# Quantum Integrability in Systems with Finite Number of Energy Levels

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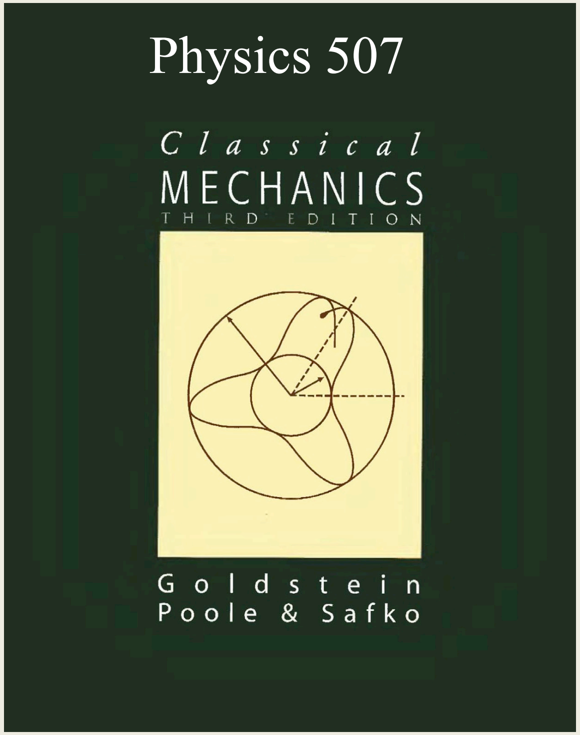
# Classical Integrability




1D Hamiltonians $H = \frac{p^2}{2m} + V(q)$

Central force, e.g. Kepler problem $H = \frac{\mathbf{p}^2}{2m} + V(r)$

Euler top $H = \frac{l_1^2}{2I_1} + \frac{l_2^2}{2I_2} + \frac{l_3^2}{2I_3}$

$$\{\mathbf{L}^2, L_z\} = \{\mathbf{L}^2, H\} = \{L_z, H\} = 0$$

Liouville
(1809 – 1882)

$H(p,q)$, where $q = (q_1, \ldots, q_n)$; $p = (p_1, \ldots, p_n)$; i.e. $n$ degrees of freedom

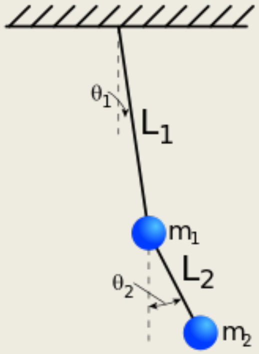
Definition: $H(p,q)$ is integrable if it has $n$ (maximum possible number) of functionally independent Poisson-commuting integrals

$$\{H_i(p,q), H_j(p,q)\} = 0, \quad i,j = 0, \ldots, n-1; \quad H_0(p,q) \equiv H(p,q)$$

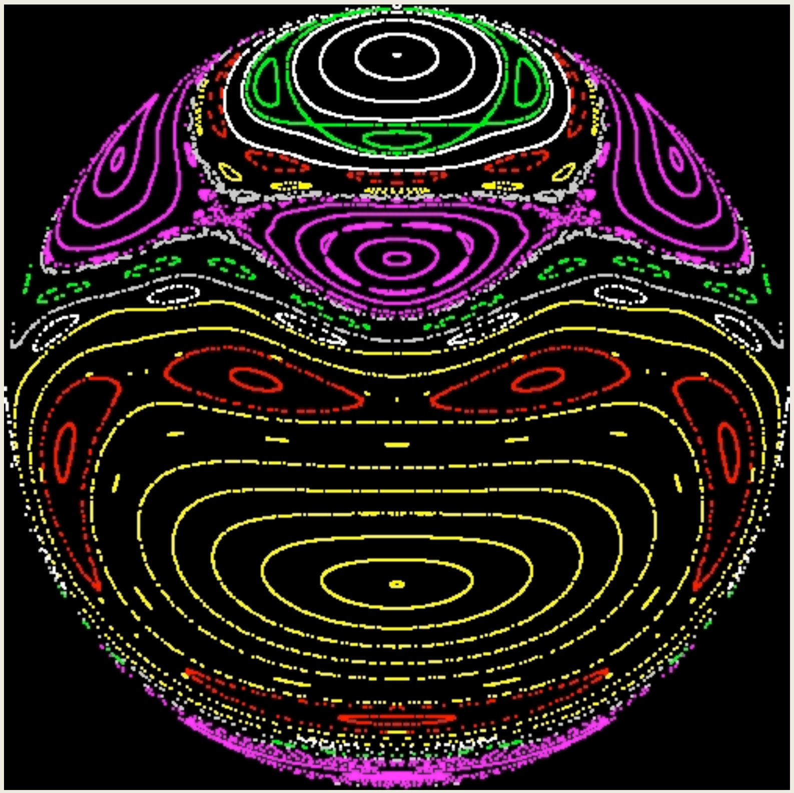
$$\Downarrow$$

- Unambiguous separation of integrable from nonintegrable (generic)
- Exact solution, various properties that don't have to be verified on a case by case basis, regular vs. chaotic dynamics

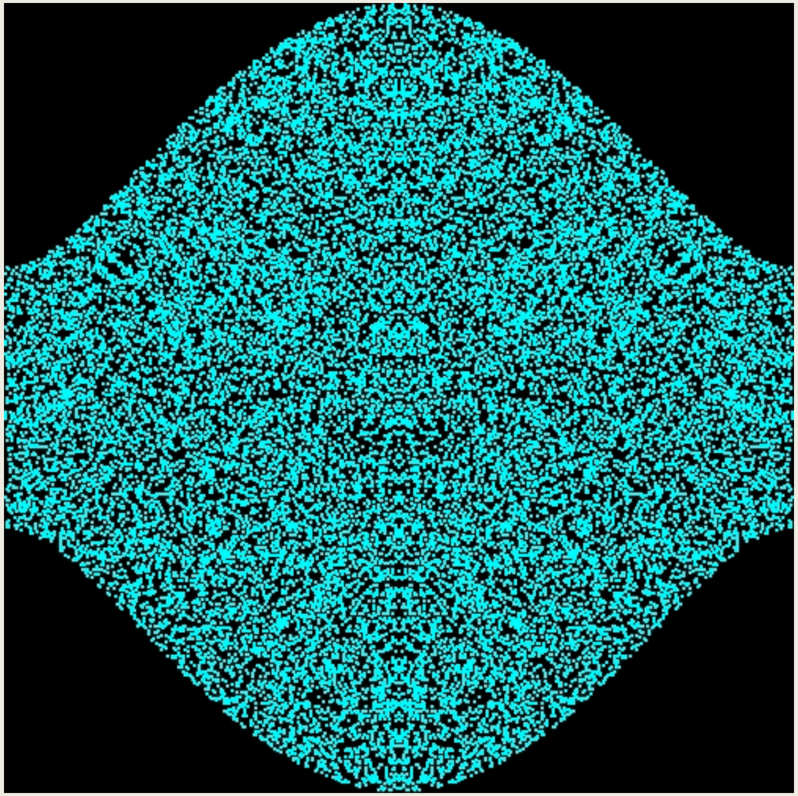
# Classical Regularity vs. Classical Chaos




Double pendulum: cross-sections of trajectories in 4D phase space



regular, quasiperiodic motion



Now "chaos is the sole ruler of the world"

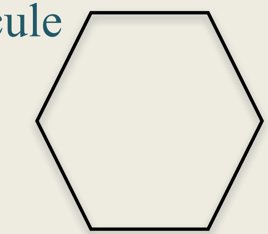
Images by wolfoerster at www.codeproject.com

# Q: What is quantum integrability? How is it defined?

Various quantum many-body lattice models (e.g. 1D Hubbard, 1D Heisenberg magnets, BCS) are called "integrable". What does it mean?

Think finite $N \times N$ matrix Hamiltonian

Example: Hubbard model for benzene molecule



$$H = \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times \end{pmatrix}$$

Given a matrix $H$ how do we tell if it's integrable?

Can we randomly generate such integrable matrices?

**No way! Not even a good definition!** [von Neumann (1931), Weigert (1992), Sutherland, *Beautiful Models* (2004), Caux & Mossel (2011), Yuzbashyan & Shastry (2013)]

