unique $\mathrm{SL}_2(\mathbb{C})$ invariant ground state $|0\rangle$, with $h_0 = \bar{h}_0 = 0$. Since all other contributions vanish when $q = 0$, the partition function on the compactification divisor, $Z(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}})$, is the ground state contribution $Z_0(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}})$.

The intermediate states $|\varphi\rangle\langle\varphi|$ correspond to the complete set of scaling fields in the conformal field theory. The sum (21) can be rewritten as a sum over the highest weight states $|\phi\rangle\langle\phi|$ of the Virasoro algebra, corresponding to the primary conformal fields $\phi(\bar{z}, z)$:

$$Z(\bar{m}, m)=\sum_{\phi} Z_{\phi}(\bar{m}_{\mathscr{D}}, m_{\mathscr{D}}, \bar{q}, q), \tag{24}$$

where Z_{ϕ} is the contribution to the partition function from the entire Virasoro representation generated by the highest weight state $|\phi\rangle$, corresponding to the field ϕ and all its descendents. As $q \to 0$, the leading behavior of Z_{ϕ} is $q^{h_{\phi}}\bar{q}^{\bar{h}_{\phi}}$. Each Z_{ϕ} is individually a section of $\bar{\mathrm{E}}_c \otimes \mathrm{E}_c$, which cannot be further decomposed into sections of $\bar{\mathrm{E}}_c \otimes \mathrm{E}_c$. This is the principle by which primary fields are extracted from the partition function.

Only states which are charge neutral, for all internal symmetries of the theory, contribute to expansions of the partition function in a coordinate q for a node which separates the surface into two components, because there must be charge neutrality on either side of the node. All states contribute to the expansion for a node in a handle, since charge flowing through the node can be neutralized through another channel in the surface.

The coefficients in the expansion eq. (21) are given by correlation functions of local fields on the surface(s) left when the node is removed. For $\mathscr{D}_{g,0}$, the moduli are $m_{\mathscr{D}}=(m^{(g-1)}, x_1, x_2)$, where $m^{(g-1)}$ ranges over the surfaces of genus $g-1$ and (x_1, x_2) is an unordered pair of distinct punctures on the surface $m^{(g-1)}$. Inserting the projection $|\varphi\rangle\langle\varphi|$ at the node is equivalent to inserting the field φ at the punctures on $m^{(g-1)}$, giving the unnormalized two-point function of φ on the surface $m^{(g-1)}$:

$$Z_{\varphi}(\bar{m}_{\mathscr{D}}, m_{\mathscr{D}})=Z(\bar{m}^{(g-1)}, m^{(g-1)})\langle\varphi(\bar{x}_1, x_1)\varphi(\bar{x}_2, x_2)\rangle_{m^{(g-1)}}. \tag{25}$$

The field associated with the vacuum state $|0\rangle$ is the identity operator, whose correlations are independent of the location of the punctures, so the leading term in the q-expansion, which is the partition function on the compactification divisor, is exactly the partition function for genus $g-1$:

$$Z(\bar{m}_{\mathscr{D}}, m_{\mathscr{D}})=Z(\bar{m}^{(g-1)}, m^{(g-1)}). \tag{26}$$

For $\mathscr{D}_{g,k}$, $k>0$, the moduli are $m_{\mathscr{D}}=(m_1^{(k)}, x_1, m_2^{(g-k)}, x_2)$ where $m_1^{(k)}$ is a genus k surface, with puncture at x_1, and $m_2^{(g-k)}$ is a genus $g-k$ surface, with puncture at x_2. The expansion coefficient Z_{φ} is the product of unnormalized one point functions:

$$Z_{\varphi}(\bar{m}_{\mathscr{D}}, m_{\mathscr{D}})=Z(\bar{m}_1^{(k)}, m_1^{(k)})\langle\varphi(\bar{x}_1, x_1)\rangle_{m_1^{(k)}} \times Z(\bar{m}_2^{(g-k)}, m_2^{(g-k)})\langle\varphi(\bar{x}_2, x_2)\rangle_{m_2^{(g-k)}}. \tag{27}$$

The leading term is the ground state contribution

$$Z(m_{\mathscr{D}}, m_{\mathscr{D}}) = Z\left(\overline{m}_1^{(k)}, m_1^{(k)}\right) Z\left(\overline{m}_2^{(g-k)}, m_2^{(g-k)}\right), \tag{28}$$

The product of the genus k and genus $g-k$ partition functions.

Near a surface with multiple nodes, the partition function has a joint expansion in the coordinates q_i parametrizing the opening of the nodes labelled by i. The coefficients of the joint q-expansion are products of correlation functions of the local fields of the conformal field theory. Writing $m = (m_{\mathscr{D}}, q_1, q_2, \dots)$,

$$Z(\overline{m}, m) = \sum_{\{\varphi_i\}} \left(\prod_i q_i^{h_{\varphi_i}} \bar{q}_i^{\bar{h}_{\varphi_i}} \right) Z_{\{\varphi_i\}}(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}}). \tag{29}$$

$Z_{\{\varphi_i\}}(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}})$ is a product of unnormalized correlation functions of the fields φ_i at the punctures on the components of the surface $m_{\mathscr{D}}$ obtained by removing the nodes. Exactly which correlation functions depends on the configuration of nodes. Write $m_{\mathscr{D}} = (m', x_1, y_1, x_2, y_2, \dots)$, where m' is the smooth surface, possibly disconnected, obtained from $m_{\mathscr{D}}$ by removing the nodes, and (x_i, y_i) is the pair of punctures on m' left by the ith node. Then

$$Z_{\{\varphi_i\}}(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}}) = Z(\overline{m}', m') \left\langle \prod_i \varphi_i(\bar{x}_i, x_i) \varphi_i(\bar{y}_i, y_i) \right\rangle_{m'}, \tag{30}$$

where the partition function $Z(\overline{m}', m')$ of a disconnected surface is the product of the partition functions of its connected components.

For surfaces with multiple nodes, factorization on the ground state expresses the partition function of a surface $m_{\mathscr{D}}$ on multiple intersections of the $\mathscr{D}_{g,k}$ as the product of the partition functions of the component surfaces into which $m_{\mathscr{D}}$ decomposes. These are not additional factorization conditions on the partition function, but follow from successive applications of single node ground state factorization, since the transversal coordinates q_i are independent.

The factorization identities (26), (28) allow us to determine the nature of the holomorphic line bundle E_c. First, they imply that E_c is trivial over the fiber (x_1, x_2) parametrizing the punctures. That is, local sections of E_c over $\mathscr{D}_g$ are actually *functions* of the locations of the punctures. By results of Wolpert [37] on the two-cohomology of $\overline{\mathscr{M}}_g$, this implies that $c_1(\mathrm{E}_c)$ is a multiple of $c_1(\lambda_{\mathrm{H}})$, (λ_{H}) being the Hodge line bundle. The argument is based on the explicit generators of $\mathrm{H}_2(\overline{\mathscr{M}}_g, \mathbb{Q})$ (sect. 2 of ref. [37]), the intersection matrix between $\mathrm{H}_2(\overline{\mathscr{M}}_q, \mathbb{Q})$ and $H_{6g-8}(\overline{\mathscr{M}}_g, \mathbb{Q})$ (subsect. 5.1), and the proof of lemma 5.4, in which $c_1(\lambda_{\mathrm{H}})$ is calculated. Ground state factorization of the partition function implies that E_c over $\mathscr{D}_g$ is determined by E_c on lower genus moduli spaces. That is, E_c restricted to $\mathscr{D}_{g,0}$ is E_c on $\overline{\mathscr{M}}_{g-1}$; and, for $k > 0$, E_c restricted to $\mathscr{D}_{g,k}$ is $\mathrm{E}_c \times \mathrm{E}_c$ on $\overline{\mathscr{M}}_k \times \overline{\mathscr{M}}_{g-k}$. By the results of Wolpert, these are exactly the defining properties of (λ_{H}). The factorization of E_c reduces the calculation of the constant of proportionality between $c_1(\mathrm{E}_c)$ and $c_1(\lambda_{\mathrm{H}})$ to the genus 1 case. This calculation is carried out in the next section.

The fact that the partition function along the compactification divisor must factorize on the ground state removes the ambiguity in the construction of the partition function $Z(\overline{m}, m)$ from the stress-energy tensor T by integrating the equation $T = -Z^{-1}\partial Z$. The constant of integration on $\overline{\mathscr{M}}_g$ is given by the value of $Z(\overline{m}, m)$ on $\mathscr{D}_g$, which is determined by $Z(\overline{m}, m)$ on lower genus surfaces through the factorization identities (26), (28). The overall normalization is fixed by setting $Z = 1$ on the 2-sphere. This makes sense because E_c is just a number at the single point which constitutes M_0.

The basic premise of the present work is that the correlation functions of the conformal field theory can be reconstructed from the partition function, via the q-expansions at the compactification divisor. We will not prove this reconstruction theorem, but we sketch the basic strategy of a proof. Suppose $m_{\mathscr{D}}$ is a surface with nodes, which is split by the nodes into components, one of which is m. Write the rest of the components as m'. Write x_i for the punctures on m and y_i for the punctures on m'. Let q_i be transversal coordinates for the nodes separating m from m'. We want to use the q_i expansion at $m_{\mathscr{D}}$ to determine correlation functions on m. Let $Z_{\bar{h},h}(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}})$ be the coefficient of $\prod_i q_i^{h_i}\bar{q}_i^{\bar{h}_i}$ in the q_i expansion of the partition function. Expand the normalized coefficient $Z_{\bar{h},h}/Z$ in real analytic functions in the form

$$Z_{\bar{h},h}/Z(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}}) = \sum_k F_k(\overline{m}, m, \bar{x}_i, x_i) G_k(\overline{m}', m', \bar{y}_i, y_i). \tag{31}$$

The functions F_i are to be interpreted as correlation functions of scaling fields $\varphi_i(x_i)$ of weight $h_i, \bar{h}_i$. The sum over k accounts for possible multiplicities of fields of the same weights.

