Integrable Matrix Theory

(Theory of integrable Hamiltonians with finite number of levels)

What is quantum integrability and who cares?

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Classical Mechanics

Definition: A classical Hamiltonian $H_0(p, q)$ with n degrees of freedom (n coordinates) is integrable if it has the maximum possible number (n) of functionally independent Poisson-commuting integrals $\{H_i(p, q), H_j(p, q)\} = 0;\ i,j = 0,1\ldots n$

⇩

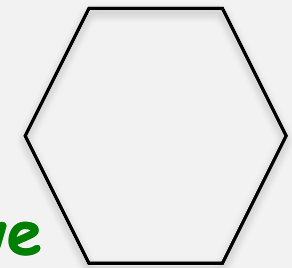
- ✓ *Unambiguous separation of integrable from nonintegrable (generic)*

- ✓ *Various properties that don't have to be verified on a case by case basis*

Q: What is quantum integrability? How is it defined?

Think finite, **N x N**, matrix even with very large **N**

Example: Hubbard model on a ring



$$H = \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times \end{pmatrix}$$

Given matrix H how do we tell if it's integrable?

How do we generate (an ensemble of) integrable matrices?

No way! Not even a definition! (See e.g. B. Sutherland, *Beautiful Models* (2004), Caux & Mossel (2011), E.Y. & Shastry (2013) for review)

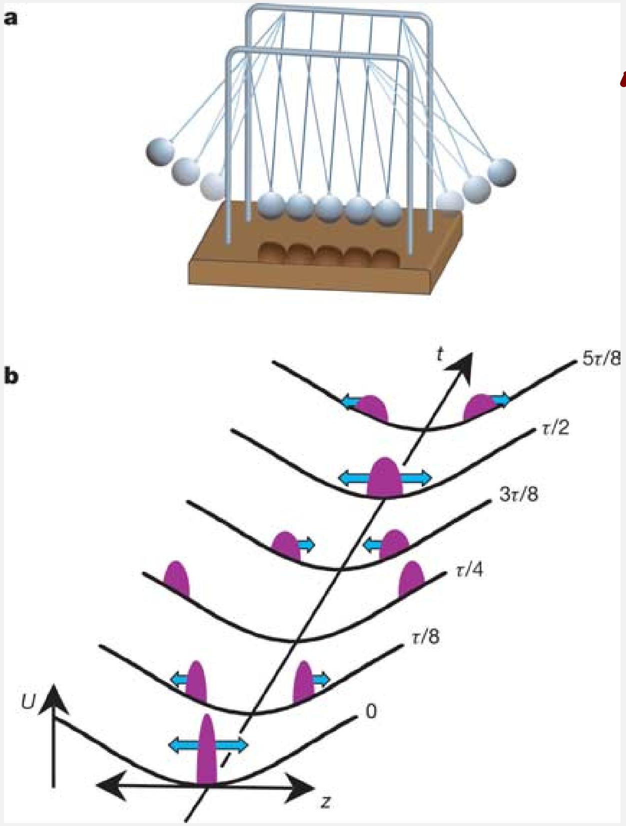
no natural notion of an integral of motion: for any H can find a full set of H_k such that $[H, H_k]=0$

$$H = \sum_{1}^{N} E_n |n\rangle\langle n|, \qquad H_k = |k\rangle\langle k|$$

Alternatively, can consider powers of H_0

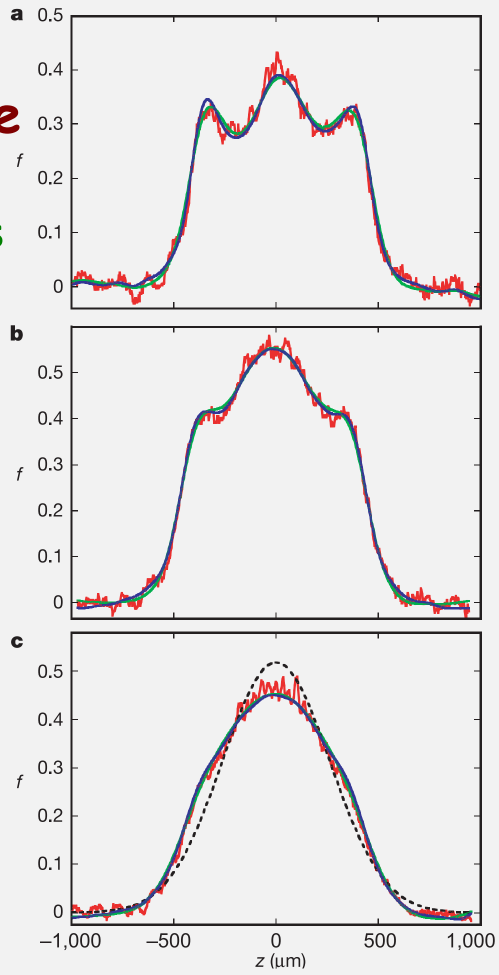
$$H_k = \sum_{n=1}^{N} a_n H_0^n$$

Who cares? – rise of integrability



A quantum Newton's cradle

T. Kinoshita, T. Wenger, D. Weiss
Nature (2006)

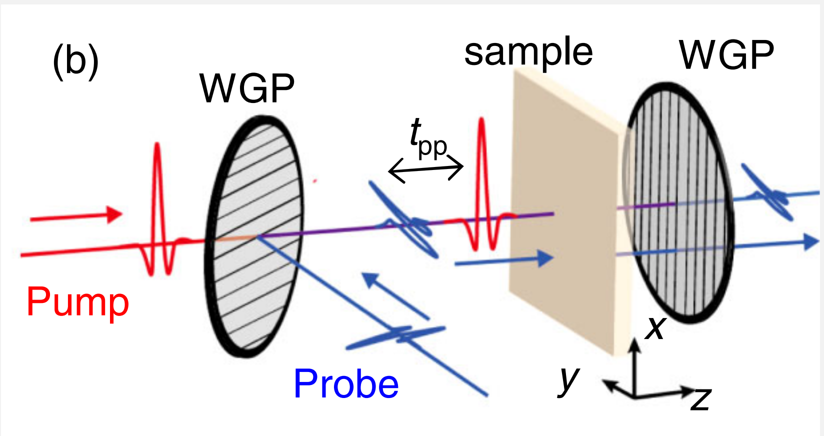


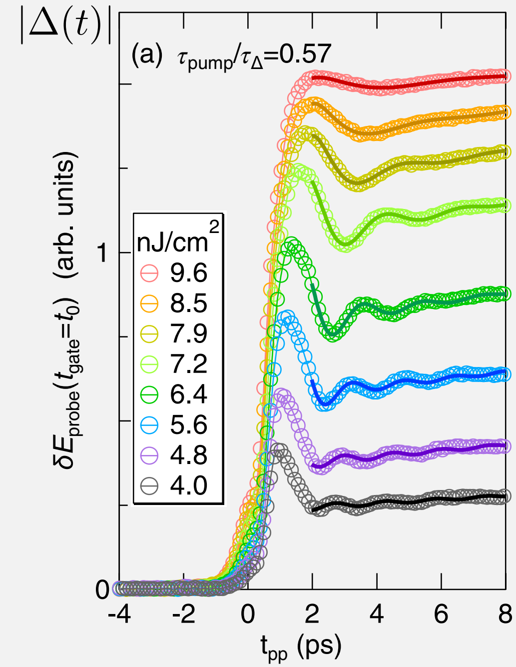
"^{87}Rb atoms … do not noticeably equilibrate even after thousands of collisions. Our results are probably explainable by the well-known fact that a homogeneous 1D Bose gas with point-like collisional interactions is *integrable*."

PRL 111, 057002 (2013) PHYSICAL REVIEW LETTERS week ending 2 AUGUST 2013

Higgs Amplitude Mode in the BCS Superconductors $Nb_{1-x}Ti_xN$ Induced by Terahertz Pulse Excitation

Ryusuke Matsunaga,[1] Yuki I. Hamada,[1] Kazumasa Makise,[2] Yoshinori Uzawa,[3] Hirotaka Terai,[2] Zhen Wang,[2] and Ryo Shimano[1]





$\tau_\Delta = \hbar/\Delta_0 \approx 3\text{ps}$ – timescale on which $|\Delta(t)|$ evolves

$$|\psi(0)\rangle = |\text{noneq. state produced by the pulse}\rangle$$

$$\hat{H}_{\text{BCS}} = \sum_{i,\sigma} \epsilon_i \hat{c}^\dagger_{i\sigma}\hat{c}_{i\sigma} - u \sum_{i,j} \hat{c}^\dagger_{i\uparrow}\hat{c}^\dagger_{i\downarrow}\hat{c}_{j\downarrow}\hat{c}_{j\uparrow}$$

$$i\frac{d|\psi\rangle}{dt} = \hat{H}_{\text{BCS}}|\psi\rangle$$

$$|\Delta(t)| = \Delta_\infty + a\frac{\cos(2\Delta_\infty t + \alpha)}{\sqrt{\Delta_\infty t}}$$

Yuzbashyan, Tsyplyatyev, Altshuler, PRL (2006)

Integrable systems follow Generalized Gibbs Ensemble?

$$\rho = Z^{-1} e^{-\sum_i \beta_i H_i} \qquad \langle O(t) \rangle_{t\to\infty} = \mathrm{Tr}\,\rho O$$

$$\langle \mathrm{in} | H_i | \mathrm{in} \rangle = \mathrm{Tr}\,\rho H_i$$

Does it work?