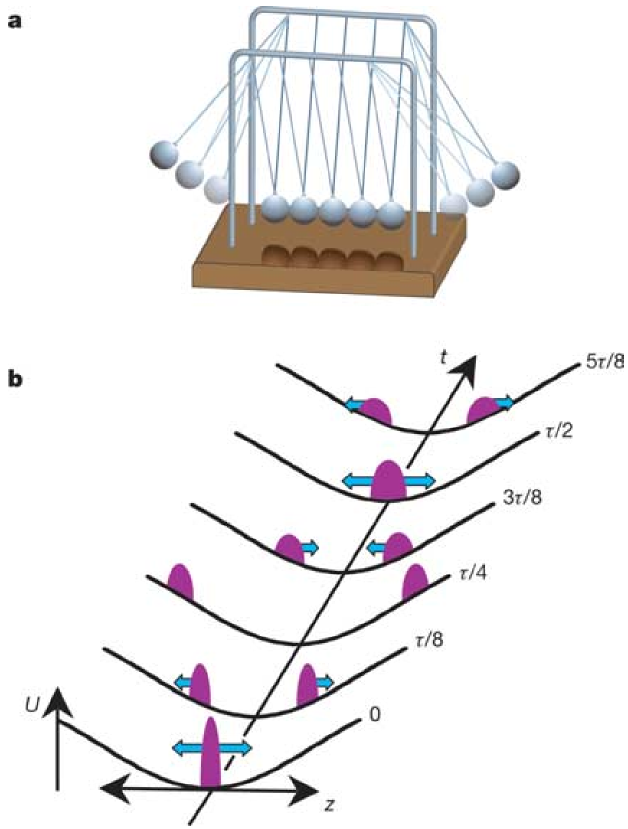
No natural notion of a nontrivial integral of motion: for any $H$ there is a full set of $H_k$ such that $[H_i, H_k] = [H_k, H] = 0$

$$H = \sum_{n=1}^{N} E_n |n\rangle\langle n|, \qquad H_k = |k\rangle\langle k|$$

Alternatively, can consider powers of $H$

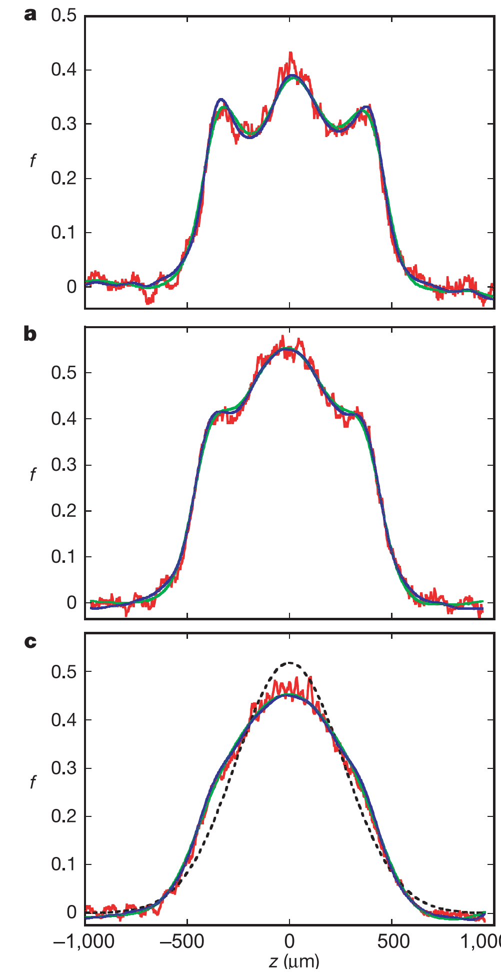
$$H_k = \sum_{n=1}^{N} a_n H^n$$

# Why is it important? – Integrability enters mainstream




Quantum Newton's cradle

Kinoshita, Wenger, Weiss,
Nature (2006)

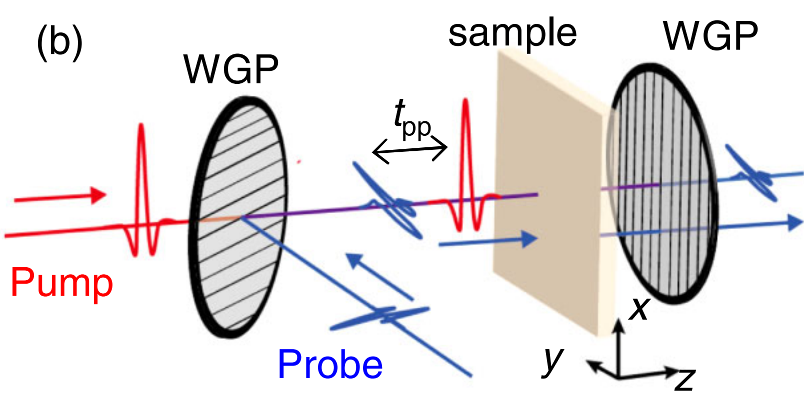


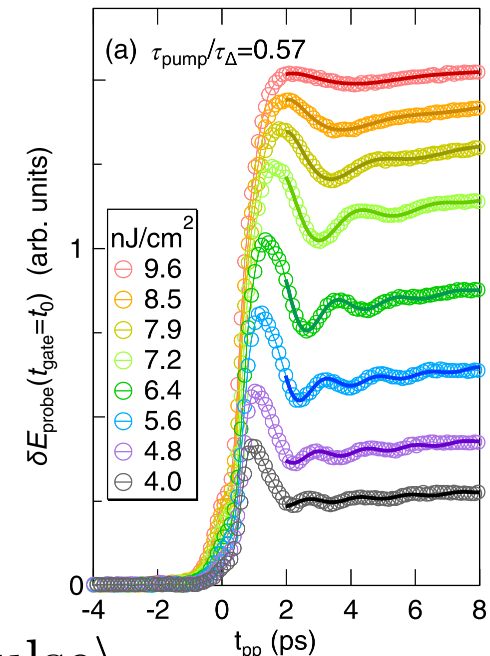

"[87]Rb atoms … do not noticeably equilibrate even after thousands of collisions. Our results are probably explainable by the well-known fact that a homogeneous 1D Bose gas with point-like collisional interactions is *integrable*."

PRL **111,** 057002 (2013) PHYSICAL REVIEW LETTERS week ending 2 AUGUST 2013

**Higgs Amplitude Mode in the BCS Superconductors $Nb_{1-x}Ti_xN$ Induced by Terahertz Pulse Excitation**

Ryusuke Matsunaga,[1] Yuki I. Hamada,[1] Kazumasa Makise,[2] Yoshinori Uzawa,[3]
Hirotaka Terai,[2] Zhen Wang,[2] and Ryo Shimano[1]




$$|\psi(0)\rangle = |\text{nonequilibrium state produced by the pulse}\rangle$$

$$\hat{H}_{\mathrm{BCS}} = \sum_{i,\sigma} \epsilon_i \hat{c}^\dagger_{i\sigma} \hat{c}_{i\sigma} - u \sum_{i,j} \hat{c}^\dagger_{i\uparrow} \hat{c}^\dagger_{i\downarrow} \hat{c}_{j\downarrow} \hat{c}_{j\uparrow}$$

$$i \frac{d|\psi\rangle}{dt} = \hat{H}_{\mathrm{BCS}} |\psi\rangle$$

Higgs mode
(order parameter)

$$\Delta(t) = u \sum_i \langle \hat{c}_{i\downarrow}(t) \hat{c}_{i\uparrow}(t) \rangle$$

Previous RU Physics Colloquium, September 2008: “New superfluid states of fermionic matter in and out of (far from) equilibrium”

PRL **96,** 097005 (2006) PHYSICAL REVIEW LETTERS week ending 10 MARCH 2006

**Relaxation and Persistent Oscillations of the Order Parameter in Fermionic Condensates**

Emil A. Yuzbashyan,[1] Oleksandr Tsyplyatyev,[2] and Boris L. Altshuler[3,4]

We determine the limiting dynamics of a fermionic condensate following a sudden perturbation

Integrability of $\hat{H}_{\mathrm{BCS}}$

⇩

$$\frac{|\Delta(t)|}{\Delta_\infty} = 1 + a\frac{\cos(2\Delta_\infty t + \phi)}{\sqrt{\Delta_\infty t}}. \qquad (1)$$

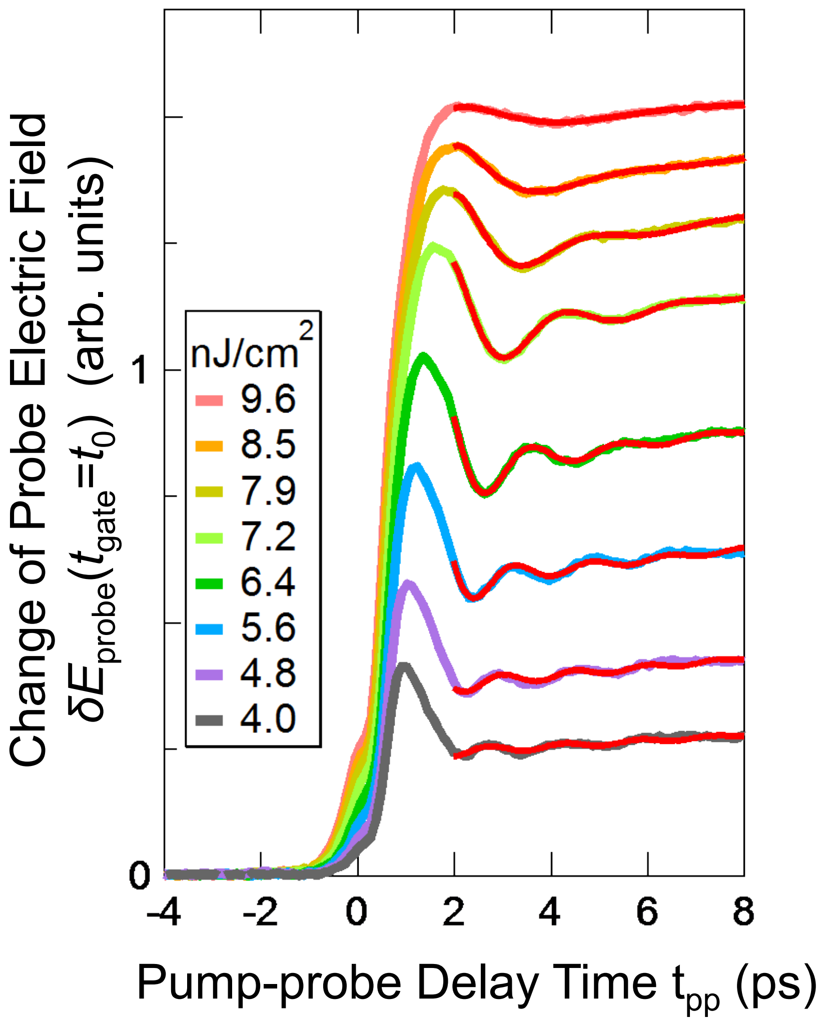
$|\Delta(t)| \to \mathrm{const}$, but the condensate (BCS superconductor) doesn’t equilibrate