A number of consistency conditions must be satisfied by the F_k in order to justify the correlation function interpretation. First, the same weights and multiplicities should appear in each joint q expansion, so that the F_k can be interpreted as correlation functions of one set of fields. Second, the same functions $F_k(\overline{m}, m, \bar{x}_i, x_i)$ should appear, independent of the rest of the surface, (m', y_i). The key to satisfying both conditions is factorization on the ground state. Two topologically distinct surfaces m'_1, m'_2 can be obtained from a single surface m'_3 by forming and removing nodes. If the partition function satisfies the fundamental factorization conditions (26), (28), then, because of the independence of the transversal coordinates q for each node, the F_k obtained using m'_3 are identical to those obtained with m'_1 and m'_2. A similar argument relates the weights and multiplicities associated with q-expansions at different subcomponents of the compactification divisor.

Once the correlation functions are reconstructed, their crossing symmetry follows from modular invariance of the partition function, that is, from the fact that the partition function is single-valued on moduli space. In particular, SL_2 invariance of correlation functions on the two-sphere follows from modular invariance, so that the field theory reconstructed from the correlation functions is conformally invariant.

We thus arrive at the necessary and sufficient conditions on the partition function which permit reconstruction of a conformal field theory. They are the real analyticity of the normalized expansion coefficients $Z_{\bar{h},h}/Z(\overline{m}_{\mathscr{D}}, m_{\mathscr{D}})$, along with the factorization identities (26), (28). The equations of factorization on excited states, eqs. (29), (30), do not give additional constraints on the partition function, since they do not close on the partition function itself. Factorization on excited states is automatic in the reconstructed conformal field theory.

The primary field decomposition, eq. (24), is recovered from the partition function by a maximal decomposition of $Z(\overline{m}, m)$ near the compactification divisor into a sum of sections of $\overline{\mathrm{E}}_c \otimes \mathrm{E}_c$. The normalized contributions Z_ϕ/Z are functions of $\overline{\mathscr{M}}_g$ locally defined near $\mathscr{D}_g$. The leading coefficient in the q-expansion of Z_ϕ/Z, the coefficient of $q^{h_\phi}\bar{q}^{\bar{h}_\phi}$, can be shown to be a conformal tensor of weight h_ϕ, $\bar{h}_\phi$ in the puncture parameters (x_1, x_2), using only the structure of E_c. Thus the correlations of the primary fields obtained from the q-expansions of the partition function are independent of the surface geometry, and transform as conformal tensors on the surface.

The correlation functions of the stress-energy tensor are obtained by differentiating the partition function with respect to the moduli. In order to confirm the canonical operator products of the stress-energy tensor, it is useful to represent variations of the moduli as Beltrami differentials supported near an almost closed node. Use the standard coordinate z, $|q|^{1/2} < |z| < |q|^{-1/2}$, on an annular neighborhood of an almost closed node, as describe in eq. (2). As usual, write $m = (m_{\mathscr{D}}, q)$. Each of the $3g-3$ tangent vectors $\partial/\partial m^k$ to $\overline{\mathscr{M}}_g$ at m can be represented as a distributional Beltrami differential μ_k supported on the boundary circles $|z| = |q|^{1/2}$ or $|z| = |q|^{-1/2}$ of the z-annulus. Recall that a 1-form ν in $\mathrm{T}^*_m\overline{\mathscr{M}}{}^{0,1}_g$ is a quadratic differential $\nu(z)(\mathrm{d}z)^2$ on the surface. In terms of the distributional Beltrami differentials μ_k,

$$\sum_k \mathrm{d}m^k \nu_k = \sum_k \mathrm{d}m^k(\nu, \mu_k) = \sum_k \mathrm{d}m^k \frac{1}{2\pi i}\oint \nu(z)(\mathrm{d}z)^2 \mu_k(z)(\mathrm{d}z)^{-1}. \quad (32)$$

Here the distributional Beltrami differentials μ_k are written as holomorphic vector fields $\mu_k(z)(\mathrm{d}z)^{-1}$ on the annulus. In particular, $\partial/\partial q$ corresponds to the conformal vector field $q^{-1}z(\mathrm{d}z)^{-1}$, $\partial/\partial x_1$ to $-q^{-1/2}(\mathrm{d}z)^{-1}$ and $\partial/\partial x_2$ to $q^{-1/2}z^2(\mathrm{d}z)^{-1}$, where (x_1, x_2) are the punctures. The remaining $3g-6$ tangent vectors at m are represented by a complementary subspace of the holomorphic vector fields on the annulus. In the limit $g \to \infty$ the infinitesimal moduli become dense in the space of holomorphic vector fields on the annulus. Thus variations of the moduli can approximate arbitrary local conformal transformations in the neighborhood of a scaling field, at a puncture. The same argument for surfaces with multiple nodes shows that local conformal transformations around any number of punctures can be represented by variations of the moduli, in the limit $g \to \infty$. The stress-energy tensor $T(z)(\mathrm{d}z)^2$ near the scaling fields is reconstructed by using the $\mu_k(z)(\mathrm{d}z)^{-1}$ to represent the variations of the moduli. The canonical operator products of the stress-energy tensor with the scaling fields [5] follow from the representation of the variations of x_1, q, and x_2 by the conformal vector fields 1, z, and z^2.

5. Genus 1

The genus 1 partition function is usually written as a function $Z_\beta(\bar{\tau}, \tau)$ on the Teichmüller space $\mathscr{T}_1$, invariant under the modular group Γ_1. Writing $q = \mathrm{e}^{2\pi i\tau}$,

$$Z_\beta(\bar{\tau}, \tau) = \sum_{a, b} \bar{q}^{h_a} q^{h_b} \overline{\chi_{h_a}(q)} h_{\bar{a}b} \chi_{h_b}(q), \tag{33}$$

where the analytic function $\chi_h(q) = \mathrm{tr}(q^{\varepsilon_0 + L_0})$ is the character of the irreducible Virasoro representation of weight h; $\varepsilon_0 = -\frac{1}{24}c$ is the universal ground state energy [41]; and $h_{\bar{a}b}$ is the integer multiplicity of the weights $(h_{\bar{a}}, h_b)$ in the Hilbert space of the conformal field theory. The universal ground state energy $\varepsilon_0 = -\frac{1}{24}c$ is derived from eq. (7). The schwarzian derivative of the mapping $z = \mathrm{e}^{2\pi i w}$ from the coordinate $w \sim w + m\tau + n$ on the torus to the coordinate $z \sim qz$ is $S(z : w)(\mathrm{d}w)^2 = -\frac{1}{2}z^{-2}(\mathrm{d}z -)^2$. The stress-energy tensor on the torus is then given by $T(w)(\mathrm{d}w)^2 = (T(z) + \varepsilon_0)(\mathrm{d}z)^2$ where

$$T(z) = z^{-2}\langle L_0 \rangle = z^{-2} q \frac{\partial}{\partial q} \ln \mathrm{tr}\left(q^{L_0} \bar{q}^{\bar{L}_0}\right). \tag{34}$$

Cardy [15] investigated the modular invariance of expression (33) for $c < 1$ in the unitary discrete series [8]. Using formulas for the characters $\chi_h(q)$ allowed by unitary, which were calculated by Rocha-Caridi [42] from results of Feigin-Fuchs [43], Cardy found that the requirement of modular invariance on eq. (33) puts strong constraints on the integer multiplicities $h_{\bar{a}b}$.

The function $Z_\beta(\bar{\tau}, \tau)$ on $\mathscr{M}_1$ is related to the abstract partition function $Z(\overline{m}, m)$ by $Z = Z_\beta |e_\beta|^2$, where e_β is a nowhere zero section of E_c over $\mathscr{M}_1$, which does not extend to $\overline{\mathscr{M}}_1$. The section e_β is defined, as in sect. 3, in terms of a flat c-connection A_β on $\mathscr{M}_1$: $A_\beta = -e_\beta^{-1}\, \partial e_\beta$. A_β is given by $A_\beta(w) = 0$ in the coordinate system $w \sim w + m\tau + n$ on the torus. A_β is invariant under the modular group Γ_1, so it does define a flat c-connection on $\mathscr{M}_1$. But A_β does not extend to $\overline{\mathscr{M}}_1$ since, in the limit $q \to 0$ or $\tau \to i\infty$, w does not become a coordinate for the two-sphere.

A section of E_c in a neighborhood of $q=0$ is constructed from another c-connection A_α, given by $A_\alpha(z)=0$ in the coordinate system $z=e^{2\pi i w}$. A_α *is* regular near $q=0$ because z goes to a good coordinate on the two-sphere at $q=0$. The section e_α defined by $A_\alpha=-e_\alpha^{-1}\partial e_\alpha$ is a local section of E_c near $q=0$. Applying eq. (7) to A_α, we get the overlap 1-form $A_\beta - A_\alpha = \frac{1}{24}c(\mathrm{d}z/z)^2$. The quadratic differential $-(\mathrm{d}z/z)^2$ on the torus corresponds to the 1-form $\mathrm{d}q/q$ on $\overline{\mathscr{M}}_1$. Therefore $A_\beta - A_\alpha = \varepsilon_0 \mathrm{d}q/q$, and the transition function for E_c is $e_\alpha = q^{-c/24}e_\beta$.