Sometimes yes, sometimes no – depends on the system, observable and the the set of integrals

- ✓ **Works for simple models, e.g. 1D hard-core bosons & Luttinger liquids** Rigol et. al. PRL (2007); Cazalilla PRL (2006)
- ✓ **Fails for models with bound states, e.g. XXZ or attractive Lieb-Liniger** Pozsgay et. al. PRL (2014); Goldstein & Andrei, arXiv:1405.4224
- ✓ **Fails for global observables except for uncorrelated free fermions** Gurarie, J. Stat. Mech. (2013)
- ✓ **Does work for XXZ if new integrals are added** Ilievski et. al. PRL (2015)

Integrable systems follow Generalized Gibbs Ensemble?

$$\rho = Z^{-1} e^{-\sum_i \beta_i H_i}$$

$$\langle \mathrm{in} | H_i | \mathrm{in} \rangle = \mathrm{Tr}\, \rho H_i$$

$$\langle O(t) \rangle_{t \to \infty} = \mathrm{Tr}\, \rho O$$

Does it work?

Sometimes yes, sometimes no – depends on the system, observable and the the set of integrals

How do we determine if we have the "right" set of integrals and the criteria for the validity of GGE?

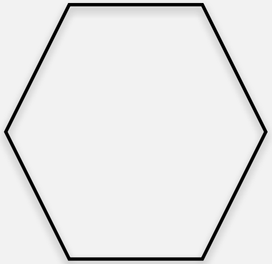
Need to know what quantum integrability is! Otherwise, GGE is a mysterious, essentially unfalsifiable conjecture.

Do Classical Mechanics first before going Quantum?!

Properties (??) of quantum integrable models

- ✓ Generalized Gibbs Ensemble: *does it work?*
- ✓ Exact solution via Bethe's Ansatz: *but any matrix can be "exactly solved"* $\det(H - \lambda I) = 0$
- ✓ Commuting integrals: *any matrix has them*
- ✓ Energy level crossings in violation of Wigner-v. Neumann non-crossing rule: *often, but not always. Can have crossings without integrability.*

Example: Hubbard model on a ring



- ✓ Poisson level statistics: *not always – e.g. BCS model. Non-integrable models can be Poisson.*

In the absence of a clear notion, have to verify every property separately on a case by case basis

Properties of quantum integrable models: Exact Solution

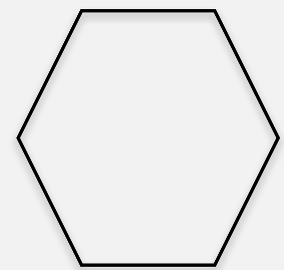
Example: Hubbard model

$$\hat{H} = T \sum_{j,s=\uparrow\downarrow} (c^{\dagger}_{j\,s} c_{j+1\,s} + c^{\dagger}_{j+1\,s} c_{j\,s}) + U \sum_{j} \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

H depends linearly on one parameter ***u=U/T***

tight-binding + onsite interactions, electrons on a ring

N=6 cites, 3 spin-up, ***M=3*** spin-down



Exact Solution (Bethe's Ansatz):

E.H. Lieb and F.Y.Wu (1969)

$$e^{6ik_j} = \prod_{\alpha=1}^{3} \frac{\Lambda_\alpha - \sin k_j - iu/4}{\Lambda_\alpha - \sin k_j + iu/4}, \qquad \prod_{\alpha=1}^{3} \frac{\Lambda_\alpha - \Lambda_\beta + iu/2}{\Lambda_\alpha - \Lambda_\beta + iu/2} = -\prod_{j=1}^{6} \frac{\Lambda_\beta - \sin k_j - iu/4}{\Lambda_\beta - \sin k_j - iu/4}$$

9 coupled nonlinear equations

$$E = -\sum_{j=1}^{6} 2\cos k_j, \qquad P = \sum_{j=1}^{6} k_j, \qquad |P, S, S_z, \ldots\rangle = \ldots$$

But cf. $\det(H - \lambda I) = 0$

Commuting integrals (conservation laws)
Example: Hubbard model

$$\hat{H} \equiv \hat{H}_0(u) = \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{js} c_{j+1\,s} + c^\dagger_{j+1\,s} c_{js}) + u \sum_{j=1}^{N} \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

$$\hat{n}_{j\sigma} = c^\dagger_{js} c_{j\,s}$$

$$\hat{H}_1(u) = -i \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{j+2s} c_{js} - c^\dagger_{js} c_{j+2s}) - iu \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{j+1s} c_{js} - c^\dagger_{js} c_{j+1s})(\hat{n}_{j+1,-s} + \hat{n}_{j,-s} - 1)$$

$$[\hat{H}_0(u), \hat{H}_1(u)] = 0 \quad \text{for all } u$$

B. S. Shastry, PRL (1986)

$H_2(u)$, $H_3(u)$, $H_4(u)$,… **- in principle, infinitely many integrals of motion can be found from Shastry's transfer matrix (but not all of them are nontrivial for finite *N*)**

But any Hamiltonian has commuting integrals. So what's special about Hubbard?

The Hamiltonian and the first integral are linear in a real parameter *u*.
Higher integrals are polynomial in *u*.

Properties of quantum integrable models: Level crossings

Example: Hubbard model

$$\hat{H} = T \sum_{j,s=\uparrow\downarrow} (c^\dagger_{j s} c_{j+1\, s} + c^\dagger_{j+1\, s} c_{j s}) + U \sum_j \hat{n}_{j\uparrow} \hat{n}_{j\downarrow}$$

H depends linearly on one parameter ***u=U/T***

Q: How do eigenvalues look like as functions of u?

For a typical *H(u)* energy levels with same quantum numbers (spin, momentum etc.) never cross – noncrossing rule

Hund (1927), Neumann & Wigner (1929)

Properties of quantum integrable models: Level crossings

Example: Hubbard model

$$\hat{H} = T \sum_{j,s=\uparrow\downarrow} (c^\dagger_{js} c_{j+1\,s} + c^\dagger_{j+1\,s} c_{js}) + U \sum_j \hat{n}_{j\uparrow} \hat{n}_{j\downarrow}$$

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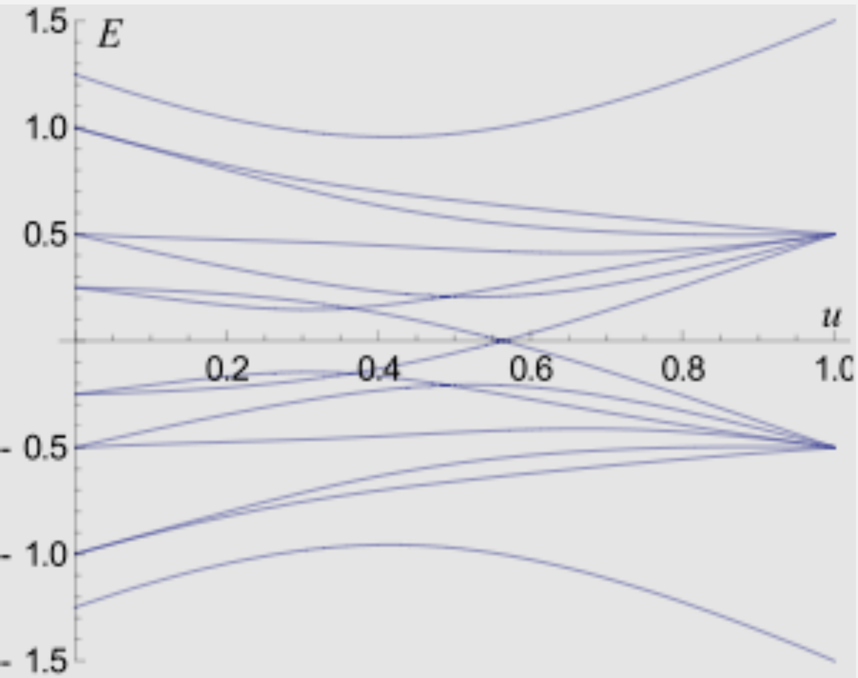