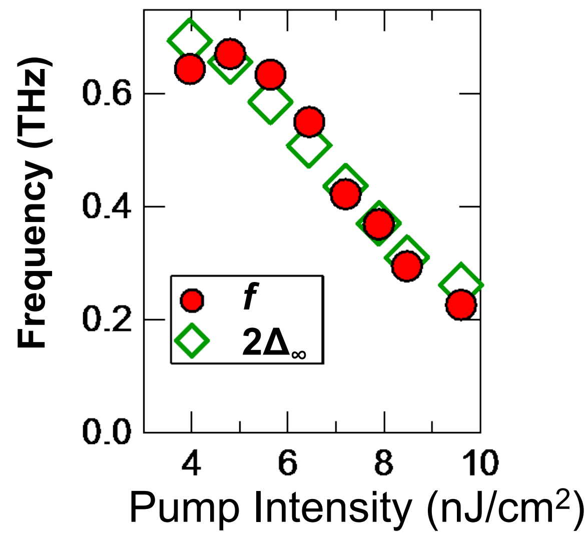
e.g. other degrees of freedom continue to oscillate and $\Delta_\infty < \Delta_0 =$ ground state gap

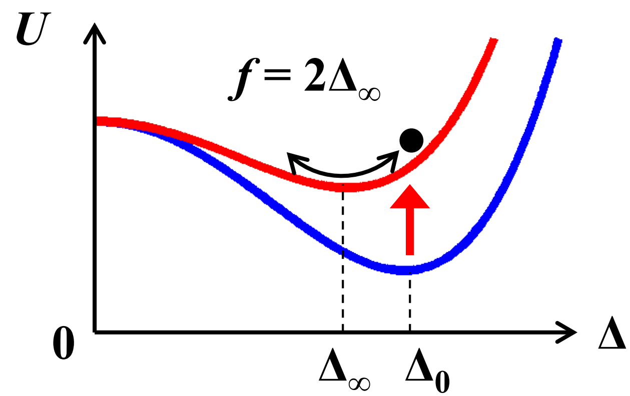
# Order parameter dynamics

$$\delta\Delta(t_{pp}) = C_1 + C_2 t_{pp} + \frac{a}{(t_{pp})^b}\cos(2\pi f t_{pp} + \phi)$$

E. Yuzbashyan et al., PRL **96**, 230404 (2006).




R. Matsunaga et al., PRL**111**, 057002 (2013)

Slide from R. Shimano's talk at Rice conference "Interacting quantum systems driven out of equilibrium", May, 2016

# Integrable systems do not equilibrate. Do they follow Generalized Gibbs Ensemble (GGE)?

$$\hat{\rho} = \frac{1}{Z} e^{-\sum_k \beta_k \hat{H}_k}, \quad [H_i, H_k] = [H, H_k] = 0$$

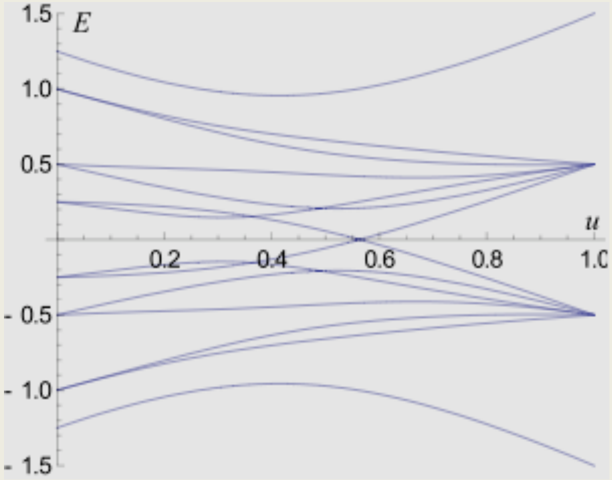
- GGE fails for 1D Heisenberg spin chains
  Goldstein & Andrei, Phys. Rev. A (2014); Pozsgay et. al. PRL (2014)

- Does work for 1D Heisenberg spin chains if newly discovered integrals are added
  Ilievski et. al. PRL (2015)

Need to know what quantum integrability is, i.e. what is a complete set of allowed $H_k$! Otherwise, GGE is essentially unfalsifiable

# Signatures (?) of quantum integrability

- Integrals of motion
- No equilibration: Generalized Gibbs Ensemble
- Exact solution for the energy spectrum via Bethe's Ansatz
- Crossings of energy levels as functions of interaction or external field strength (seen as resonances in e.g. relaxation rates)
- Energy levels $\{E_n\}$ have Poisson statistics, i.e. behave as independent random numbers (observed directly in small systems)


In the absence of a clear notion, have to verify each property independently on a case by case basis.

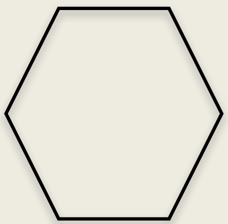
Cannot predict them as in classical integrability

# Properties of quantum integrable systems: Exact Solution

## Example: 1D Hubbard model for benzene molecule

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

hopping + onsite interaction



Electrons on a hexagon,
6 sites, 3 spin-up, 3 spin-down

$400 \times 400$ matrix linear in $u$

Exact Solution via Bethe's Ansatz: Lieb and Wu (1969)

$$e^{6ik_j} = \prod_{\alpha=1}^{3} \frac{\Lambda_\alpha - \sin k_j - iu/4}{\Lambda_\alpha - \sin k_j + iu/4}, \quad \prod_{\alpha=1}^{3} \frac{\Lambda_\alpha - \Lambda_\beta + iu/2}{\Lambda_\alpha - \Lambda_\beta + iu/2} = -\prod_{j=1}^{6} \frac{\Lambda_\beta - \sin k_j - iu/4}{\Lambda_\beta - \sin k_j - iu/4}$$

9 coupled nonlinear equations

$$E = -\sum_{j=1}^{6} 2\cos k_j$$

But cf. $\det(H - EI) = 0$

# Properties of quantum integrable systems: Integrals of motion
## Example: 1D Hubbard model

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

$$\hat{H}_1(u) = -i\sum_{j,s=\uparrow\downarrow}(\hat{c}^\dagger_{j+2s}\hat{c}_{js} - \hat{c}^\dagger_{js}\hat{c}_{j+2s}) - iu\sum_{j,s=\uparrow\downarrow}(\hat{c}^\dagger_{j+1s}\hat{c}_{js} - \hat{c}^\dagger_{js}\hat{c}_{j+1s})(\hat{n}_{j+1,-s} + \hat{n}_{j,-s} - 1)$$

Shastry, PRL (1986)

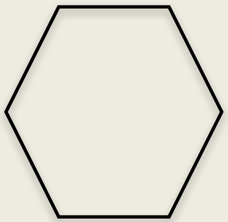
$$[\hat{H}(u), \hat{H}_1(u)] = 0, \quad \text{for all } u$$

$\hat{H}_2(u), \hat{H}_3(u), \hat{H}_4(u), \ldots$ – infinitely many integrals from Shastry's transfer matrix

The Hamiltonian and the first integral are linear in a real parameter $u$

# Properties of quantum integrable systems: Level crossings
## Example: 1D Hubbard model for benzene molecule

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$



Electrons on a hexagon,
6 cites, 3 spin-up, 3 spin-down
$400 \times 400$ matrix linear in $u$

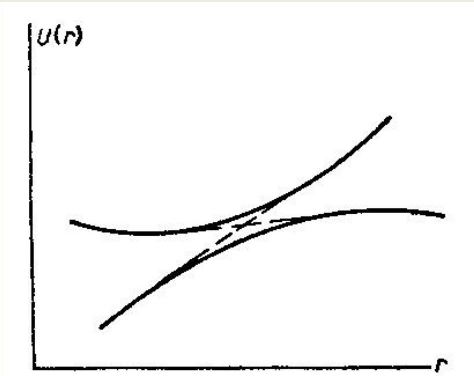



FIG. 27

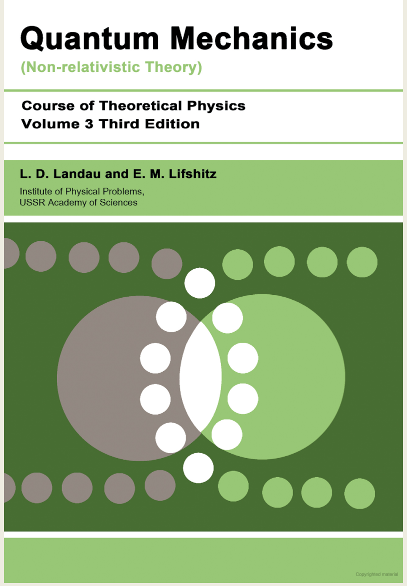
**Q**: How do eigenvalues look as functions of $u$?

Hund (1927)

Noncrossing rule:

"Thus we reach the result that…the intersection of terms of like symmetry is impossible (E. Wigner and J. von Neumann 1929)"

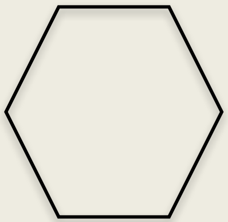
In other words, for a typical *H(u)* energy levels with the same quantum numbers (spin, momentum etc.) never cross.