We now can write the abstract partition function $Z(\overline{m},m)$ near $q=0$ by changing basis in E_c:

$$Z(\overline{m},m)=Z_\alpha|e_\alpha|^2\,, \tag{35}$$

where $Z_\alpha(\bar{q},q)=\bar{q}^{c/24}q^{c/24}Z_\beta$ is regular at $q=0$, in accord with the general result that $Z(\overline{m},m)$ is regular at the compactification divisor.

It is now an easy exercise to complete the identification $E_c=(\lambda_{\mathrm{H}})^{c/2}$ by calculating $c_1(E_c)$ on $\overline{\mathscr{M}}_1$, given the transition function $q^{c/24}$. But it is more interesting to make the calculation in conformal field theory. $\overline{\mathscr{M}}_1$ is one-dimensional, so it suffices to calculate the integral

$$c_1=\int_{\overline{\mathscr{M}}_1} c_1(E_c)=\frac{i}{2\pi}\int_{\overline{\mathscr{M}}_1} F_A\,, \tag{36}$$

for any smooth connection A in E_c over $\overline{\mathscr{M}}_1$. In particular, for the physical connection T:

$$c_1=\frac{1}{2\pi i}\int_{|q|>\varepsilon}\bar{\partial}\partial \ln Z_\beta = \tfrac{1}{24}c\,. \tag{37}$$

The Hodge line bundle on $\overline{\mathscr{M}}_1$ has $c_1(\lambda_{\mathrm{H}})=\frac{1}{12}\mathscr{D}_{1,0}$, and $c_1=\frac{1}{12}$, so $c_1(E_c)=\frac{1}{2}cc_1(\lambda_{\mathrm{H}})$ and $E_c=(\lambda_{\mathrm{H}})^{c/2}$.

We now translate the character expansion (33) into geometric language, to prepare the way for generalization to higher genus, in the next section. We will do this in the context of a concrete example, using methods which apply to the general case. The simplest example is the Ising model, which has $c=\frac{1}{2}$, $\varepsilon_0=-\frac{1}{48}$. The unitary weights at $c=\frac{1}{2}$ are $h=0,\frac{1}{16},\frac{1}{2}$. The genus-1 partition function is

$$Z_\beta(\bar{\tau},\tau)=|\chi_0(q)|^2+|\chi_{1/16}(q)|^2+|\chi_{1/2}(q)|^2\,. \tag{38}$$

The characters $\chi_h(q)$ are:

$$\chi_0(q)\pm\chi_{1/2}(q)=q^{\varepsilon_0}\prod_{n=1}^{\infty}(1\pm q^{n-1/2})\,,$$

$$\chi_{1/16}(q)=q^{\varepsilon_0+1/16}\prod_{n=1}^{\infty}(1+q^n)\,. \tag{39}$$

In the following, we write $h_0 = 0$, $h_1 = \frac{1}{16}$, $h_2 = \frac{1}{2}$ and $\chi^a = \chi_{h_a}$, $a = 0, 1, 2$.

The three $c = \frac{1}{2}$ characters form a three-dimensional unitary representation of the modular group Γ_1, so $Z_\beta(\bar{\tau}, \tau)$ is indeed invariant under Γ_1. Γ_1 is generated by $T\tau = \tau + 1$ and $S\tau = -1/\tau$. The characters satisfy $\chi^a(\gamma\tau) = \gamma^a_b \chi^b(\tau)$ where $\gamma \to \gamma^a_b$ is the unitary representation of Γ_1 defined by

$$T^a_b = \exp(2\pi i(\varepsilon_0 + h_a))\delta^a_b, \qquad S^a_b = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}. \tag{40}$$

The three characters $\chi^a(q)$ can be thought of as the single function $\chi^0(q)$ analytically continued over all of Teichmüller space.

Such a unitary representation of the modular group Γ_1 in a vector space $\mathscr{H}_1$ is equivalent to a holomorphic, flat, hermitian vector bundle W_1 over $\bar{\mathscr{M}}_1$. To be concrete, let $\{\xi_a\}$ be a basis for $\mathscr{H}_1$, with inner product $h_{\bar{a}b} = \langle \bar{\xi}_{\bar{a}}, \xi_b \rangle$. Define the representation matrices γ^a_b by $\gamma\xi_b = \gamma^a_b \xi_a$. The matrices γ^a_b are unitary with respect to the inner product $h_{\bar{a}b}$. The flat vector bundle W_1 is $\mathscr{T}_1 \times \mathscr{H}_1$ divided by Γ_1 acting on both spaces simultaneously, $\mathrm{W}_1 = \mathscr{T}_1 \times_{\Gamma_1} \mathscr{H}_1$. Equivalently, a local holomorphic section w of W_1 is a local holomorphic function from Teichmüller space $\mathscr{T}_1$ to the representation vector space $\mathscr{H}_1$, $\tau \mapsto w(\tau)$, satisfying $w(\gamma\tau) = \gamma w(\tau)$ for all $\gamma \in \Gamma_1$. Over the smooth points in $\mathscr{M}_1$, the fibers of W_1 are isomorphic to $\mathscr{H}_1$. But over an orbifold point the fibers degenerate to the subspace in $\mathscr{H}_1$ of invariant vectors of the little group at the orbifold point. A vector bundle over an orbifold or V-manifold which degenerates in this manner at the orbifold points is called a V-*bundle* [44].

The flat holomorphic connection D in W_1 is defined by describing a basis of locally flat holomorphic sections $w_a(m)$ near any smooth point $m_0 \in \mathscr{M}_1$. Pick a representative τ_0 of m_0, and let $\tau(m)$ be any local analytic lifting map from $\mathscr{M}_1$ to $\mathscr{T}_1$ such that $\tau(m_0) = \tau_0$. The local lifting map $\tau(m)$ is uniquely determined by τ_0. The locally flat holomorphic sections w_a are the vector valued functions $w_a(\gamma\tau(m)) = \gamma\xi_a$. The hermitian metric h in W_1 is defined by $h(\bar{w}_{\bar{a}}, w_b) = h_{\bar{a}b}$, where $h_{\bar{a}b}$ is the inner product in $\mathscr{H}_1$. Clearly the hermitian metric is compatible with the flat connection.

The representation $\mathscr{H}_1$ can be recovered from the flat hermitian bundle W_1, because parallel transport in W_1 using the flat connection determines a representation of Γ_1 in the fibers of W_1 over smooth points in $\mathscr{M}_1$. This representation is given explicitly, in terms of a local basis of covariant constant sections, by $\gamma w_b = \gamma^a_b w_a$. The hermitian metric is invariant under parallel transport, so this representation is unitary. At a branch point, only the normalizer of the little group acts on the fiber. The normalizer of the little group is the subgroup of modular transformations γ which satisfy the condition that $\gamma\gamma_0\gamma^{-1}$ is in the little group whenever γ_0 is. The normalizer preserves the subspace of vectors invariant under the little group.

The characters $\chi^a(\tau)$ define a single holomorphic section χ of W_1 over $\mathscr{M}_1$ by the function χ: $\tau \mapsto \chi^a(\tau)\xi_a$ from $\mathscr{T}_1$ to $\mathscr{H}_1$. The vector valued function $\chi(\tau)$ is a section of W_1 because

$$\chi(\gamma\tau) = \chi^a(\gamma\tau) = \gamma^a_b\chi^b(\tau)\xi_a = \gamma\chi(\tau). \tag{41}$$

Note that the section χ is *not* covariant constant, i.e. not invariant under the action of Γ_1 by parallel transport, $\gamma\chi \neq \chi$. Using the metric $h_{\bar{a}b}$ in W_1 and the holomorphic section $\chi(m)$, we can write the partition function $Z_\beta = h(\bar{\chi}, \chi)$ on $\mathscr{M}_1$. But this is not yet what we want. We are looking for a geometric formulation of the *abstract* partition function $Z(\bar{m}, m)$, defined over all of $\overline{\mathscr{M}}_1$.

Let us first examine the extension of W_1 to $\overline{\mathscr{M}}_1$. The little group at $q = 0$ is the group of twists T^n, $n \in \mathbb{Z}$. Thus W_1 degenerates at $q = 0$. For the Ising model, the fiber of W_1 at $q = 0$ contains only the zero vector, since none of the eigenvalues of the matrix T^a_b are equal to 1. On the other hand, $q = 0$ is a smooth point in $\overline{\mathscr{M}}_1$, so W_1 over $\overline{\mathscr{M}}_1$ cannot be a V-bundle in the ordinary sense. We must modify the definition of V-bundle over $\overline{\mathscr{M}}_1$, and over $\overline{\mathscr{M}}_g$ in general, to require that the fibers on the compactification divisor degenerate to the subspace of twist invariant vectors. The strict definition of a V-bundle over an orbifold is based on modelling a neighborhood of each orbifold point as a neighborhood of the origin in a complex vector space, modulo the action of the finite little group. For our purposes, a neighborhood of a point $q = e^{2\pi i\tau} = 0$ on the compactification divisor should be modelled on the half-space $\mathrm{Im}\,\tau > 1/\varepsilon$ in a complex vector space, modulo the action $\tau \to \tau + n$ of the twists.