Energies for a **14 x 14** block of 1d Hubbard on six sites characterized by a complete set of quantum numbers

H(u)=A+uB is a **14 x 14** Hermitian matrix linear in real parameter ***u***

"The noncrossing rule is apparently violated in the case of the 1d Hubbard Hamiltonian for benzene molecule *[six sites]*…"

Heilmann and Lieb (1971)

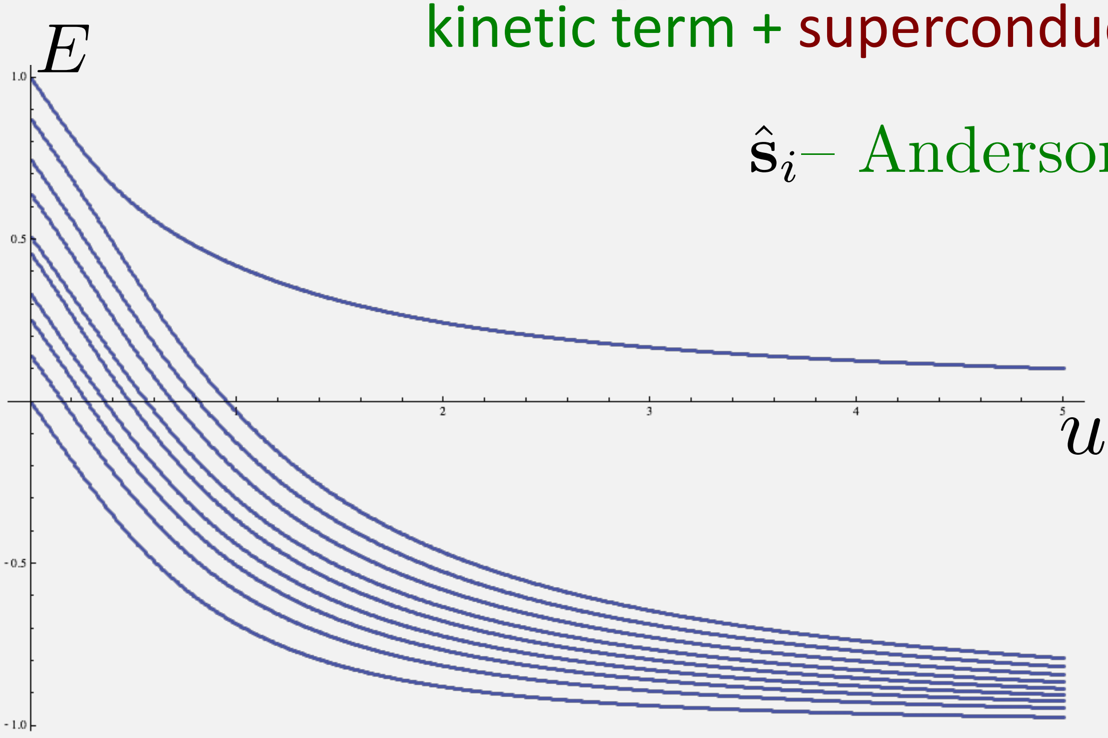
Properties of quantum integrable models: Level crossings

Counterexample: BCS (Richardson) model

$$\hat{H}_{\mathrm{BCS}} = \sum_i 2\varepsilon_i \hat{s}_i^z - u \sum_{i,j} \hat{s}_i^- \hat{s}_j^+ = \sum_i 2\varepsilon_i \hat{H}_i$$

kinetic term + superconducting interactions

$\hat{\mathbf{s}}_i$ – Anderson pseudospins



Energies for a 10 x 10 block of the BCS model for 10 levels characterized by a complete set of quantum numbers

Gaudin magnet integrable family

$$\hat{H}_i(u) = \hat{s}_i^z - u \sum_{j \neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j}$$