Physics 502

# Properties of quantum integrable systems: Level crossings
## Example: 1D Hubbard model for benzene molecule

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

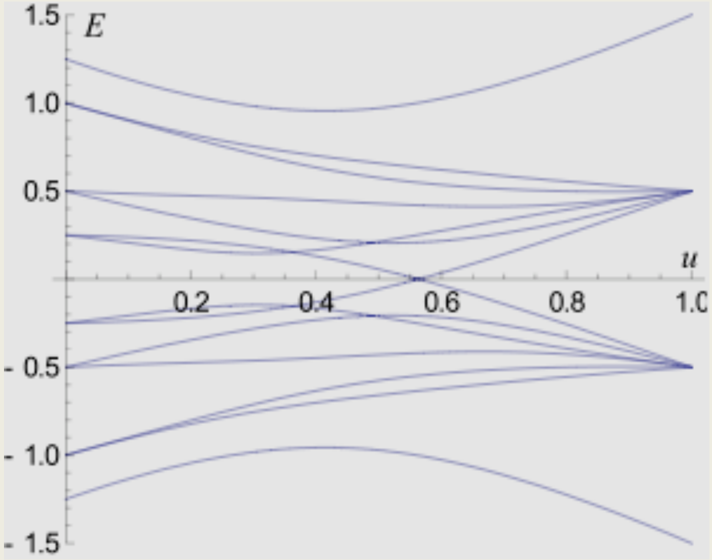


Electrons on a hexagon,
6 cites, 3 spin-up, 3 spin-down
$400 \times 400$ matrix linear in $u$

**Q**: How do eigenvalues look as functions of $u$?

"The noncrossing rule is apparently violated in the case of the 1d Hubbard Hamiltonian for benzene molecule…"

Heilmann and Lieb (1971)




(All) energy levels (14) for a certain complete set of quantum numbers

# Properties of quantum integrable systems: Level crossings
# Counterexample: BCS model

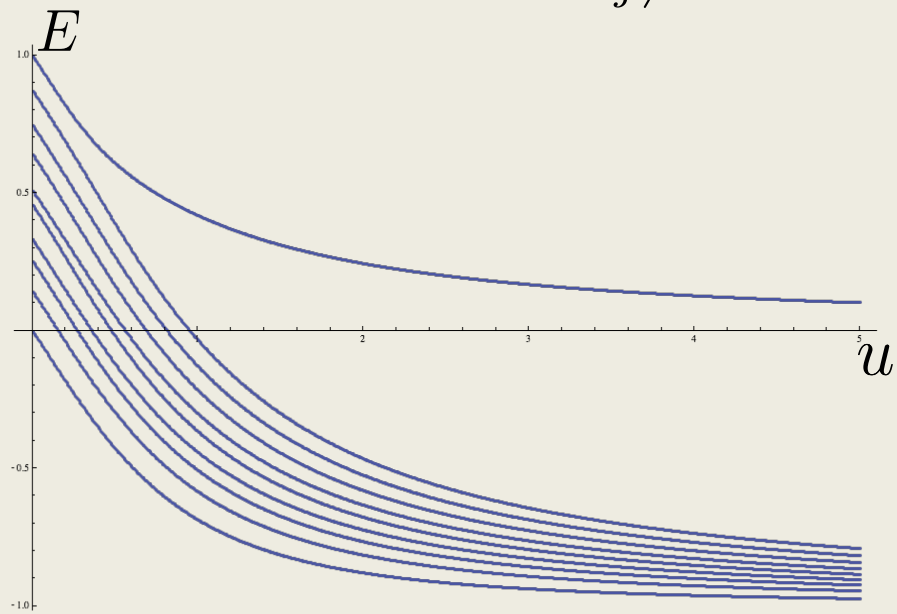
$$\hat{H}_{\rm BCS} = \sum_i 2\varepsilon_i \hat{s}_i^z - u \sum_{i,j} \hat{s}_i^- \hat{s}_j^+ = \sum_i 2\varepsilon_i \hat{H}_i$$

single-particle + superconducting interactions

Integrals of motion for BCS (Gaudin magnets)

$$[\hat{H}_i(u), \hat{H}_j(u)] = [\hat{H}_{\rm BCS}(u), \hat{H}_i(u)] = 0 \qquad \hat{H}_i(u) = \hat{s}_i^z - u \sum_{j \neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j}$$

**Q:** Are crossings really a signature of integrability? Why don't they always happen? Can we predict them? their number?




(All) energy levels (10) for a certain complete set of quantum numbers for the BCS model

# Statistics of energy levels $\{E_n\}$ – what to expect?

$\hat{H}$ – complex system, e.g. heavy nucleus, disordered metal, quantum dot, generic many-body interacting system etc.

**Q:** What can we say about its energy levels $\{E_n\}$?


Wigner


Dyson

Wigner (1950s): model by a random matrix $H$ consistent with basic space-time symmetries, i.e. choose the Hamiltonian "at random"

"We picture a complex nucleus as a black box…we shall consider an ensemble of Hamiltonians, each of which can describe a different nucleus." (Dyson 1962)

Statistical independence of $H_{ij}$ plus invariance of $P(H)$ with respect to arbitrary change of basis $P(O^T H O) = P(H)$ ⇒

$$P(H) = C \exp(-a \, \mathrm{tr} H^2)$$

Gaussian ensemble of random matrices

Time reversal inv. (no $B$-field) – $H_{ij}$ are real: Gaussian Orthogonal Ensemble (GOE)

# Statistics of energy levels $\{E_n\}$ – Random Matrix Theory (RMT)

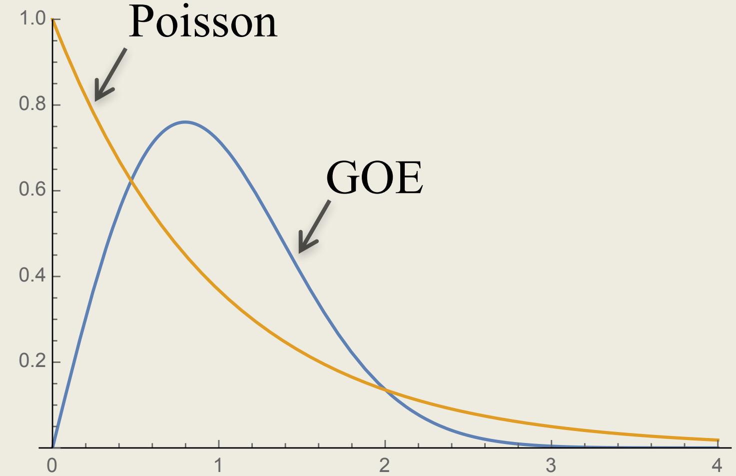
Nearest-neighbor level spacing:

$$s_n = \frac{E_{n+1} - E_n}{\delta},$$

$$\delta = \langle E_{n+1} - E_n \rangle$$

Time reversal inv. (GOE): Wigner surmise, Wigner-Dyson statistics

$$P(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4} s^2}$$




GOE & Poisson: two universal distributions

$\{H_{ij}\}$ random uncorrelated $\Longrightarrow$ $\{E_n\}$ correlated, level repulsion: $P(0) = 0$

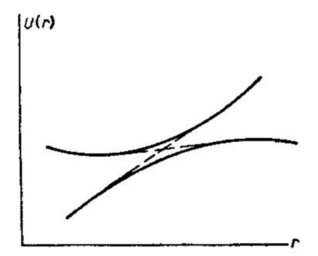
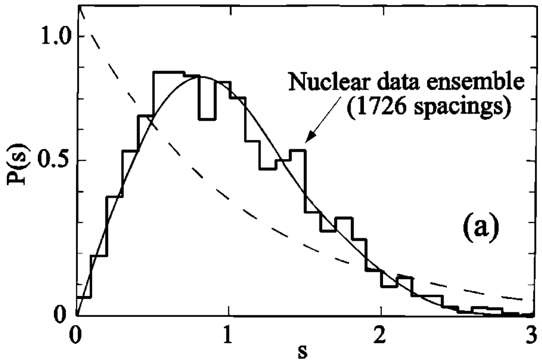



Fig. 27

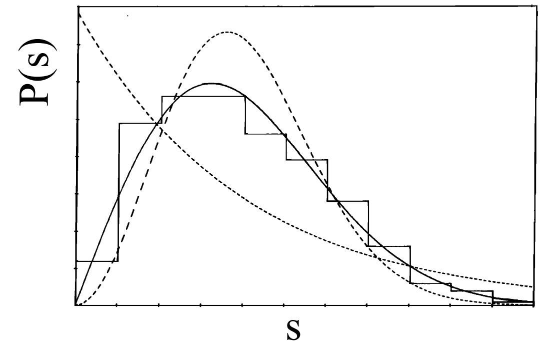
$\{E_n\}$ random uncorrelated $\Longrightarrow$ Poisson statistics, $P(s) = e^{-s}$, no repulsion