The flat hermitian metric $h_{\bar{a}b}$ clearly extends to the flat bundle W_1 over $\overline{\mathscr{M}}_1$. But W_1 over $\overline{\mathscr{M}}_1$ is not the appropriate setting for the analytic section $\chi(m)$, since $\chi(m)$ diverges as $q^{-c/24}$ near $q = 0$. The solution is to define a holomorphic section $\psi(m)$ of $V_1 = E_c \otimes W_1$ by

$$\psi = \chi e_\beta. \tag{42}$$

This section of V_1 extends to $\overline{\mathscr{M}}_1$ because, near $q = 0$, $\chi e_\beta = \chi q^{c/24} e_\alpha$ and $\chi q^{c/24}$ *is* regular at $q = 0$.

The geometric expression for the abstract genus-1 partition function, over all of $\overline{\mathscr{M}}_1$, is given in terms of the flat hermitian bundle W_1 over $\overline{\mathscr{M}}_1$, and the holomorphic section ψ of the holomorphic bundle $V_1 = E_c \otimes W_1$ over $\overline{\mathscr{M}}_1$:

$$Z(\bar{m}, m) = h(\bar{\psi}, \psi) = \overline{\psi^a(m)} h_{\bar{a}b} \psi^b(m), \tag{43}$$

which is manifestly a section of $\bar{E}_c \otimes E_c$. Note that V_1 is actually a vector bundle in the extended sense described above, even though E_c is a projective line bundle. This is possible because the Chern class of E_c in genus 1 can be concentrated at the compactification divisor. The multi-valued transition function $q^{c/24}$ of E_c can be reinterpreted to give a single-valued transition function for V_1, by modifying the twist operator of W_1. The eigenvalues of the twist T, acting in the fiber of V_1 at $q=0$, are then $e^{2\pi i h_a}$. In particular, for the Ising model in genus 1, the fiber of V_1 at $q=0$ is one dimensional. If V_1 were not a vector bundle over $\overline{\mathcal{M}}_1$, but only a projective vector bundle, there could be no holomorphic section $\psi(m)$.

6. The gauge system

We now begin to assemble the abstract geometric formulation of two-dimensional conformal field theory on all Riemann surfaces. We call this mathematical structure the *gauge system* of the conformal field theory. If we were to extrapolate from the geometric formulation of the genus-1 partition function, we would define a gauge system as a hermitian metric h in a flat vector bundle W_g over $\overline{\mathcal{M}}_g$, along with a holomorphic section ψ, sometimes written $\psi_{(g)}$, of $V_g = E_c \otimes W_g$ over $\overline{\mathcal{M}}_g$. The partition function would be the section $Z(\overline{m}, m) = h(\overline{\psi(m)}, \psi(m))$ of $\overline{E}_c \otimes E_c$. This definition turns out to be slightly naive, but, before correcting it, we discuss some further motivation for the gauge system construction.

First, given a conformal field theory, there is in principle a vector bundle V_g for each genus g, and a holomorphic section $\psi(m)$ of V_g, constructed purely from invariants of the Virasoro algebra. There is also a flat metric $\tilde{h}_{\bar{a}b}$ in V_g over $\mathcal{M}_g$, constructed from the operator product coefficients of the conformal field theory, such that the partition function on $\mathcal{M}_g$ is $Z = \tilde{h}(\bar{\psi}, \psi)$. This construction uses a particular choice of surface metric g, so the partition function is an actual function on moduli space. The abstract partition function, and the metric $h_{\bar{a}b}$ is recovered by writing $\tilde{h} = h^g h$, where h^g is the metric associated with the surface geometry g, eq. (18). The section ψ is built from generalized characters $\chi^a(t)$ of the Virasoro algebra. In genus 1, the character $\chi_h(\tau)$ is analogous to the open string partition function, modified by inserting a projection on the irreducible Virasoro representation of weight h. Because of the projection, the character can be calculated from the Virasoro algebra alone. The reference to the open string alludes to the fact that only the L_n participate, without the second Virasoro algebra $\overline{L}_n$. The generalized characters are analogous to the open string partition function for genus g, modified by projecting on irreducible Virasoro representations inserted in a maximal collection of channels, and setting the operator product coefficients of the primary fields, the vertex coefficients, identically equal to 1. The generalized characters can again be calculated from the Virasoro algebra alone, using an abstract version of the dual model sewing construction for the open string [45]. The sewing construction uses surface geometries which are flat except for a finite set of points with delta function curvature. The generalized characters can also be regarded as generalizations of the conformal blocks, which are projected n-point functions of primary fields on the two sphere [1, 6, 46]. The genus g partition function is calculated by summing in each channel over highest weight states for both Virasoro algebras of the conformal field theory, the L_n and the $\bar{L}_n$, giving a hermitian product $\tilde{h}(\bar{\psi}, \psi)$ of generalized traces. The matrix elements $\tilde{h}_{\bar{a}b}$ are products of the operator product coefficients of the field theory, and are independent of the moduli. The generalized traces which participate in a given conformal field theory must form a representation of the mapping class groups Γ_g, unitary in the metric $\tilde{h}_{\bar{a}b}$, in order for the partition function to be single-valued on moduli space. The vector bundle V_g over $\mathscr{M}_g$ is formed from this representation, as in the genus-1 example, and extended to $\overline{\mathscr{M}}_g$ by a generalization of eq. (42). The matrix elements $\tilde{h}_{\bar{a}b}$ form a flat hermitian metric in V_g.

The simplest tractable nontrivial examples in higher genus are gaussian models, the nonlinear models whose target spaces are multidimensional tori. The generalized characters are theta functions. The Ising model can also be represented explicitly as a gauge system. The Ising partition function in genus g is written in terms of pfaffians or square roots of determinants of chiral Dirac operators:

$$Z = 2^{-g} \sum_{s} |\mathrm{pfaffian}\ \partial\!\!\!/_s|^2 . \tag{44}$$

The sum is over the $2^{2g-1} + 2^{g-1}$ spin structures in genus g which generically have no zero modes, i.e. for which the Z_2 index vanishes. For $c < 1$, the rank of the vector bundle V_g is finite, and grows exponentially in g. For $c \geq 1$, the rank of V_g is infinite in general, because an infinite number of highest weights appear in the theory. In exceptional cases the metric of the infinitely many generalized characters is highly degenerate, so the rank of W_g is much smaller than the number of highest weights in the theory. In chiral theories this collapse is taken to the extreme; the rank of W_g is 1, for all g.

We present the generalized character argument as motivation rather than as a proof that all conformal field theories are gauge systems, because the details have not been worked out. In particular, the modular transformation properties of the generalized traces are far from obvious. Even the modular transformation properties of conformal blocks are mysterious for $c \geq 1$ [46]. Still, it should eventually be possible to prove, through the generalized character construction, that every conformal field theory can be represented as a gauge system.

Additional motivation for the gauge system construction comes from the two fundamental properties of conformal field theory: the correlation functions should be real analytic in moduli space, and should be single-valued under analytic continuation around the orbifold points. That is, a conformal field theory is completely determined by its behavior in any open neighborhood of moduli space, and its analytic continuation to all of moduli space is unambiguous. In more physical language, these are the properties of locality and crossing symmetry. The gauge system construction implements these basic properties by a division of labor.

The complex analyticity of the section $\psi(m)$, combined with the local flatness of the metric $\tilde{h}$, provides real analyticity. The flatness is crucial, since it allows the metric to be expressed locally as a constant matrix. The flat metric is invariant under global parallel transport and $\psi(m)$ is a global section of V_g, which guarantee that the analytic continuation of the partition function is single-valued.

The problem with the naive gauge system construction described above is that, for genus $g>1$, the Chern class $c_1(E_c)=\frac{1}{24}c([\omega_{wp}/2\pi^2]+\delta)$ is not entirely concentrated on the compactification divisor, since $[\omega_{wp}/2\pi^2]$ restricts to a nontrivial cohomology class on $\mathscr{M}_g$. In fact, $[\omega_{wp}/2\pi^2]$ generates $H^2(\mathscr{M}_g, \mathbb{Q})$ [37]. If W_g were to be flat, then $V_g = E_c \otimes W_g$ would be a projective bundle on $\mathscr{M}_g$. It could not have a global holomorphic section $\psi(m)$. The division of labor between $h_{\bar{a}b}$ and $\psi(m)$ must be more subtle.

W_g is a *projective* holomorphic vector bundle, with projectively flat hermitian metric $h_{\bar{a}b}$. A projective vector bundle differs from an ordinary vector bundle in that the matrix valued transition functions $g_{\alpha\beta}(m)^a_b$ satisfy, on intersections $U_\alpha \cap U_\beta \cap U_\gamma$,

$$g_{\alpha\gamma}(m)^a_b = g_{\alpha\beta}(m)^a_c\, g_{\beta\gamma}(m)^c_b\, \sigma_{\alpha\beta\gamma}\,. \tag{45}$$

where $\sigma_{\alpha\beta\gamma}\in U(1)$. A projectively flat hermitian connection D is a hermitian connection for which the matrix valued curvature $(1,1)$-form is a multiple η of the identity, where η is an ordinary $(1,1)$-form on $\overline{\mathscr{M}}$:

$$\begin{aligned} D &= \partial + A\,, \qquad & Dh_{\bar{a}b} &= \partial h_{\bar{a}b} - h_{\bar{a}c}A^c_a = 0\,,\\ \bar{D} &= \bar{\partial}\,, \qquad & F^a_b &= \bar{\partial}A^a_b = \eta\delta^a_b\,. \end{aligned} \tag{46}$$

The vanishing of the $(2,0)$ component of the curvature, which is equivalent to local hermiticity of the connection, takes the form

$$[\partial, A] + [A, A] = 0\,. \tag{47}$$

The projective flatness of D can be regarded as a geometric formulation of the operator product relations of the stress-energy tensor [5, 30]. The cohomology class of the curvature form must be $[\eta] = -\frac{1}{24}c[\omega_{wp}/2\pi^2]$, so that $V_g = E_c \otimes W_g$ is a vector bundle, and can have a global holomorphic section $\psi(m)$. As in the genus-1 case, the contribution $\frac{1}{24}c\delta$ in $c_1(E_c)$ gets absorbed into the action of the twist operator in the fibers of V_g over the compactification divisor. The eigenvalues of the twist operator are modified from their values $\varepsilon_0 + h_a$ in the fibers of W_g over $\mathscr{D}_g$ to their values h_a in the fibers of V_g over $\mathscr{D}_g$. The section $\psi(m)$ can then lie in the nontrivial twist invariant subspace $h_a = 0$.