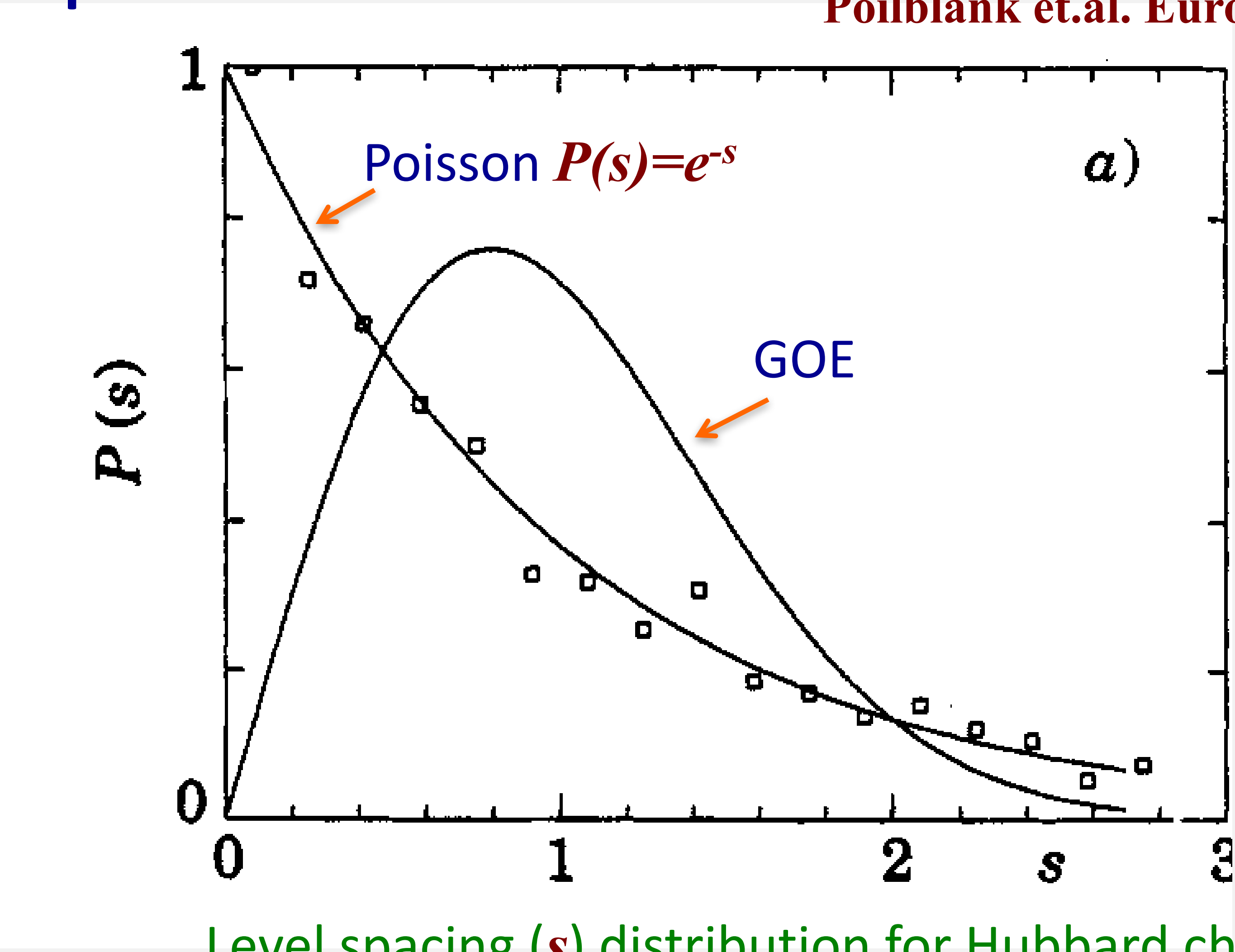
$$[\hat{H}_i(u), \hat{H}_j(u)] = 0$$

$$[\hat{H}_{\mathrm{BCS}}(u), \hat{H}_i(u)] = 0$$

Properties of quantum integrable models: Poisson statistics

Example: Hubbard model

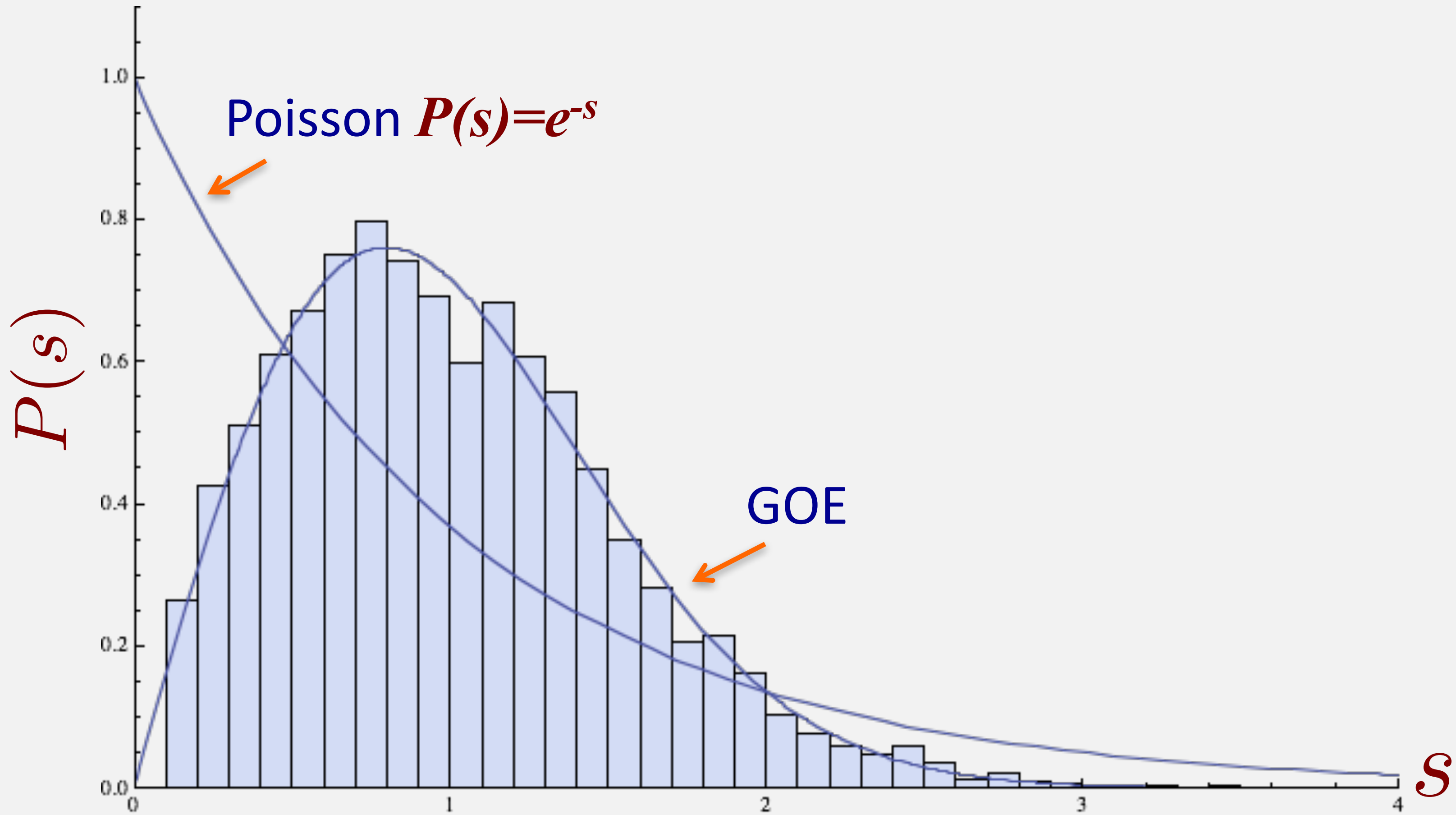
Poilblank et.al. Europhys. Lett. (1993)



Level spacing (*s*) distribution for Hubbard chain with 12 sites at ¼ filling, total momentum $P=\pi/6$, spin $S=0$

Properties of quantum integrable models: Poisson statistics

Counterexample: BCS (Richardson) model



Level spacing (s) distribution for the BCS model for $N=5000$ levels and *1* Copper pair

See also **Relano, Dukelsky et. al. PRE (2004)**

Notion of Quantum Integrability: What are we looking for?

Definition: Quantum Hamiltonian H_0 is integrable if…

Consequences:

1. Exact Solution
2. Generate (ensembles of) integrable models
3. Commuting integrals $[H_i, H_j]=0;\ i, j=0,1\ldots$
4. Energy level crossings?
5. Poisson level statistics *and exceptions*
6. Generalized Gibbs Ensemble for dynamics?

Classical integrability has it all

Definition: A classical Hamiltonian $H_0(p, q)$ with n degrees of freedom (n coordinates) is integrable if it has the maximum possible number (n) of functionally independent Poisson-commuting integrals $\{H_i, H_j\}=0;\ i,j=0,1\ldots n$

⇓

Consequences:

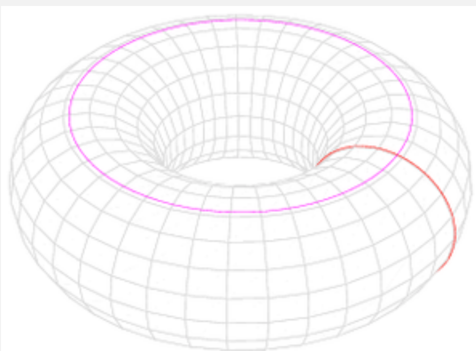
1. **Exact solution:** the dynamics of $H_i(p, q)$ is exactly solvable by quadratures **(Liouville-Arnold theorem)**
2. Poisson level statistics semi-classically *[Berry & Tabor (1976)]* except when $E(n_1, n_2, \ldots)$ is flat in $n_1, n_2, \ldots$, i.e. decoupled harmonic oscillators
3. **Generalized Microcanonical Ensemble typically holds for dynamics** *[Arnold, Math. Methods of CM, E.Y. ArXiv:1509.06351]*

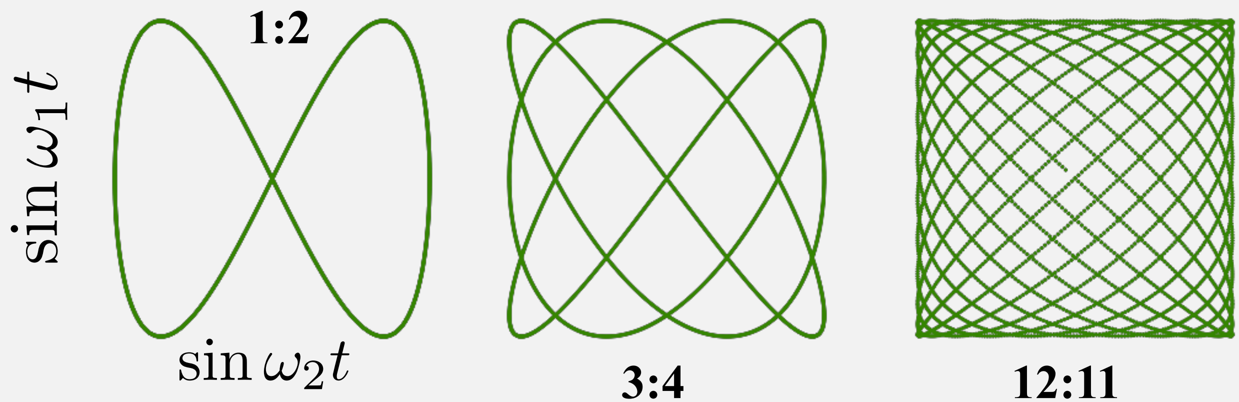
Generalized Gibbs Ensemble DeMystified in Classical Mechanics

Dynamics is on "invariant torus" – n-dim portion of $2n$-dim phase-space cut out by integrals of motion $H_1(p,q,)$=const, $H_2(p,q,)$ =const, …, $H_n(p,q,)$=const

There are n typically incommensurate frequencies $\omega_1, \omega_2, \ldots, \omega_n$ (non-resonant torus)

Lissajous figures





Theorem about averages (Arnold, *Math. Methods of CM*):

For a non-resonant torus and any "reasonable" observable $O(p,q,)$

time average = phase-space average over the torus

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T O\left(t\right) dt = \int O(\varphi)\frac{d\varphi}{(2\pi)^n}$$

Generalized Gibbs Ensemble DeMystified in Classical Mechanics

Theorem about averages (Arnold, *Math. Methods of CM*):

For a non-resonant torus and any "reasonable" observable *O(p,q,)*

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Going back to the original variables *p & q* and using the fact that this is a canonical transform can prove Generalized Microcanonical distribution

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T O\left(t\right)dt = \int O(p,q)\rho(p,q)dpdq$$

E.Y. ArXiv:1509.06351

$$\rho(p,q) = V^{-1}\prod_{k=1}^{n}\delta\left(H_k(p,q)-\alpha_k\right)$$

Works for any system size (any *n*)!

Exceptions: resonant tori

Additive integrals, thermodynamic limit

$$H_k \propto n$$

$$n \to \infty$$

not always the case???