# Universality of Random Matrix Theory (RMT)

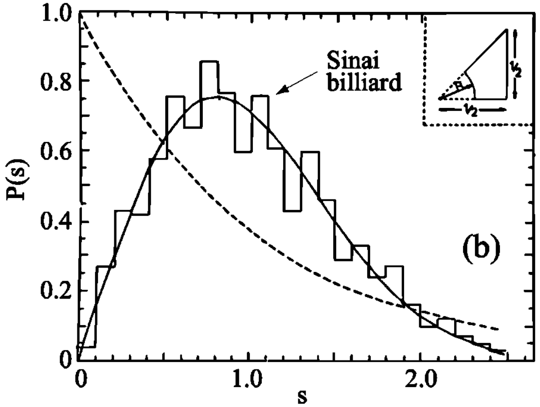
Smooth solid line in all graphs: GOE



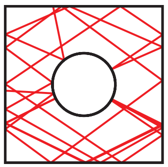

Neutron & proton resonances measured in several heavy nuclei



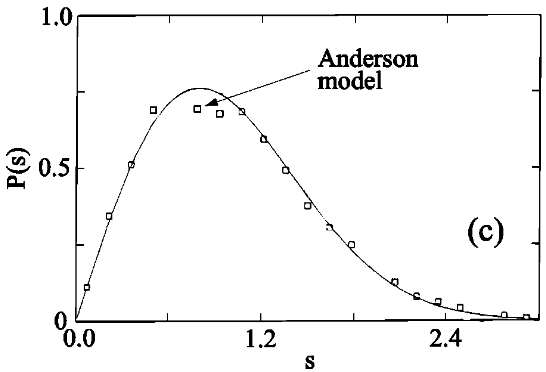

Acoustic resonances in Al blocks




2D chaotic motion



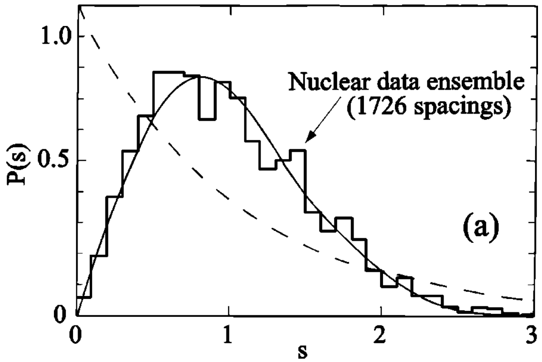
Exp. realization: microwave cavity, Sridhar, PRL (1991)



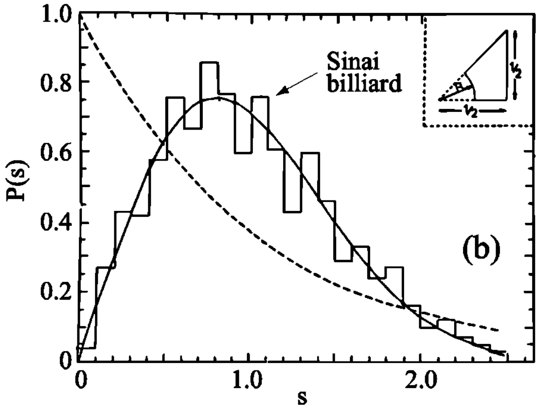

Disordered metal (3D Anderson model at $w/t = 2$)

Pictures from: Y. Alhassid, Rev. Mod. Phys. 72, 850 (2000); R.L. Weaver, J. Acoust. Soc. Am. 85, 1005 (1989); N. Anantharaman & A. Bäcker, IAMP News Bulletin, April 2013

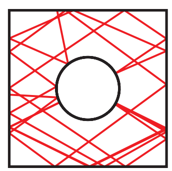
# Universality of Random Matrix Theory (RMT)

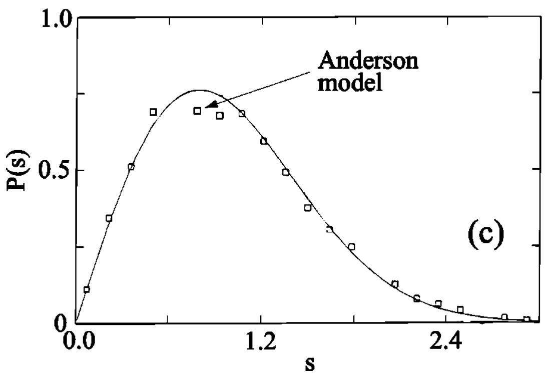



Neutron & proton resonances measured in several heavy nuclei




2D chaotic motion

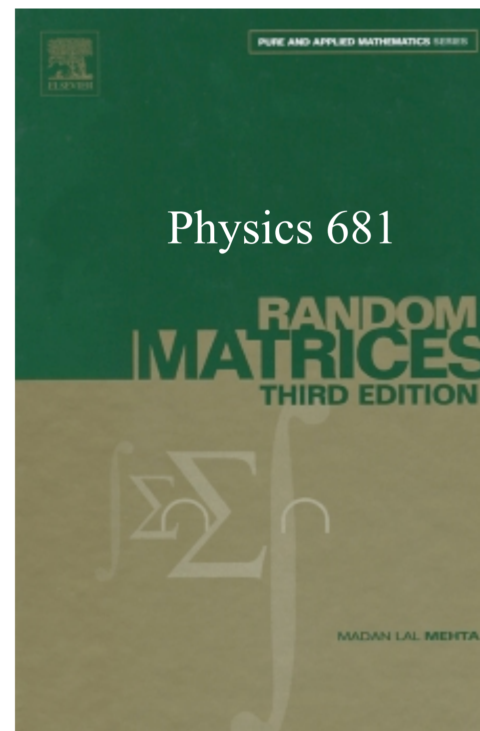





Disordered metal (3D Anderson model at $w/t = 2$)

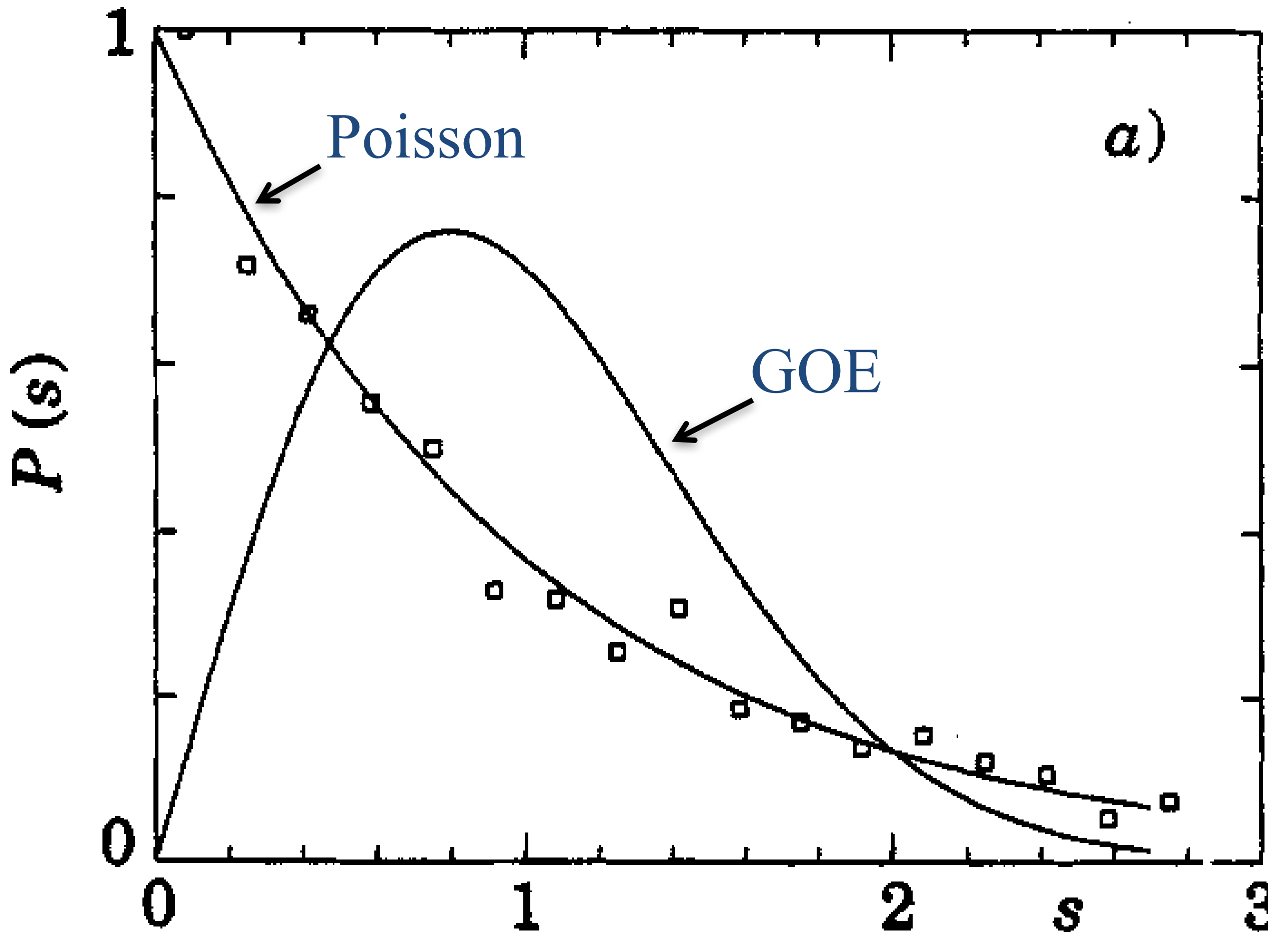
RMT: quantum chaos, long-time dynamics of interacting many-body Hamiltonians, quantum chromodynamics, fractional quantum Hall, superconductivity, number theory, neuroscience, finance etc.

Do Swedish pines diagonalize random matrices? Le Caer (1989).