Parallel transport by the projectively flat connection D does not give a unitary representation of the mapping class group Γ_g in the fibers of W_g, because the phase of the parallel transport operator π_p depends on the path p. Choose a base point m, and choose, in each homotopy class $\gamma \in \Gamma_g$ of closed paths at m, a representative path $p(\gamma)$. Parallel transport along $p(\gamma)$ gives a projective unitary representation of Γ_g, $\gamma(m) = \pi_{p(\gamma)}$, in the fiber of W_g at m:

$$\gamma_1(m)\gamma_2(m) = \gamma_2(m)\gamma_1(m)\sigma(\gamma_1, \gamma_2)(m). \tag{48}$$

The operators $\gamma(m)$ are unitary because the covariant constant hermitian metric h is globally invariant under parallel transport. Conversely, a projectively flat hermitian vector bundle can be reconstructed from a projective unitary representation $\mathscr{H}_g$ of Γ_g, by essentially the same technique used to make a flat hermitian bundle from a unitary representation of Γ_g.

Away from orbifold points and the compactification divisor, we can choose a local basis for W_g of holomorphic sections $w_a(m)$ which are covariant constant up to a common phase,

$$Dw_a(m) = \partial f(\bar{m}, m) w_a(m), \tag{49}$$

where f is a locally defined function satisfying $\bar{\partial}\partial f = \eta$. The components of the metric, $h_{\bar{a},b} = h(\overline{w_a}, w_b)$, are not constant, since $\partial(h_{\bar{a}b}) = (\partial f)h_{\bar{a}b}$. But we can write, locally,

$$h_{\bar{a}b} = \mathrm{e}^f h^0_{\bar{a}b}, \tag{50}$$

with $h^0_{\bar{a}b}$ constant. Now define $\mathrm{V}_g = \mathrm{E}_c \otimes \mathrm{W}_g$. Let $\psi(m)$ be a holomorphic section of V_g. In components, $\psi(m) = \psi^a(m) w_a$ where $\psi^a(m)$ is a local section of E_c. A conformal field theory in the strict sense requires $\psi(m)$ nowhere zero. The partition function of the gauge system is

$$Z = h(\bar{\psi}, \psi) = \overline{\psi^a(m)} h_{\bar{a}b} \psi^b(m), \tag{51}$$

which is manifestly a section of $\bar{\mathrm{E}} \otimes \mathrm{E}$ over $\bar{\mathscr{M}}_g$. Locally, the partition function has the form

$$Z(\bar{m}, m) = \mathrm{e}^f\, \overline{\psi^a(m)} h^0_{\bar{a}b} \psi^b(m), \tag{52}$$

so it is real analytic up to the factor e^f. Note that the gauge system introduces an arbitrary element into the construction of the partition function, namely the particular choice of (1,1)-form η representing the cohomology class $[\eta] = -\frac{1}{24}c[\omega_{\mathrm{wp}}/2\pi^2]$. We interpret η as the abstract version of the renormalization scheme. We will see below that the correlation functions of the primary fields are independent of η. But the choice of η affects the definition of the stress-energy tensor and thus of the descendent fields. Different choices of η gives different, but equivalent, presentations of the underlying conformal field theory. Given a choice of surface geometry g, with its associated hermitian metric h^g in E_c over $\mathscr{M}_g$, we can fix η so that the metric $\tilde{h} = h^g h$ in V_g is flat over $\mathscr{M}_g$, but singular on the compactification divisor. The flatness, as opposed to projective flatness, of $\tilde{h}$ over $\mathscr{M}_g$ allows the partition function to be written as a manifestly single-valued and real analytic function on moduli space, which is singular on the compactification divisor, diverging as $q^{\varepsilon_0}\bar{q}^{\varepsilon_0}$. The real analyticity of correlation functions is then also manifest. This is the situation we envision in the construction of the gauge system from the conformal field theory using generalized characters.

As an aside, we mention that one possible way to make W_g and $\psi(m)$ at the same time is to follow the pattern in the genus-1 Ising model. Start with one or more sections $\tilde{\psi}_i(t)$ of E_c over Teichmüller space. Γ_g acts on the sections $\tilde{\psi}_i$ to generate a vector bundle V_g over $\overline{\mathscr{M}}_g$, in which the $\tilde{\psi}_i(t)$ can be interpreted as a single section $\psi(m)$. W_g is then defined as the projectively flat vector bundle $\mathrm{E}_{-c} \otimes \mathrm{V}_g$. Then a projectively flat metric $h_{\bar{a}b}$ must be found in W_g. It is not at all obvious under what conditions the $\tilde{\psi}_i(t)$ yield a projectively flat vector bundle W_g which is hermitian.

Now let us examine the behavior of the abstract partition function of the gauge system near the compactification divisor. Let $m_{\mathscr{D}}$ be a point in $\mathscr{D}_g$. Let t be a point on the boundary of $\mathscr{T}_g$ in the equivalence class of $m_{\mathscr{D}}$. Let $\gamma^{tw} \in \Gamma$ be the Dehn twist $q \to \mathrm{e}^{2\pi i} q$ around a node in t. Off the compactification divisor, near $m_{\mathscr{D}}$, W_g decomposes into subbundles $\mathrm{W}_g(h)$ whose fibers are the eigenspaces $\mathscr{H}_g(h)$ on which γ^{tw} has eigenvalue $\mathrm{e}^{-2\pi i h}$. Regard h as a real number, so $\mathscr{H}_g(h)$ depends only on h modulo integers. Clearly $h_{\bar{a}b}$ is block diagonal with respect to this decomposition. V_g decomposes into the local sub-bundles $\mathrm{V}_g(h) = \mathrm{E}_c \otimes \mathrm{W}_g(h)$. $\psi(m)$ can be decomposed near $m_{\mathscr{D}}$ in the form

$$\psi(m_{\mathscr{D}}, q) = \sum_k \psi_k(m_{\mathscr{D}}, q), \tag{53}$$

where $\psi_k(m_{\mathscr{D}}, q) \sim q^h k$ is a local section of $\mathrm{V}_g(h_k)$, such that ψ cannot be further decomposed into a sum of local sections of $\mathrm{V}_g(h_k)$. By convention, $h_0 = 0$. The partition function now has the expansion

$$Z(\bar{m}_{\mathscr{D}}, m_{\mathscr{D}}, \bar{q}, q) = \sum_{j,k} Z_{j,k}(\bar{m}_{\mathscr{D}}, m_{\mathscr{D}}, \bar{q}, q),$$

$$Z_{j,k}(\bar{m}_{\mathscr{D}}, m_{\mathscr{D}}, \bar{q}, q) = \overline{\psi_j^a(m_{\mathscr{D}}, q)}\, h_{\bar{a}b} \psi_k^b(m_{\mathscr{D}}, q), \tag{54}$$

where $Z_{j,k} \sim \bar{q}^{h_j} q^{h_k}$. Each index (j, k) labels a primary conformal field $\phi_{j,k}$ of conformal weights $\bar{h} = h_j$, $h = h_k$. $Z_{j,k}$ is the contribution of $\psi_{j,k}$ and its descendents in the channel passing through the node. If we choose a local frame such that $h_{\bar{a}b} = \mathrm{e}^f h^0_{\bar{a}b}$ with $h^0_{\bar{a}b}$ constant, as in eq. (50), then the normalized contribution of $\phi_{j,k}$,

$$Z_{j,k}/Z = \left(\overline{\psi_j^a(m_{\mathscr{D}}, q)}\, h^0_{\bar{a}b} \psi_k^b(m_{\mathscr{D}}, q)\right) \Big/ \left(\overline{\psi^a(m_{\mathscr{D}}, q)}\, h^0_{\bar{a}b} \psi^b(m_{\mathscr{D}}, q)\right) \tag{55}$$

is real analytic. This shows the real analyticity of the correlation functions of the primary fields reconstructed from the partition function of a gauge system.

7. The factorization algebra

We must now ensure that the partition function of the gauge system, eq. (51), factorizes on the compactification divisor. This is a necessary condition for the gauge system to be a conformal field theory, and we have argued in sect. 4 that this should also be a sufficient condition.

We want conditions on $h_{\bar{a}b}$ and on $\psi(m)$ which will imply factorization on the compactification divisor. The form of the generalized characters provides guidance. When a node is formed, a generalized character decomposes into a sum of products of generalized characters on the component surface(s) left by removal of the node. The matrix elements $\tilde{h}_{\bar{a}b}$, for the surface with node, determine the $\tilde{h}_{\bar{a}b}$ for the component surfaces. That is, both ψ and $h_{\bar{a}b}$ should individually satisfy factorization identities on the compactification divisor.