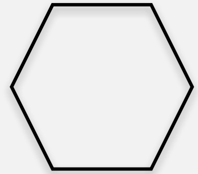
Generalized (canonical) Gibbs

See e.g. Ruelle, Stat. Mech.: Rigorous Results (1999)

$$\rho(p,q) = Z^{-1}\exp\left(-\sum_k \lambda_k H_k(p,q)\right)$$

Can we develop a similar sound notion of integrability in Quantum Mechanics – for N x N Hermitian matrices (Hamiltonians)?

Hints from Hubbard study, *u=U/T*:
Yuzbashyan, Altshuler, Shastry (2002)



$$H(u) = T + uV$$

u – real parameter,
T,V – *N* x *N* Hermitian matrices

Nontrivial integrals depend on a real parameter (interaction or external field) in a certain fixed way. *Always at least one linear integral. Same is the case for other known parameter-dependent models*

- 1d Hubbard, XXZ spin chain (*u* = anisotropy): integrals are polynomial in *u*
- Gaudin magnets (all integrable pairing models): *u*=hyperfine interaction, Hamiltonian and all integrals are linear in *u*

$$\hat{H}_i(u) = \hat{s}_i^z - u \sum_{j \neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j}$$

$$[\hat{H}_i(u), \hat{H}_j(u)] = 0$$

Proposed solution: fix parameter dependence

Let $H(u) = T + uV$ *u* – real parameter, *T,V* – *N* x *N* Hermitian matrices

Suppose we require a commuting partner also linear in *u:*

$$H_1(u) = T_1 + uV_1$$

$$[H(u), H_1(u)] = 0$$

$$\Updownarrow$$

$$[V, V_1] = 0, \quad [T, V_1] = [T_1, V], \quad [T, T_1] = 0$$

These commutation relations severely constraint matrix elements of *T*. For a generic/typical *H(u)* – no commuting partners except itself and identity. *Now can separate generic (no partners) from special (integrable).*

Proposed solution: fix parameter dependence

Let $H(u) = T + uV$ ***u*** – real parameter, ***T,V*** – ***N*** x ***N*** Hermitian matrices

Suppose we require a commuting partner also linear in ***u:***

$$H_1(u) = T_1 + uV_1$$

$$[H(u), H_1(u)] = 0$$

$$\Updownarrow$$

$$[V, V_1] = 0, \quad [T, V_1] = [T_1, V], \quad [T, T_1] = 0$$

In the simplest 3 x 3 case – single algebraic constraint on matrix elements T_{ij}

Xing condition: $\exists u_0 : \text{Discriminant}_\lambda \, |H(u_0) - \lambda I| = 0$ also single constraint

Moreover, xing condition = commutation condition, i.e.

$$[H_0(u), H_1(u)] = 0 \Longleftrightarrow \text{xings in } 3 \times 3 \text{ case!}$$

N x N Hamiltonians linear in a parameter separate into *two distinct classes = good notion of integrability*

$H(u) = T + uV$

No commuting partners linear in u other than itself and identity (typical) – nonintegrable, need $N^2/2$ real parameters to specify $H(u)$

Nontrivial commuting partners $H_k(u)=T_k+uV_k$ exist – integrable, turns out need less than $4N$ parameters – measure zero in the space of linear Hamiltonians

Classification by the number n of commuting partners

$n = N\text{-}1$ (maximum possible) – type 1 integrable system

$n = N\text{-}2$ – type 2

$n = N\text{-}3$ – type 3

…

$n = N\text{-}M$ – type M

…

Definition: A Hamiltonian operator $H \equiv H_0(u) = T_0 + uV_0$ is integrable if it has $n \geq 1$ nontrivial linearly independent commuting partners $H_i(u) = T_i + uV_i$

$$[H_i(u), H_j(u)] = 0 \text{ for all } u \text{ and } i, j = 0, \ldots, n-1$$

General member of the commuting family: $h(u) = \sum_{i=1}^{n} d_i H_i(u)$

Known parameter-dependent integrable models fall under this definition:

- **1d Hubbard model: *u=U/T*, Hamiltonian and first integral are linear in *u***
- **integrable XXZ spin chain: *u* = anisotropy, *H₀(u)* and *H₁(u)* are linear in *u***
- **Gaudin magnets (all integrable pairing models): *u*=spin exchange,**

  Hamiltonian and all integrals are linear in *u*

$$\hat{H}_i(u) = \hat{s}_i^z - u \sum_{j \neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j} \qquad [\hat{H}_i(u), \hat{H}_j(u)] = 0$$

$\mathbf{s}_i$ – quantum spins ϵ_i – real parameters

What can we achieve with this notion of quantum integrability? – quite a lot!!

Definition: Quantum Hamiltonian H_0 is integrable if…

Consequences:

1. Exact Solution
2. Generate (ensembles of) integrable models
3. Commuting integrals $[H_i, H_j]=0;\ i, j=0,1\ldots$
4. Energy level crossings?
5. Poisson level statistics *and exceptions*
6. Generalized Gibbs distribution for dynamics?

What can we achieve with this notion of quantum integrability? – quite a lot!!

✓ **Construct (ensembles of) integrable models with any given number *n* of integrals!**

$$[V, V_1] = 0, \quad [T, V_1] = [T_1, V], \quad [T, T_1] = 0$$

Simplest case: *n=N-1* (type 1 – max # of integrals – analog of classical integrability)

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Every type-1 family contains a "reduced" Hamiltonian

$$\Lambda(u) = E + u|\gamma\rangle\langle\gamma|$$

Hermitian matrix E Arbitrary vector $|\gamma\rangle$

N commuting $N \times N$ Hermitian matrices $H_i(u)$

General member of the commuting family: $H(u) = \sum_{i=1}^{N} d_i H_i(u) = T + uV$

$$[H(u)]_{km} = u\gamma_k\gamma_m \left(\frac{d_k - d_m}{\varepsilon_k - \varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2 \left(\frac{d_j - d_m}{\varepsilon_j - \varepsilon_m}\right)$$

ϵ_k - eigenvalues of E,γ_k - components of $|\gamma\rangle$ ($2N$ arbitrary real parameters)

d_k- eigenvalues of T - another N arbitrary real numbers to fix a linear combination *within* the family. By construction $[T, E] = 0$.

Constructed *all* *n = N-1, N-2, N-3* (types 1, 2, 3) and *some* for arbitrary other *n*

What can we achieve with this notion of quantum integrability? – quite a lot!!

✓ **Exact solution through a single algebraic equation for all types (cf. Bethe Ansatz)**

(type 1) $$\sum_j \frac{\gamma_j^2}{\lambda-\epsilon_j} = \frac{1}{u}, \quad E_k = \frac{u\gamma_k^2}{\lambda-\epsilon_k}, \quad |\lambda\rangle = \sum_j \frac{\gamma_j |j\rangle}{\lambda-\epsilon_j}$$

γ_j, ϵ_j - given; solve for λ

✓ **Number of level crossings as a function of the # (n) of commuting partners in an integrable family**

$$\#\text{ of xings} = (N^2-5N+2)/2 + n - 2k, \quad k = 1, 2, \ldots$$

Typically $\sim N^2/2$ xings

But it's also possible to have no xings

✓ **Yang-Baxter formulation**

scattering matrix $$S_{ij} = \frac{(\epsilon_j-\epsilon_i)I + 2g\Pi_{ij}}{(\epsilon_j-\epsilon_i) + g\,(\gamma_i^2+\gamma_j^2)}$$

$$S_{ik}S_{jk}S_{ij} = S_{ij}S_{jk}S_{ik}$$

Applications:1d Hubbard model (6 sites, 3 up/3 down spins

- Each block is characterized by a complete set of quantum #s (*P*, S^2, S_z…)
- We determine the type of each block

# of nontrivial integrals = Size – Type

Momenta $P = \pi/6, 5\pi/6$	
Size of the block	Its Type
8×8	Type 3
3×3	Type 1
16×16	Type 12
14×14	Type 3
3×3	Type 1

Momenta $P = \pi/3, 2\pi/3$	
Size of the block	Its Type
12×12	Type 7
14×14	Type 11
4×4	Type 1
2×2	—
16×16	Type 6

Results for Hubbard:

- In most blocks – exact solution in terms of a single equation – vast simplification over Bethe Ansatz (9 equations)!

- New symmetries in 1d Hubbard! # of nontrivial integrals linear in u=U/T is 14-3-1=10. Only one such integral was identified before

Applications: BCS (Richardson) and Gaudin models

$$\hat{H}_{\mathrm{BCS}} = \sum_i 2\varepsilon_i \hat{s}_i^z - u\sum_{i,j} \hat{s}_i^- \hat{s}_j^+ = \sum_i 2\varepsilon_i \hat{H}_i$$

Gaudin magnet integrable family

$$\hat{H}_i(u) = \hat{s}_i^z - u\sum_{j\neq i} \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j}{\epsilon_i - \epsilon_j}$$

One spin-flip sector $J_z = \{\max -1, \min +1\}$ is type-1 with $\gamma_i^2 = 2s_i$.
Other sectors – other types.