# Properties of quantum integrable systems: Poisson statistics

## Example: 1D Hubbard model

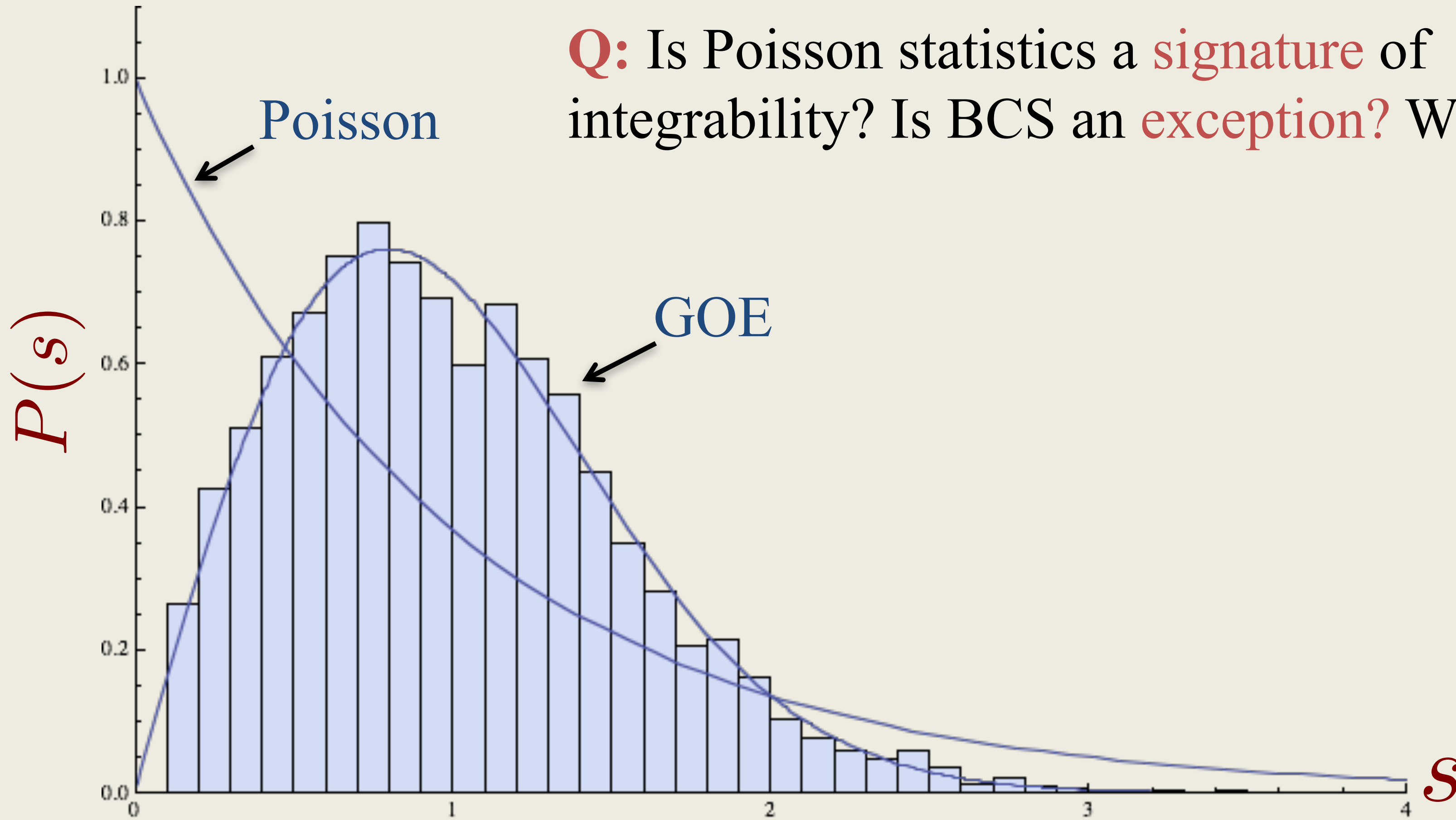



Poilblank et.al. Europhys. Lett. (1993)

Similarly to RMT observed in integrable systems

Level spacing distribution for a Hubbard chain with 12 sites at 1/4 filling, total momentum $P = \pi/6$, spin $S = 0$

# Properties of quantum integrable systems: Poisson statistics
## Counterexample: BCS Hamiltonian




**Q:** Is Poisson statistics a signature of integrability? Is BCS an exception? Why?

Level spacing distribution for the BCS Hamiltonian in a $5000 \times 5000$ same symmetry sector

See also Relano, Dukelsky et. al. PRE (2004)

# Notion of Quantum Integrability: What are we looking for?

Definition: Quantum (matrix) Hamiltonian $H$ is integrable if…

Classical integrability has it!

Consequences:

1. Exact Solution
2. Energy level crossings: why sometimes there are none? How many crossings to expect?
3. Poisson level statistics and exceptions – need ensembles of integrable models for this.
4. Generalized Gibbs Ensemble for dynamics?

Can we develop a similarly sound notion of integrability in Quantum Mechanics – for $N \times N$ Hermitian matrices (Hamiltonians)?

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{js}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{js}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

$$\hat{H}_1(u) = -i\sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j+2s}\hat{c}_{js} - \hat{c}^\dagger_{js}\hat{c}_{j+2s}) - iu\sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j+1s}\hat{c}_{js} - \hat{c}^\dagger_{js}\hat{c}_{j+1s})(\hat{n}_{j+1,-s} + \hat{n}_{j,-s} - 1)$$

The Hamiltonian and at least one other integral of motion are linear in a real parameter $u$. This integral is sufficient for explaining the level crossings. Same is the case of other parameter-dependent integrable lattice models (BCS, 1D Heisenberg).

For any given number of sites:

$$H(u) = T + uV, \quad H_1(u) = T_1 + uV_1, \quad u \text{ – real parameter}$$

$$T, V, T_1, V_1 \text{ – } N \times N \text{ Hermitian matrices}$$

# Proposed solution: introduce & fix parameter dependence

Let $H(u) = T + uV$, $u$ – real parameter, $T, V$ – $N \times N$ Hermitian matrices

Suppose we require a commuting partner also linear in $u$:

$$H_1(u) = T_1 + uV_1$$

$$[H(u), H_1(u)] = 0 \text{ for all } u$$

$$\Updownarrow$$

$$[V, V_1] = 0, \quad [T, V_1] = [T_1, V], \quad [T, T_1] = 0$$

These commutation relations severely constraint matrix elements of $T$. For a generic/typical $H(u)$ – no commuting partners except itself and identity. Now can separate generic (no integrals) from special (integrable).

# N x N Hamiltonians linear in a parameter separate into two distinct classes = good notion of integrability

$H(u) = T + uV$

No commuting partners linear in $u$ other than itself and identity (typical) – nonintegrable, need $N^2/2$ real parameters to specify $H(u)$

Nontrivial commuting partners $H_k(u)=T_k+uV_k$ exist – integrable, turns out need less than $4N$ parameters – measure zero in the space of linear Hamiltonians

Owusu & Yuzbashyan, J. Phys. A (2011)
Yuzbashyan & Shastry, J. Stat. Phys. (2013)

Classification by the number $n$ of integrals of motion

$n = N - 1$ (maximum possible) – type 1 integrable system

$n = N - 2$ – type 2

$n = N - 3$ – type 3

…

$n = N - M$ – type $M$

…

Definition: A matrix Hamiltonian $H \equiv H_0(u) = T_0 + uV_0$ is integrable if it has $n > 1$ linearly independent commuting partners $H_i(u) = T_i + uV_i$ discounting multiples of the identity.

$[H_i(u), H_j(u)] = 0$ for all $u$ and $i, j = 0, 1, \ldots, n-1$

General member of the commuting family: $H(u) = \sum_{i=0}^{n-1} d_i H_i(u)$

# What can we achieve with this notion of quantum integrability? – almost everything we wanted and more!!

- Explicitly Construct integrable models with any prescribed number $n$ of integrals!

$$[H_i(u), H_j(u)] = 0, \quad H_i(u) = T_i + uV_i, \quad H_j(u) = T_j + uV_j$$

$$\Updownarrow$$

$$[V_i, V_j] = 0, \quad [T_i, V_j] = [T_j, V_i], \quad [T_i, T_j] = 0$$

Simplest case: $n = N - 1$ (type 1 – max # of integrals – analog of classical integrability)

Simplest case: $n = N - 1$ (type 1 – max # of integrals – analog of classical integrability)

Every type 1 family is uniquely specified by a choice of a Hermitian matrix and a vector and vice versa

Hermitian matrix $E$ Arbitrary vector $|\gamma\rangle$

⇓

$N$ commuting $N \times N$ Hermitian matrices $H_i(u)$

General member of the commuting family: $H(u) = \sum_i d_i H_i(u) = T + uV$

To pick $H(u)$, pick $N$ arbitrary $d_i$ or, equivalently, pick a matrix $T$ (or $V$)

$$[H(u)]_{km} = u\gamma_k\gamma_m\left(\frac{d_k - d_m}{\varepsilon_k - \varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j - d_m}{\varepsilon_j - \varepsilon_m}\right)$$

$\varepsilon_k$ – eigenvalues of $E$, $\gamma_k$ – components of $|\gamma\rangle$
($2N$ arbitrary real parameters to pick a commuting family)

$d_k$ – eigenvalues of $T$ – another $N$ arbitrary real numbers to pick a specific Hamiltonian within the family

Constructed all $n = N$-1, $N$-2, $N$-3 (types 1, 2, 3) and some for arbitrary other $n$

## What can we achieve with this notion of quantum integrability? – almost everything we wanted and more!!

- Exact solution through a single algebraic equation for all types (cf. Bethe's Ansatz)

(type 1) $$\sum_{j=1}^{N} \frac{\gamma_j^2}{\lambda - \epsilon_j} = u, \quad E_k = \frac{\gamma_k^2}{\lambda - \epsilon_k}, \quad |\lambda\rangle = \sum_j \frac{\gamma_j |j\rangle}{\lambda - \epsilon_j}$$

$\gamma_j, \epsilon_j$ - given; solve for $\lambda$

- Number of level crossings as a function of type, i.e. the number ($n$) of integrals of motion

$$\#\text{ of crossings} = (N^2 - 5N + 2)/2 + n - 2k, \quad k = 1, 2, \ldots$$

Typically $\approx N^2/2$ crossings.