We now describe the abstract mathematical structure in which the factorization of ψ and $h_{\bar{a}b}$ are expressed. We consider factorization in the vector bundles V_g over $\overline{\mathscr{M}}_g$, but the discussion applies equally to a general vector bundle, projective vector bundle, or projective line bundle. We need only consider the generic case of factorization at a surface with a single node, because factorization at a surface with multiple nodes is given by iteration of single node factorization, since there are simultaneous transversal coordinates for all the nodes. However, factorization on multiple nodes must be independent of the sequences in which the single node factorizations are performed, so we will need to impose associativity conditions on the single node factorization. It is enough to consider surfaces with two nodes to determine the associativity conditions.

Repeating diagram (3), $\mathscr{D}_{g,0}$ is a fiber bundle over $\overline{\mathscr{M}}_{g-1}$ and $\mathscr{D}_{g,k}$ is a fiber bundle over $\overline{\mathscr{M}}_k \times \overline{\mathscr{M}}_{g-k}$, $k>0$, where the fibers are the locations (x_1, x_2) of the punctures:

$$\begin{array}{ccccc} \mathscr{D}_{g,0} & \to \overline{\mathscr{M}}_g & \qquad & \mathscr{D}_{g,k} & \to \overline{\mathscr{M}}_g \\ \big\downarrow & & & \big\downarrow & \\ \overline{\mathscr{M}}_{g-1} & & & \overline{\mathscr{M}}_k \times \overline{\mathscr{M}}_{g-k} & \end{array} \tag{56}$$

A simple example of factorization is provided by the line bundle E_c. E_c is trivial along the punctures. Therefore a local section of E_c over $\overline{\mathscr{M}}_{g-1}$ lifts to give a local section of E_c over $\mathscr{D}_{g,0}$, which is constant along the fibers. And, for $k>0$, the product of a pair of local sections of E_c, one over $\overline{\mathscr{M}}_k$ and one over $\overline{\mathscr{M}}_{g-k}$, lifts to give a local section of E_c over $\mathscr{D}_{g,k}$, constant along the fibers.

Given vector bundles V_g over $\overline{\mathscr{M}}_g$, in the extended sense of V-bundle described in sect. 5, write $\mathscr{V}_g$ for the local holomorphic sections of V_g. A factorization structure for the V_g is a collection of maps F_g and $F_{k,g-k}$ acting on local holomorphic sections:

$$\begin{array}{ccc} \mathscr{V}_{g/\mathscr{D}_{g,0}} \leftarrow \mathscr{V}_g & \qquad & \mathscr{V}_{g/\mathscr{D}_{g,k}} \leftarrow \mathscr{V}_g \\ F_g \Big\uparrow & & F_{k,g-k} \Big\uparrow \\ \mathscr{V}_{g-1} & & \mathscr{V}_k \times \mathscr{V}_{g-k} \end{array} \tag{57}$$

satisfying consistency conditions which will be described. Recall that a vector bundle V_g restricted to the compactification divisor degenerates to the twist invariant sub-bundle of V_g nearby.

For $k=0$ the map F_g takes a local section of V_{g-1} over an open set U in $\overline{\mathscr{M}}_{g-1}$ to a local section of V_g over the lift of U in $\mathscr{D}_{g,0}$. For $k>0$, $F_{k,g-k}$ is a bilinear map from local sections, one of V_k over an open set U_1 in $\overline{\mathscr{M}}_k$, the other of V_{g-k} over an open set U_2 in $\overline{\mathscr{M}}_{g-k}$ to a local section of V_g over the lift of $\mathrm{U}_1 \times \mathrm{U}_2$ in $\mathscr{D}_{g,k}$. The F_g, $F_{k,g-k}$ thus *define* what it means for sections of V_g over $\mathscr{D}_{g,0}$ or $\mathscr{D}_{k,g-k}$, in the images of F_g or $F_{k,g-k}$, to be constant in the fibers. For this to be possible, those subbundles of $V_{g/\mathscr{D}_g}$ must be trivial in the puncture variables.

Schematically, in component form, the factorization maps have the form

$$\begin{gathered} \left(F_g \eta_{(g-1)}\right)^a=\left(F_g\right)_b^a \eta_{(g-1)}^b, \\ F_{k,g-k}\left(\eta_{(k)}, \eta_{(g-k)}\right)^a=\left(F_{k,g-k}\right)_{bc}^a \eta_{(k)}^b \eta_{(g-k)}^c, \end{gathered} \tag{58}$$

where $\eta_{(g)}$ is a local section of V_g over the appropriate neighborhood, and we use indices a, b, c for all the vector bundles V_g, even though their fibers are not the same. In more detail,

$$\begin{aligned} &\left(F_g \eta_{(g-1)}\right)^a\left(m^{(g-1)}, x_1, x_2\right)=F_g\left(m^{(g-1)}\right)_b^a \eta_{(g-1)}^b\left(m^{(g-1)}\right), \\ &F_{k,g-k}\left(\eta_{(k)}, \eta_{(g-k)}\right)^a\left(m_1^{(k)}, m_2^{(g-k)}, x_1, x_2\right) \\ &\quad=F_{k,g-k}\left(m_1^{(k)}, m_2^{(g-k)}\right)_{bc}^a \eta_{(k)}^b\left(m_1^{(k)}\right) \eta_{(g-k)}^c\left(m_2^{(g-k)}\right). \end{aligned} \tag{59}$$

We omit indicating the dependence of F_g and $F_{k,g-k}$ on the puncture variables, assuming a frame in which they are constant in (x_1, x_2). If $2k=g$, $F_{k,k}$ must be consistent with exchange symmetry:

$$F_{k,k}\left(m_1^{(k)}, m_2^{(k)}\right)_{bc}^a=F_{k,k}\left(m_2^{(k)}, m_1^{(k)}\right)_{cb}^a. \tag{60}$$

It is convenient to define $F_{k,g-k}$ for $k>g-k$ so that F_{k_1,k_2} is invariant under exchange symmetry:

$$F_{k_1,k_2}\left(m_1^{(k_1)}, m_2^{(k_2)}\right)_{b_1 b_2}^a=F_{k_2,k_1}\left(m_2^{(k_2)}, m_1^{(k_1)}\right)_{b_2 b_1}^a. \tag{61}$$

The consistency conditions come from factorizing on two nodes in different orders. There are several cases. In the first case, one node separates the surface into genus k_1 and k_2+1, and the other node lies in a handle in the genus k_2+1 component:

$$\begin{tikzcd} & \mathrm{m}_{\mathscr{D}}^{(k_1+k_2+1)} \arrow[dl] \arrow[dr] & \\ \left(m_1^{(k_1)}, m_3^{(k_2+1)}\right) \arrow[dr] & & m_4^{(k_1+k_2)} \arrow[dl] \\ & \left(m_1^{(k_1)}, m_2^{(k_2)}\right). & \end{tikzcd} \tag{62}$$

The consistency condition is

$$F_{k_1+k_2+1}\left(m_4^{(k_1+k_2)}\right)^a_c F_{k_1,k_2}\left(m_1^{(k_1)}, m_2^{(k_2)}\right)^c_{b_1b_2}$$
$$= F_{k_1,k_2+1}\left(m_1^{(k_1)}, m_3^{(k_2+1)}\right)^a_{b_1c} F_{k_2+1}\left(m_2^{(k_2)}\right)^c_{b_2}. \tag{63}$$

In the second case, one node separates the surface into genus k_1 and k_2+k_3, and the other node separates the surface into genus k_1+k_2 and k_3:

$$\begin{tikzcd} & m_{\mathscr{D}}^{(k_1+k_2+k_3)} \arrow[dl] \arrow[dr] & \\ \left(m_4^{(k_1+k_2)}, m_3^{(k_3)}\right) \arrow[dr] & & \left(m_1^{(k_1)}, m_5^{(k_2+k_3)}\right) \arrow[dl] \\ & \left(m_1^{(k_1)}, m_2^{(k_2)}, m_3^{(k_3)}\right). & \end{tikzcd} \tag{64}$$

The consistency condition is:

$$F_{k_1,k_2}\left(m_1^{(k_1)}, m_2^{(k_2)}\right)^c_{b_1b_2} F_{k_1+k_2,k_3}\left(m_4^{(k_1+k_2)}, m_3^{(k_3)}\right)^a_{cb_3}$$
$$= F_{k_1,k_2+k_3}\left(m_1^{(k_1)}, m_5^{(k_2+k_3)}\right)^a_{b_1c} F_{k_2,k_3}\left(m_2^{(k_2)}, m_3^{(k_3)}\right)^c_{b_2b_3}. \tag{65}$$

The remaining configurations of two nodes give analogous consistency conditions.