General member of the commuting family: $H(u) = \sum_{i=1}^{N} d_i H_i(u) = T + uV$

$$[H(u)]_{km} = u\gamma_k\gamma_m \left(\frac{d_k - d_m}{\varepsilon_k - \varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2 \left(\frac{d_j - d_m}{\varepsilon_j - \varepsilon_m}\right)$$

Set $d_i = \varepsilon_i$ and $\gamma_i = 1$ to get BCS, $\hat{H}_{\mathrm{BCS}} = \Lambda(u) = E + |\gamma\rangle\langle\gamma|$

Every type-1 family contains a "reduced" Hamiltonian

Integrable Matrix Theory (IMT) – ensemble theory of quantum integrability

Two matrices $[T, E] = 0$ & vector $|\gamma\rangle \iff$ type 1 $H(u) = T + uV$

Other types similarly given in terms of two commuting matrices and a vector $|\gamma>$

To generate an integrable matrix with any prescribed number of integrals – generate T, E and $|\gamma>$

Integrable Matrix Theory (IMT) – ensemble theory of quantum integrability

$$\text{Two matrices } [T,E]=0 \text{ \& vector } |\gamma\rangle \iff \text{type 1 } H(u)=T+uV$$

Other types similarly given in terms of two commuting matrices and a vector $|\gamma>$

To generate an *ensemble* of integrable matrices with any prescribed number of integrals – generate an *ensemble* of *T, E* and $|\gamma>$

Type 1 in the shared eigenbasis of *T & E:*

$$[H(u)]_{km} = u\gamma_k\gamma_m\left(\frac{d_k-d_m}{\varepsilon_k-\varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j-d_m}{\varepsilon_j-\varepsilon_m}\right)$$

d_k, ε_k – eigenvalues of T, E. γ_k – components of $|\gamma\rangle$

Q: What is the natural probability density function for this ensemble? How do we generate most typical/random integrable models?

$$P(T,E,\gamma) = ?$$

Two matrices $[T, E] = 0$ & vector $|\gamma\rangle \Longleftrightarrow$ type 1 $H(u) = T + uV$

Q: **What is the natural probability density function for this ensemble? How do we generate most typical/random integrable models?**

$$P(T, E, \gamma) = ?$$

Similar to Random Matrix Theory, two ways to derive *P(T, E, γ)*

1. **Maximize the entropy of the distribution (least information, most unbiased choice. Generalized Gibbs Ensemble follows from the same principle)**

$$S[P] = -\langle \ln(P) \rangle = -\int P(T, E, \gamma) \ln(P(T, E, \gamma)) d\gamma \, dT \, dE$$

$$\langle \mathrm{Tr}\, T \rangle, \langle \mathrm{Tr}\, T^2 \rangle, \langle \mathrm{Tr}\, E \rangle, \langle \mathrm{Tr}\, E^2 \rangle = \mathrm{const}$$

Integration over constrained space: $[T, E] = 0, \quad |\gamma| = 1$

1. **Statistical independence + rotational invariance of *P(T, E, γ)*. *T, E, γ* are given by RMT results projected onto the constrained space *[T, E] = 0***

Integrable Matrix Theory (IMT)

Both approaches yield the same answer, *β =1,2* for Hermitian, real-symmetric

$$P(d,\varepsilon,\gamma) \propto \delta\left(1-|\gamma|^2\right)\prod_{i<j}|\varepsilon_i-\varepsilon_j|^\beta |d_i-d_j|^\beta e^{-\sum_k \varepsilon_k^2} e^{-\sum_k d_k^2}$$

d_k, ε_k – eigenvalues of T, E. γ_k – components of $|\gamma\rangle$

T, E - random matrices with uncorrelated eigenvalues

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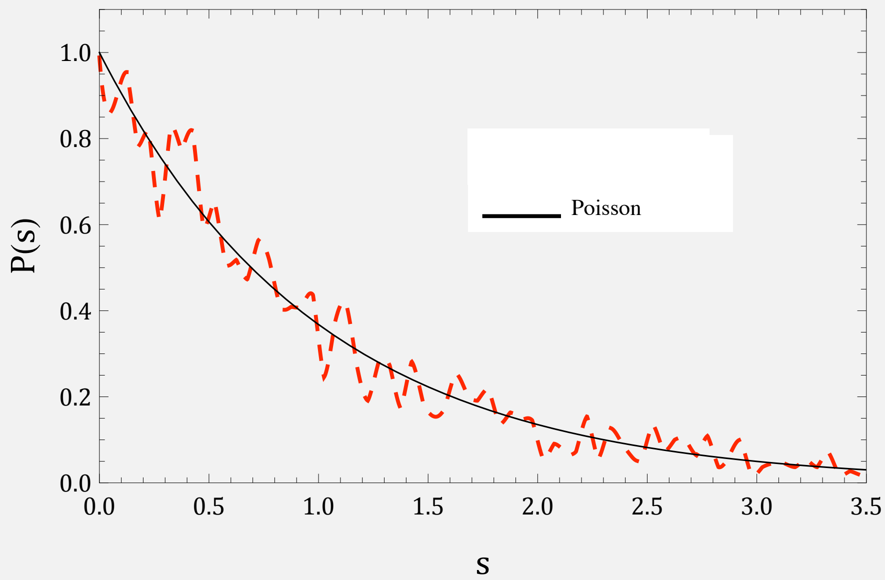
Similar but more involved construction for other types, see arXiv:1511.02446

Now can study *ensembles of integrable matrices* and obtain integrable counterparts of RMT results as opposed to only a spectral statistics of specific integrable models

Integrable Matrix Theory, Level Statistics (numerics)

I. Statistics are typically Poisson as long as the # of integrals (=size-type) isn't too small



Level spacing distribution for a 4000 x 4000 real symmetric integrable matrix $H(u)=T+uV$ at $u=1$

Integrable Matrix Theory, Level Statistics

I. Statistics are typically Poisson as long as the # of integrals (=size-type) isn't too small

II. There are two exceptions to Poisson statistics

   A. At $u=0$ the statistics is Wigner-Dyson. Can engineer any statistics in $H(u)=T+uV$ at isolated value of the coupling $u=u_0$

T, E - random matrices with uncorrelated eigenvalues d_i, ε_i

Can arbitrarily chose either T or V, but not both, i.e. can have a desired statistics e.g. at $u=0$, but not at all u

Integrable Matrix Theory, Level Statistics (numerics)

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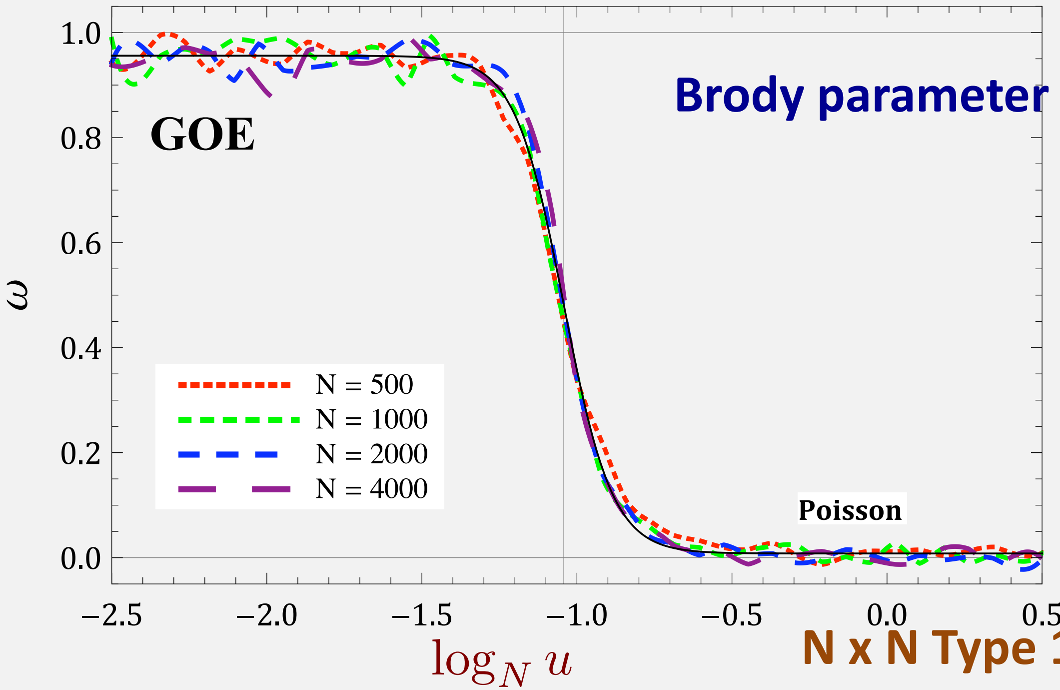
But it becomes Poisson already at $(u-u_0)\propto 1/N$