Any type 1 Hamiltonian has at least one crossing.

But for higher types it is also possible to have no crossings.

Owusu & Yuzbashyan, J. Phys. A (2011); Yuzbashyan, Shastry, Scaramazza, PRE (2016)

# Integrable Matrix Theory (IMT) – ensemble theory of quantum integrability

$$\text{Two matrices } T, E \text{ \& vector } |\gamma\rangle \Longleftrightarrow \text{type 1 } H(u) = T + uV$$

Other types arise similarly from two commuting matrices and a vector

To generate an integrable matrix with any prescribed number of integrals – generate *T*, *E* and $|\gamma>$

# Integrable Matrix Theory (IMT) – ensemble theory of quantum integrability

$\text{Two matrices } T, E \text{ \& vector } |\gamma\rangle \Longleftrightarrow \text{type 1 } H(u) = T + uV$

Other types arise similarly from two commuting matrices and a vector

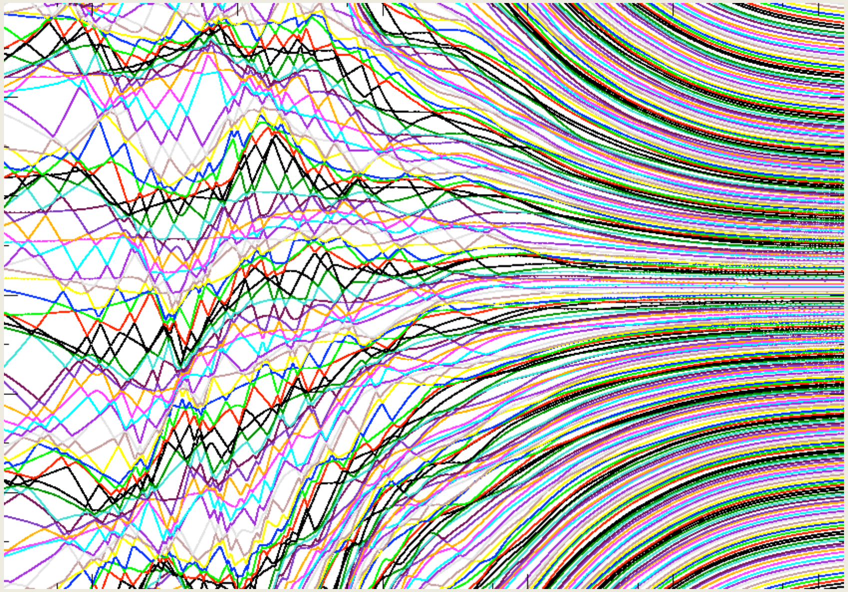
To generate an ensemble of integrable matrices with any prescribed number of integrals – generate an ensemble of $T$, $E$ and $|\gamma>$

Probability density function $P(T, E, \gamma)$ from rotational invariance as in Random Matrix Theory

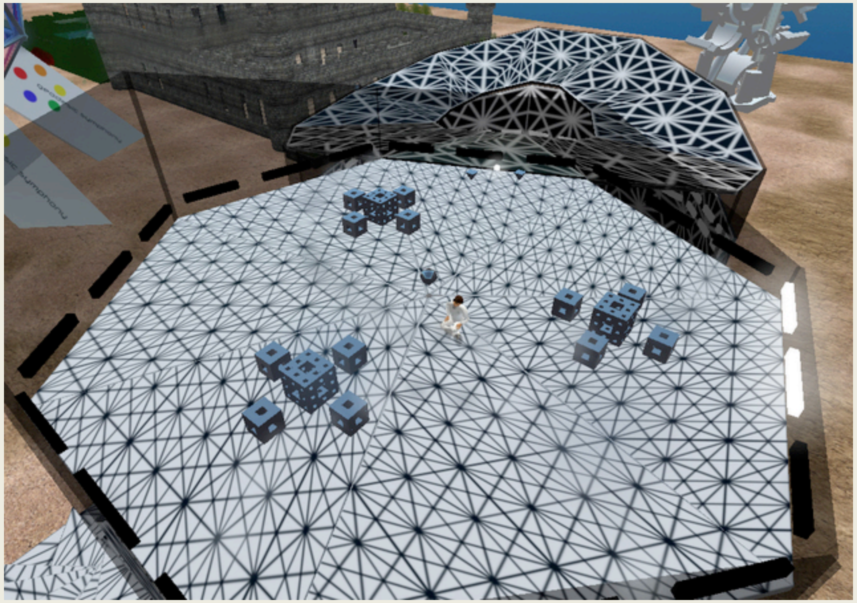
$T, E$ – random matrices, e.g. from GOE, $|\gamma\rangle$ – random vector

Now can study ensembles of integrable matrices and obtain integrable counterparts of the RMT results as opposed to only a spectral statistics of isolated integrable models!!

# Integrable Matrix Theory (IMT) – ensemble theory of quantum integrability



From Ben Simon's group website



Regularity by Jackal Ennui: "A little ode… to regularity and chaos"

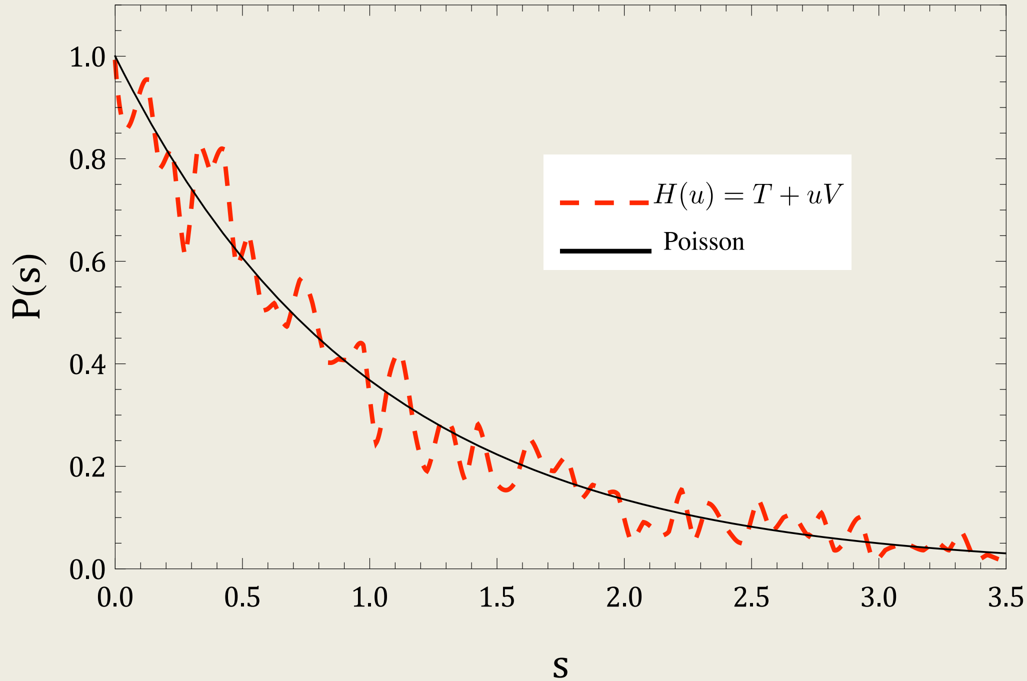
RMT – theory of quantum chaos

IMT – theory of quantum regularity

# Integrable Matrix Theory, Level Statistics

I. Statistics are typically Poisson as long as the number of integrals (= size-type) isn't too small

Scaramazza, Shastry, Yuzbashyan, PRE (2016)




Nearest neighbor level spacing distribution for a $4000 \times 4000$ time reversal invariant integrable Hamiltonian $H(u) = T + uV$ at $u = 1$

General member of the commuting family: $H(u) = \sum_{i=0}^{n-1} d_i H_i(u)$

Poisson because of superposition of many independent spectra

# Integrable Matrix Theory, Level Statistics

I. Statistics are typically Poisson as long as the number of integrals (= size-type) isn't too small

II. There are two exceptions to Poisson statistics

  A. There is a single, isolated value of the coupling $u = u_0$ where the level statistics of $H(u) = T + uV$ are Wigner-Dyson (here $u_0 = 0$).

$T, E$ – random matrices, e.g. from GOE, $|\gamma\rangle$ – random vector

# Integrable Matrix Theory, Level Statistics

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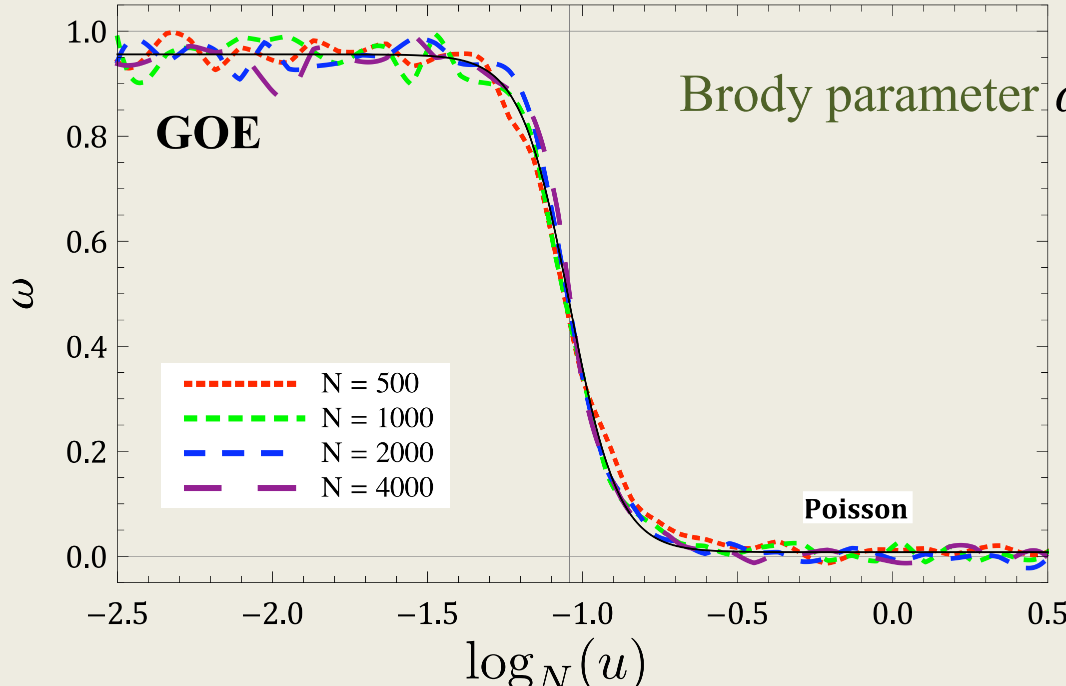
But it reverts to Poisson already at $(u - u_0) \propto 1/N$