The definition of the gauge system is now completed by imposing factorization conditions on the sections $\psi_{(g)}$ of V_g over $\overline{\mathscr{M}}_g$ and on the metric $h^{(g)}$ in W_g over $\overline{\mathscr{M}}_g$. For ψ,

$$\psi_{(g)/\mathscr{D}_{g,0}}=F_g\psi_{(g-1)},\qquad \psi_{(g)/\mathscr{D}_{g,k}}=F_{k,g-k}\big(\psi_{(k)},\psi_{(g-k)}\big).\tag{66}$$

Schematically, in components,

$$\psi^a_{(g)/\mathscr{D}_{g,0}}=\big(F_g\big)^a_b\psi^b_{(g-1)},\qquad \psi^a_{(g)/\mathscr{D}_{g,k}}=\big(F_{k,g-k}\big)^a_{bc}\psi^b_{(k)}\psi^c_{(g-k)}.\tag{67}$$

For the metric, and arbitrary sections η of the W_g,

$$h^{(g)}\Big(\overline{F_g\eta_{(g-1)}},F_g\eta_{(g-1)}\Big)_{/\mathscr{D}_{g,0}}=h^{(g-1)}\Big(\overline{\eta_{(g-1)}},\eta_{(g-1)}\Big),$$

$$\begin{aligned}&h^{(g)}\Big(\overline{F_{k,g-k}\big(\eta_{(k)},\eta_{(g-k)}\big)},F_{(k,g-k)}\big(\eta_{(k)},\eta_{(g-k)}\big)\Big)_{/\mathscr{D}_{g,k}}\\&\quad=h^{(k)}\Big(\overline{\eta_{(k)}},\eta_{(k)}\Big)h^{(g-k)}\Big(\overline{\eta_{(g-k)}},\eta_{(g-k)}\Big),\end{aligned}\tag{68}$$

where we have left implicit the factorization maps for E_c. Schematically, in components,

$$h^{(g)}_{\bar{a}b}\big(\overline{F_g}\big)^{\bar{a}}_{\bar{c}}\big(F_g\big)^b_d=h^{(g-1)}_{\bar{c}d},$$

$$h^{(g)}_{\bar{a}b}\big(\overline{F_{k,g-k}}\big)^{\bar{a}}_{\bar{c}_1\bar{c}_2}\big(F_{k,g-k}\big)^b_{d_1d_2}=h^{(k)}_{\bar{c}_1d_1}h^{(g-k)}_{\bar{c}_2d_2}.\tag{69}$$

It follows from factorization of the metric that the metric connection D is compatible with the factorization structure F. This means that the connection D on the compactification divisor $\mathscr{D}_g$ agrees via F with the connection D on the component surface(s), or equivalently that F is covariant constant.

From the factorization conditions (66), (68) on ψ and h it follows immediately that

$$h^{(g)}\big(\bar{\psi}_{(g)},\psi_{(g)}\big)_{/\mathscr{D}_{g,0}}=h^{(g-1)}\big(\bar{\psi}_{(g-1)},\psi_{(g-1)}\big),$$

$$h^{(g)}\big(\bar{\psi}_{(g)},\psi_{(g)}\big)_{/\mathscr{D}_{g,k}}=h^{(k)}\big(\bar{\psi}_{(k)},\psi_{(k)}\big)h^{(g-k)}\big(\bar{\psi}_{(g-k)},\psi_{(g-k)}\big),\tag{70}$$

which are exactly the required factorization identities (26), (28) for the partition function of the gauge system.

A local holomorphic function f defined on a neighborhood U of $\bar{\mathscr{R}}$ near the compactification divisor $\mathscr{D}$ and also on $\pi(\mathrm{U} \cap \mathscr{D})$ must satisfy $f_{/\mathscr{D}} = f \circ \pi$. In other words, $f_{/\mathscr{D}}$ must be constant in the puncture variables and must be consistent, via removal of nodes, with its value at lower genus. It is easy to verify, by induction on n in the diagrams (75) or (76), that the only globally holomorphic functions on $\bar{\mathscr{R}}$ are the constants. In this sense, $\bar{\mathscr{R}}$ is a connected, compact analytic space.

The diagrams (75) or (76) give a natural factorization structure for analytic functions on $\bar{\mathscr{R}}$. The factorization condition described in sect. 7, for the special case of holomorphic functions, expresses exactly the requirement of analyticity on $\bar{\mathscr{R}}$.

A holomorphic vector bundle V over $\bar{\mathscr{R}}$ consists of ordinary vector bundles over the components of $\mathscr{R}$, patched together over the compactification divisor by maps π^* analogous to those in diagrams (76). In terms of the local analytic sections $\mathscr{V}(\bar{\mathscr{R}})$ of V over $\bar{\mathscr{R}}$, $\mathscr{V}(\mathscr{D})$ over $\mathscr{D}$, $\mathscr{V}(\mathscr{R})$ over $\mathscr{R}$, V is defined by the commuting diagrams

$$\cdots \begin{tikzcd} \mathscr{V}(\mathscr{D}_n) & \mathscr{V}(\bar{\mathscr{R}}_n) \arrow[l] \arrow[dl] \\ \mathscr{V}(\mathscr{R}_{n-1}) \arrow[u, "\pi^*"] & \end{tikzcd} \qquad \cdots \begin{tikzcd} \mathscr{V}(\mathscr{D}) & \mathscr{V}(\bar{\mathscr{R}}) \arrow[l] \arrow[dl] \\ \mathscr{V}(\mathscr{R}) \arrow[u, "\pi^*"] & \end{tikzcd} \tag{77}$$

A local holomorphic section η of V must satisfy $\eta_{/\mathscr{D}} = \pi^* \eta$. This is exactly a factorization structure in bundles V_g over $\bar{\mathscr{M}}_g$ as described in sect. 7. Connections and hermitian metrics in V must respect the maps π^*.

One possible approach to studying the universal moduli space is to describe its universal analytic covering space and covering group, the universal modular group. The covering space cannot be the universal Teichmüller space described in the mathematics literature, because that universal Teichmüller space is the same as its universal modular group; the quotient, the moduli space, is a single point [36].

In the analytic formulation of string theory [29], nonperturbative string effects are to be described in terms of world surfaces of infinite genus. It might also be useful to study conformal field theory extended to some appropriate class of infinite genus surfaces. Here we only make some formal comments about infinite genus surfaces. There are two universal moduli spaces of interest. The larger space is

$$\bar{\mathscr{R}}_\infty = \mathscr{R} \times \bigcup_{k=0}^{\infty} \mathrm{Sym}^k(\bar{\mathscr{M}}_\infty), \tag{78}$$

where $\bar{\mathscr{M}}_\infty$ is the space of connected, stable Riemann surfaces of infinite genus. We think of these infinite genus surfaces as essentially compact, almost all of the infinitely many handles being very small. Except for the finite genus parts of $\bar{\mathscr{R}}_\infty$, the surfaces with nodes should be dense in $\bar{\mathscr{M}}_\infty$, since adding or subtracting a very small handle, or splitting off a very small component surface, should be a very small disturbance. Except for the finite genus parts, the analytic structure on $\overline{\mathscr{R}}_\infty$ should be very smooth. The crucial question is whether it is possible to extend a conformal field theory, and a gauge system, to $\overline{\mathscr{R}}_\infty$.

The second infinite genus universal moduli space is a subspace of $\overline{\mathscr{R}}_\infty$:

$$\overline{\mathscr{R}}^0_\infty = \bigcup_{k=0}^{\infty} \mathrm{Sym}^k\left(\overline{\mathscr{M}}^0_\infty\right), \tag{79}$$

where $\overline{\mathscr{M}}^0_\infty$ is the space of connected, stable Riemann surfaces of infinite genus, which have the property that removing a finite number of nodes leaves only infinite genus connected components. $\overline{\mathscr{R}}^0_\infty$ should be extremely regular. If the extension to infinite genus is possible, it would be interesting to know the relation among gauge systems on $\overline{\mathscr{R}}$, on $\overline{\mathscr{R}}_\infty$ and on $\overline{\mathscr{R}}^0_\infty$. It might be conjectured that any gauge system on $\overline{\mathscr{R}}$ would extend to $\overline{\mathscr{R}}_\infty$ and thus to $\overline{\mathscr{R}}^0_\infty$, and that every gauge system on $\overline{\mathscr{R}}^0_\infty$ would come this way. But to reduce a surface in $\overline{\mathscr{R}}^0_\infty$ to finite genus, an infinite number of nodes would have to be formed and removed. We might suppose that there could be more than one way of extending a gauge system on $\overline{\mathscr{R}}^0_\infty$ to $\overline{\mathscr{R}}_\infty$ and $\overline{\mathscr{R}}$. The gauge systems on $\overline{\mathscr{R}}^0_\infty$ might serve as universal approximations to the more various collection of ordinary conformal field theories, the gauge systems on $\overline{\mathscr{R}}$. We might even speculate that the factorization algebra, and projectively flat hermitian metric, associated with the projective vector bundle W over $\overline{\mathscr{R}}^0_\infty$ might have an especially simple structure, perhaps equivalent to a Fock space. The holomorphic section ψ would be an analytic field over $\overline{\mathscr{R}}^0_\infty$ of free field ground states in this Fock space.

9. Discussion

We have formulated two-dimensional conformal field theories as gauge systems on universal stable moduli space $\overline{\mathscr{R}}$. A gauge system is a vector bundle W over universal stable moduli space, a projectively flat metric h in W, and a holomorphic section ψ of $\mathrm{V}=\mathrm{E}_c\otimes \mathrm{W}$, where E_c is the line bundle $(\lambda_{\mathrm{H}})^{c/2}$. The partition function is $Z(\overline{m}, m) = h(\overline{\psi(m)}, \psi(m))$. The factorization conditions which are equivalent to analyticity of ψ and smoothness of h on $\overline{\mathscr{R}}$ imply factorization of the partition function and allow reconstruction of the correlation functions of the conformal field theory from the gauge system. Projective flatness of h and global analyticity of ψ imply modular invariance and real analyticity of the partition function, which in turn imply crossing symmetry and real analyticity of the reconstructed correlation functions.

The description of two-dimensional conformal field theory as geometry on universal moduli space exposes an underlying gauge invariance, which is the projective unitary gauge group of the projective vector bundle W. Note that these

8. Universal moduli space

The gauge system, in order to be equivalent to a conformal field theory, must obey the factorization conditions described in the previous section. These factorization conditions are natural from the point of view of conformal field theory. In this section we show that they are also mathematically natural.