Brody parameter ω as a function of $\log_N(u)$

Brody distribution:

$$P(s,\omega) = as^{\omega}e^{-bs^{\omega+1}}$$

$P(s,1) = \frac{\pi}{2}se^{-\frac{\pi}{4}s^2}$ - Wigner

$P(s,0) = e^{-s}$ - Poisson

N x N Type 1, # of integrals = N - 1

Exceptions to Poisson Statistics in IMT

A. **At $u=0$ the statistics is Wigner-Dyson. Can engineer any statistics in $H(u)=T+uV$ at isolated value of the coupling $u=u_0$**

T, E - random matrices with uncorrelated eigenvalues d_i, ε_i

A. **Statistics is non-Poisson when normally uncorrelated parameters become correlated (atypical integrable models)**

$T=f(E)$, $d_i=f(\varepsilon_i)$ - non-Poisson with strong level repulsion, e.g. BCS model has $d_i=\varepsilon_i$

General member of the commuting family: $H(u)=\sum_{i=1}^{N} d_i H_i(u)=T+uV$

Type 1 in the shared eigenbasis of T & E:

$$[H(u)]_{km}=u\gamma_k\gamma_m\left(\frac{d_k-d_m}{\varepsilon_k-\varepsilon_m}\right), \quad [H(u)]_{mm}=d_m-u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j-d_m}{\varepsilon_j-\varepsilon_m}\right)$$

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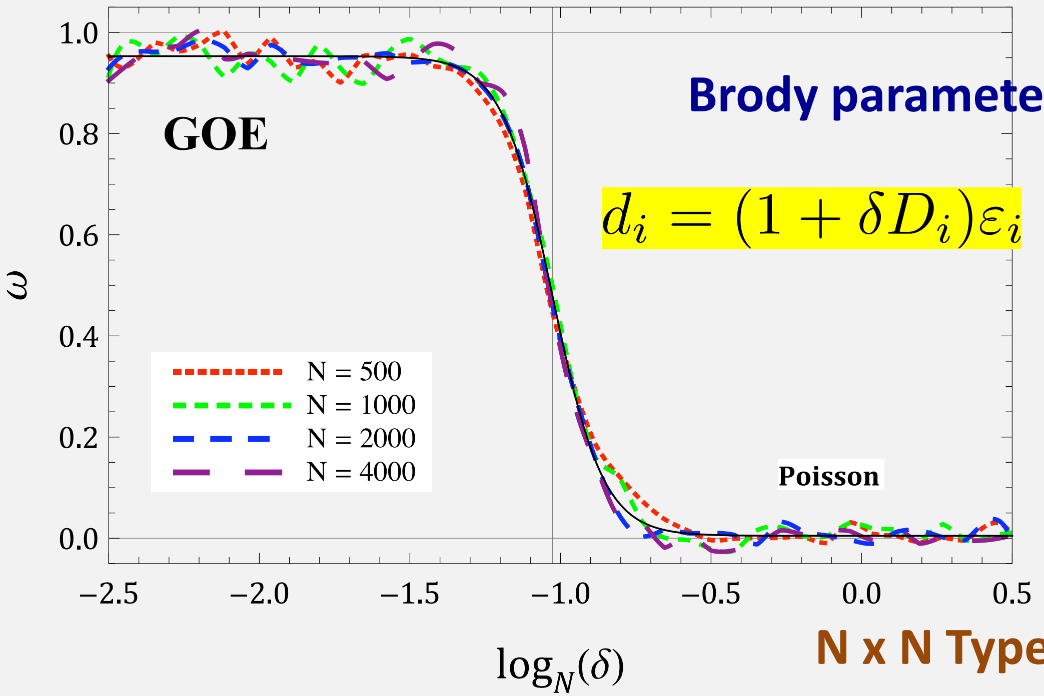
A. **Statistics is non-Poisson when normally uncorrelated parameters become correlated (atypical integrable models)**

Reverts to Poisson at deviations $\delta \propto 1/N$ from such points



Brody parameter ω as a function of $\log_N(\delta)$

$d_i = (1 + \delta D_i)\varepsilon_i$ $D_i - \mathcal{O}(1)$ random number

N x N Type 1, # of integrals = N – 1, *u=1*

Exceptions to Poisson Statistics in IMT

A. At $u=0$ the statistics is Wigner-Dyson. Can engineer any statistics in $H(u)=T+uV$ at isolated value of the coupling $u=u_0$

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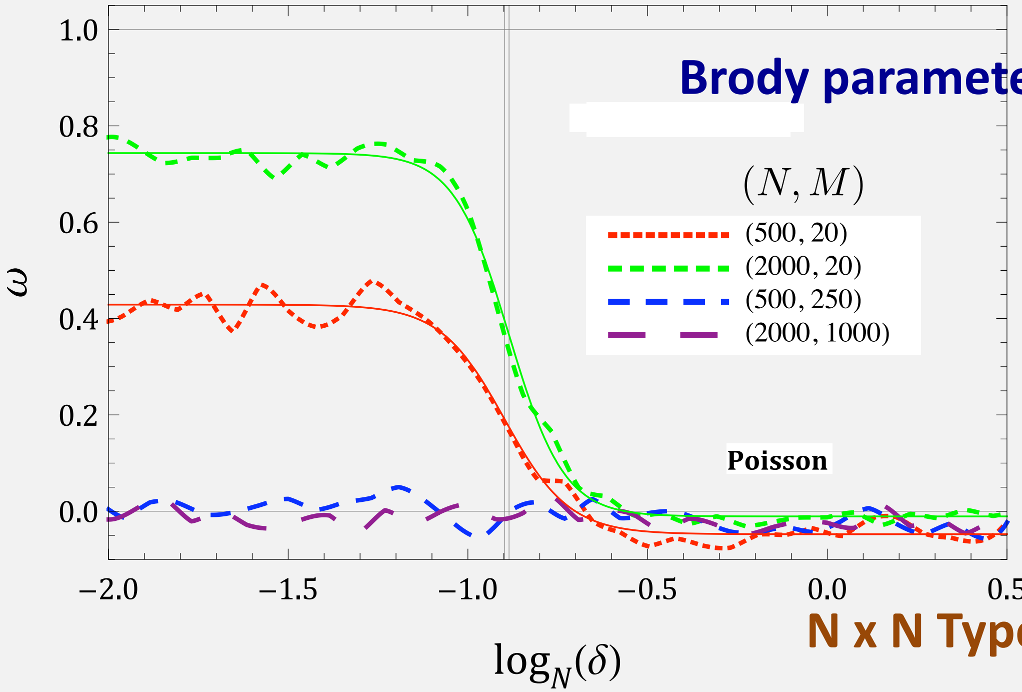


Brody parameter ω as a function of $\log_N(\delta)$

$$d_i = (1+\delta D_i)\varepsilon_i$$

D_i – $\mathcal{O}(1)$ random number

N x N Type M, # of integrals = N - M, $u=1$

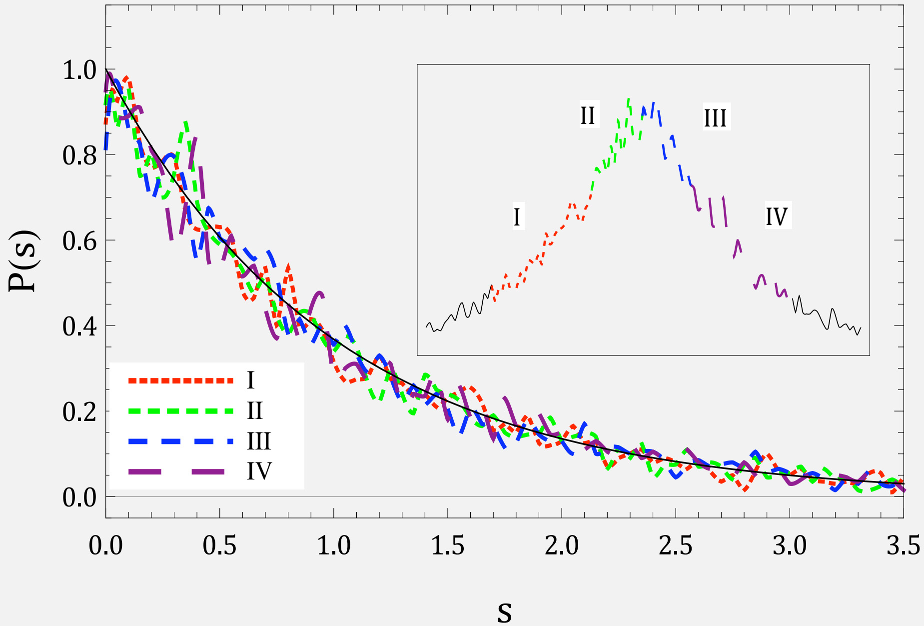
Integrable Matrix Ensembles are ergodic (numerics)