Brody parameter $\omega$ as a function of $\log_N(u)$

Brody distribution:

$$P(s,\omega) = as^{\omega}e^{-bs^{\omega+1}}$$

$P(s,1) = \frac{\pi}{2}se^{-\frac{\pi}{4}s^2}$ – Wigner

$P(s,0) = e^{-s}$ – Poisson

$N \times N$ type 1, number of integrals $= N-1$

# Exceptions to Poisson Statistics in IMT

A. There is a single, isolated value of the coupling $u = u_0$ where the level statistics of $H(u) = T+uV$ are Wigner-Dyson.

$T, E$ – random matrices, e.g. from GOE, $|\gamma\rangle$ – random vector

A. Statistics are non-Poisson when normally uncorrelated parameters become correlated (atypical integrable model, special member of the family)

$T = f(E)$, $d_i = f(\varepsilon_i)$ – non-Poisson with strong level repulsion, e.g. BCS model has $d_i = \varepsilon_i$ (all-to-all energy-independent interactions)

General member of the commuting family: $H(u) = \sum_i d_i H_i(u) = T + uV$

Most general type 1 integrable model:

$$[H(u)]_{km} = u\gamma_k\gamma_m \left(\frac{d_k - d_m}{\varepsilon_k - \varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j - d_m}{\varepsilon_j - \varepsilon_m}\right)$$

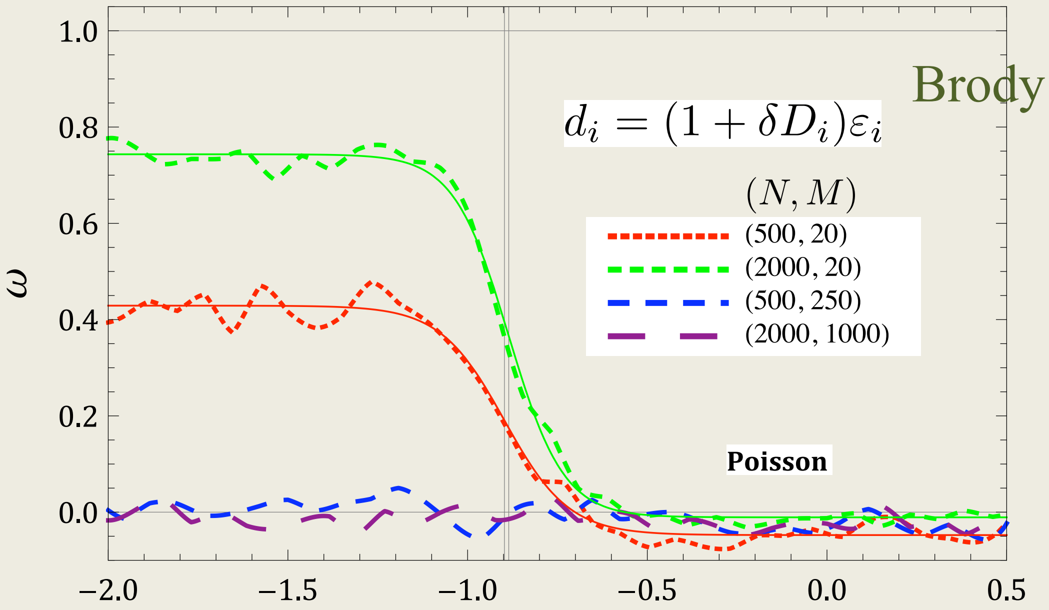
# Exceptions to Poisson Statistics in IMT

A. There is a single, isolated value of the coupling $u = u_0$ where the level statistics of $H(u) = T + uV$ are Wigner-Dyson.

$T, E$ – random matrices, e.g. from GOE, $|\gamma\rangle$ – random vector

A. Statistics are non-Poisson when normally uncorrelated parameters become correlated (atypical integrable model, special member of the family)

Reverts to Poisson at deviations $\delta \propto 1/N$ from such special members




Brody parameter $\omega$ as a function of $\log_N(\delta)$

Brody distribution:

$$P(s, \omega) = a s^{\omega} e^{-b s^{\omega+1}}$$

$P(s,1) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2}$ – Wigner

$P(s,0) = e^{-s}$ – Poisson

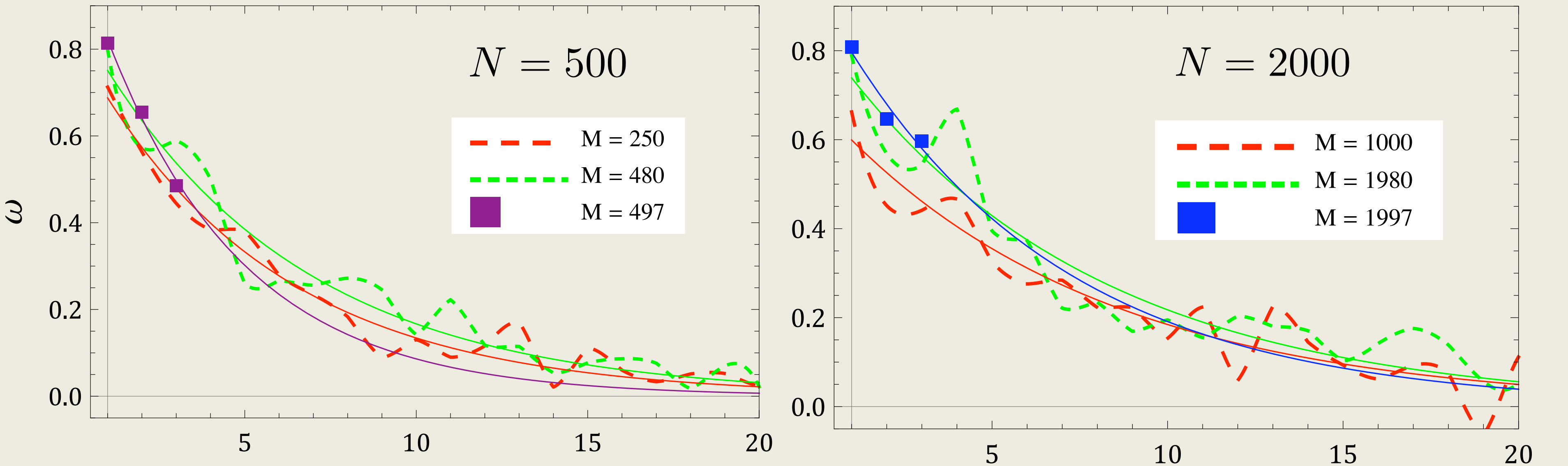
$D_i$ – $\mathcal{O}(1)$ random number

$N \times N$ type M, number of integrals $= N - M$, $u = 1$

**Q:** How many nontrivial integrals of motion must a system have so that its level statistics are Poisson?

# of nontrivial integrals = Size – Type = $N$ - $M$

$$H(u) = \sum_{i=1}^{k} d_i H_i(u), \quad k \le N - M$$




Brody parameter $\omega$ as a function of $k$ for $N \times N$ type $M$ matrices.
Fit: $a\exp(-bk/\ln N)$. $b = (1.13, 1.04; 0.99, 1.03)$ for $M = (250, 480; 1000, 1980)$

$\omega = 1$ – GOE, $\omega = 0$ – Poisson

# of integrals needed $\approx \ln N =$ log of Hilbert space dim $\propto$ particle #

Scaramazza, Shastry, Yuzbashyan, PRE (2016)

Proposed a simple notion of integrability for parameter-dependent N x N Hamiltonians

$$[H(u), H_1(u)] = 0 \text{ for all } u$$

Consequences:

1. Exact solution in terms of a single algebraic equation

1. \# of level xings as function of size and # of integrals. # of xings varies within the commuting family. Typically $N^2/2$ xings, but can also have no xings when the # of integrals is less then maximal

1. Integrable Matrix Theory – theory of quantum regularity. Typical statistics are Poissonian when the # of integrals > ln N. Guaranteed Wigner-Dyson at isolated value of the parameter and for special, “correlated members” of the commuting family (explains BCS). Further: ergodicity etc.

1. Generalized Gibbs Ensemble works when the # of integrals are maximal. Has to do with localization of the eigenstates of H(u). Does it work for fewer integrals?

1. Solvable multi-state Landau-Zener problems are integrable matrices. Can we solve new such problems?

$H(t) = A + tB$, where $A, B$ - $N \times N$ Hermitian matrices.
$t$ goes from $-\infty$ to $+\infty$. Determine $p(i \to k)$

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