We define the *universal moduli space* $\overline{\mathscr{R}}$ of stable Riemann surfaces. We show that a holomorphic vector bundle V on the universal moduli space $\overline{\mathscr{R}}$ is exactly a collection of holomorphic vector bundles V_g on the moduli spaces $\overline{\mathscr{M}}_g$ along with a factorization structure at the compactification divisor. In particular, the projective line bundles $\mathrm{E}_c = (\lambda_{\mathrm{H}})^{c/2}$ on the moduli spaces $\overline{\mathscr{M}}_g$ define a projective line bundle E_c on $\overline{\mathscr{R}}$. The partition function $Z(\overline{m}, m)$, satisfying factorization at the compactification divisor, defines a section of $\overline{\mathrm{E}}_c \otimes \mathrm{E}_c$ over $\overline{\mathscr{R}}$. The holomorphic sections $\psi_{(g)}$ of the vector bundles V_g, satisfying the factorization conditions, define a holomorphic section ψ of V over $\overline{\mathscr{R}}$. The metrics $h^{(g)}$ on $\mathrm{W}_g = \mathrm{E}_{-c} \otimes \mathrm{V}_g$, satisfying the factorization conditions, define a metric h on $\mathrm{W} = \mathrm{E}_{-c} \otimes \mathrm{V}$.

A gauge system, or two dimensional conformal field theory, is simply a holomorphic vector bundle V over universal moduli space $\overline{\mathscr{R}}$, a holomorphic section ψ of V, and a projectively flat hermitian metric h in $\mathrm{W} = \mathrm{E}_{-c} \otimes \mathrm{V}$. The partition function is $Z = h(\overline{\psi}, \psi)$.

The definition of universal moduli space is motivated by the intuitive picture of factorization. From the point of view of two-dimensional conformal field theory, when a smooth Riemann surface of genus g develops a node and the node is removed, the surface becomes a possibly disconnected Riemann surface of genus $g-1$. More generally, if k nodes are removed the genus becomes $g-k$. For disconnected Riemann surfaces, the Euler number is $\chi = 2 - 2g = 2(1 - \#_{\mathrm{h}} - \#_{\mathrm{c}})$, where $\#_{\mathrm{h}}$ is the number of handles and $\#_{\mathrm{c}}$ is the number of components. The genus is therefore $g = \#_{\mathrm{h}} - \#_{\mathrm{c}} + 1$. Removing a node raises the Euler number by two and lowers the genus by one.

The natural setting for the factorization conditions is the space of all compact stable Riemann surfaces, connected and disconnected. The moduli space $\mathscr{R}$ of all compact, smooth Riemann surfaces consists of infinitely many disconnected components:

$$\mathscr{R} = \prod_{g=0}^{\infty} \left(\bigcup_{k=0}^{\infty} \mathrm{Sym}^k(\mathscr{M}_g) \right) = \bigcup_{\{n_g\}} \prod_{g=0}^{\infty} \mathrm{Sym}^{n_g}(\mathscr{M}_g), \tag{71}$$

where Sym^k is the k-fold symmetric product. The connected components of $\mathscr{R}$ are indexed by the multiplicities $\{n_g\}$, where $n_g \geq 0$ is the number of connected components of genus g in the Riemann surface. On the other hand, the moduli space of all compact *stable* Riemann surfaces,

$$\overline{\mathscr{R}} = \prod_{g=0}^{\infty} \left(\bigcup_{k=0}^{\infty} \mathrm{Sym}^k(\overline{\mathscr{M}}_g) \right), \tag{72}$$

is a connected space. Given any two surfaces, there is a third surface of higher genus which can be continuously, even analytically, deformed into either of the two original surfaces by forming nodes and forgetting the punctures. The universal compactification divisor is $\mathscr{D} = \overline{\mathscr{R}} - \mathscr{R}$.

We define an analytic structure on $\overline{\mathscr{R}}$ which makes this intuition precise. The naive analytic structure on $\overline{\mathscr{R}}$ is the product structure inherited from the $\overline{\mathscr{M}}_g$. But we impose a stronger analytic structure by requiring that the removal of nodes be analytic. Define, for $n \geq 0$,

$$\mathscr{R}_n = \bigcup_{\Sigma_g g n_g \leq n} \prod_{g=0}^{\infty} \mathrm{Sym}^{n_g}(\mathscr{M}_g),$$

$$\overline{\mathscr{R}}_n = \bigcup_{\Sigma_g g n_g \leq n} \prod_{g=0}^{\infty} \mathrm{Sym}^{n_g}(\overline{\mathscr{M}}_g),$$

$$\mathscr{D}_n = \overline{\mathscr{R}}_n - \mathscr{R}_n . \tag{73}$$

The universal moduli space $\overline{\mathscr{R}}$ is the limit of the inclusions

$$\overline{\mathscr{R}}_0 \to \overline{\mathscr{R}}_1 \to \overline{\mathscr{R}}_2 \to \cdots \to \overline{\mathscr{R}} . \tag{74}$$

Let π: $\mathscr{D}_n \to \mathscr{R}_{n-1}$ be the map which takes a surface $m_{\mathscr{D}}$ with nodes to the smooth surface $\pi(m_{\mathscr{D}})$ which results from removing the nodes and healing the punctures that are left. Along with the inclusions $\mathscr{D}_n \to \overline{\mathscr{R}}_n$ and $\mathscr{R}_n \to \overline{F}_n$, the maps π form diagrams,

$$\begin{tikzcd} \cdots & \mathscr{D}_n \arrow[r] \arrow[d, "\pi"'] & \overline{\mathscr{R}}_n & \cdots & \mathscr{D} \arrow[r] \arrow[d, "\pi"'] & \overline{\mathscr{R}} \\ & \mathscr{R}_{n-1} \arrow[ur] & & & \mathscr{R} \arrow[ur] & \end{tikzcd} \tag{75}$$

which do *not* commute in the naive analytic analytic structure on $\overline{\mathscr{R}}$. We define the analytic structure on $\overline{\mathscr{R}}$ to be the strongest analytic structure, having the fewest local holomorphic functions, which includes the local holomorphic functions on $\mathscr{R}$ and for which the diagrams (75) commute as diagrams of analytic maps. In terms of the local analytic functions $\mathscr{O}(\overline{\mathscr{R}})$ on $\overline{\mathscr{R}}, \mathscr{O}(\mathscr{D})$ on $\mathscr{D}, \mathscr{O}(\mathscr{R})$ on $\mathscr{R}$, there are the commuting diagrams

$$\begin{tikzcd} \cdots & \mathscr{O}(\mathscr{D}_n) & \mathscr{O}(\overline{\mathscr{R}}_n) \arrow[l] \arrow[dl] & \cdots & \mathscr{O}(\mathscr{D}) & \mathscr{O}(\overline{\mathscr{R}}) \arrow[l] \arrow[dl] \\ & \mathscr{O}(\mathscr{R}_{n-1}) \arrow[u, "\pi^*"] & & & \mathscr{O}(\mathscr{R}) \arrow[u, "\pi^*"] & \end{tikzcd} \tag{76}$$

gauge transformations do not necessarily leave the metric invariant, but can change the curvature form η in its $(1,1)$-cohomology class. These extended gauge transformations change the partition function, but only by changing the overall factor e^{f} in eq. (52). The correlation functions of the primary fields are unchanged, so these gauge transformations take a conformal field theory to an equivalent one. Presumably this gauge symmetry includes the fundamental redundancy in the field algebra of a conformal field theory, since the redundant operators [47] are perturbations of the system which do not change the partition function.

The main technical task in establishing the foundations of this analytic formulation of conformal field theory is to make rigorous the arguments outlined in sects. 4 and 6 that a conformal field theory can be reconstructed from the partition function of every gauge system and that the partition function of every conformal field theory is given by a gauge system.

It would be interesting to have a natural way of constructing projectively flat bundles, for example from natural analytic functions on the universal analytic covering space of the universal moduli space, in analogy to the character $\chi_0(q)$, eq. (39), for the Ising model in genus 1. In any case, effective means of constructing gauge systems are certainly needed.

The classification of gauge systems seems a worthwhile mathematical enterprise. Present knowledge of conformal field theory puts interesting constraints on the structure of gauge systems. We list some of these known facts. For $c<1$, unitarity forces c to be in the discrete series, the highest weights h_a are restricted to a finite list, and the multiplicities are highly constrained. In particular, the rank of each vector bundle W_g is finite, for finite g. In all known conformal field theories, the weights h_a form a discrete set, with multiplicities which can be calculated or estimated under some circumstances, e.g. in the $c<1$ discrete series or in the infinite volume, $\alpha' \to \infty$, limit of Calabi-Yau spaces. The identity operator is the only field with conformal weights $0, 0$. In general, the multiplicities $h_{\bar{a}b}$ of the genus-1 partition function are integers. When a symmetry group S acts, these integers $h_{\bar{a}b}$ become the multiplicities of the representations of S in the Hilbert space of the two-dimensional conformal field theory. It is especially important to understand how the constraints which follow from the structure of the Virasoro algebra arise in the gauge systems.

There is a straightforward generalization of gauge systems to supergauge systems, which are the abstract versions of superconformal field theories. Super-Riemann surfaces [48] replace ordinary Riemann surfaces; supermoduli space replaces ordinary moduli space; and supervector bundles replace ordinary vector bundles. The supergauge systems are of interest because of their application to fermionic string theory [3] and supersymmetric critical phenomena [8, 19].

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