At large N, spectral statistics is independent of the region R of the spectrum and coincides with the ensemble distribution of j^{th} spacing

$$\lim_{N\to\infty} P_{i,N,R}(s) \approx e^{-s} \approx \lim_{N\to\infty} p_{N,j}(s)$$

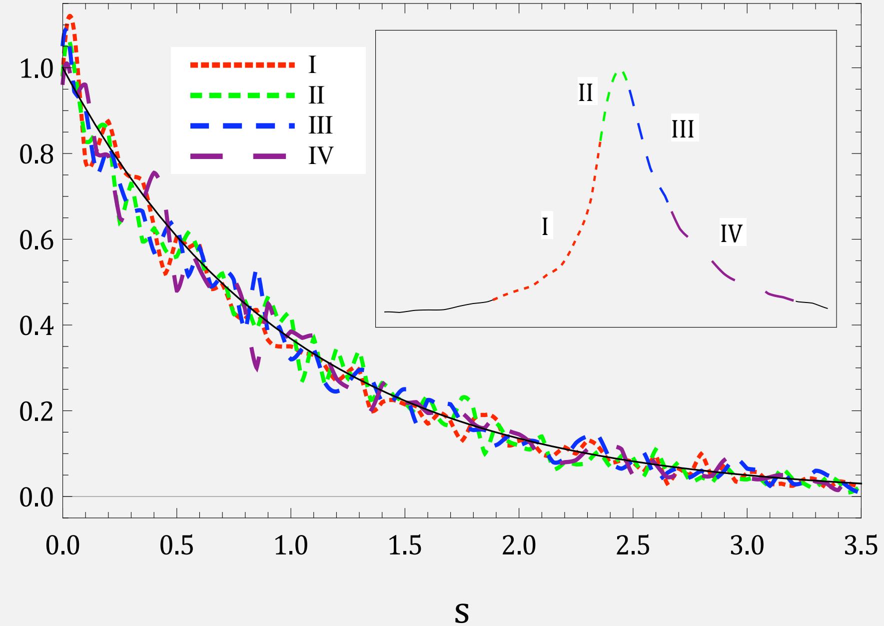
i^{th} matrix (member) of the ensemble

j^{th} spacing across the entire ensemble



Single N x N Type 1 matrix,

$N = 20000$, $u=1$, # of integrals = 19999



Single N x N Type 10000 matrix,

$N = 20000$, $u=1$, # of integrals = 10000

Integrable Matrix Ensembles are ergodic (numerics)

At large N, spectral statistics is independent of the region R of the spectrum and coincides with the ensemble distribution of j^{th} spacing

$$\lim_{N\to\infty} P_{i,N,R}(s) \approx e^{-s} \approx \lim_{N\to\infty} p_{N,j}(s)$$

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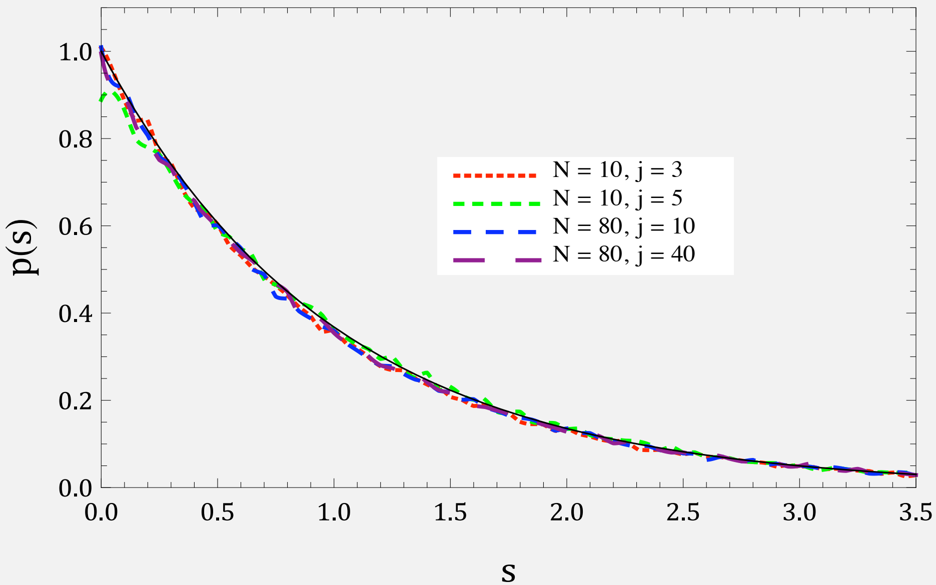
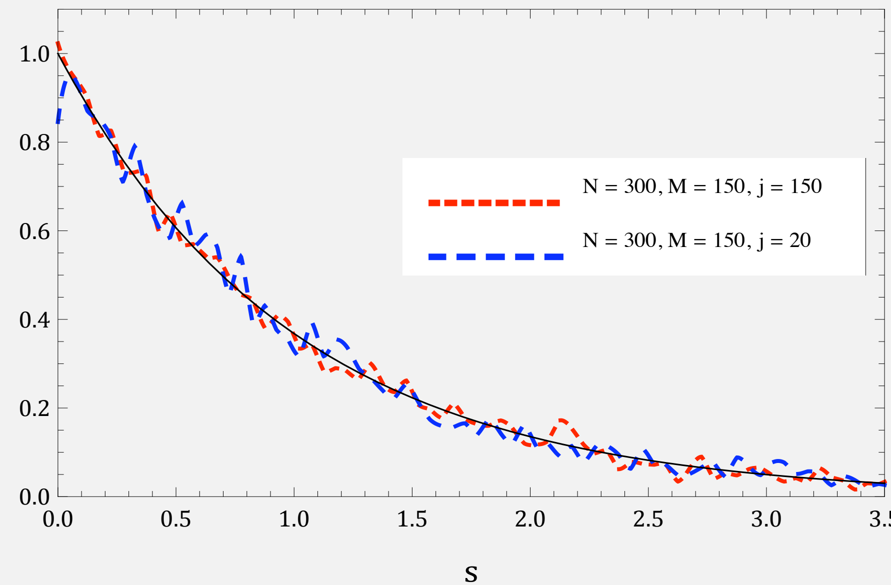


$p_{N,j}(s)$ - distribution of j^{th} spacing in $\sim 10^5$ type 1 $N\times N$ matrices



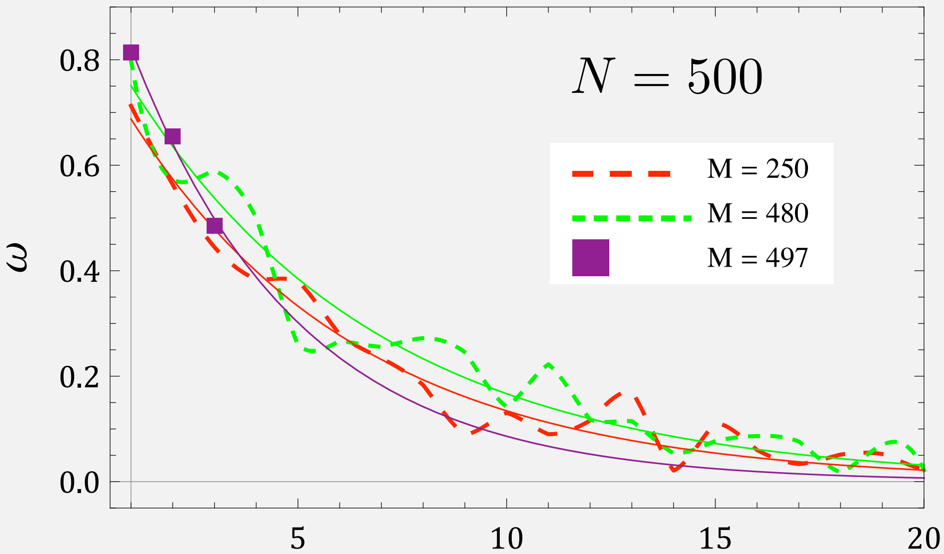
$p_{N,j}(s)$ - distribution of j^{th} spacing in $\sim 10^4$ type M $N\times N$ matrices

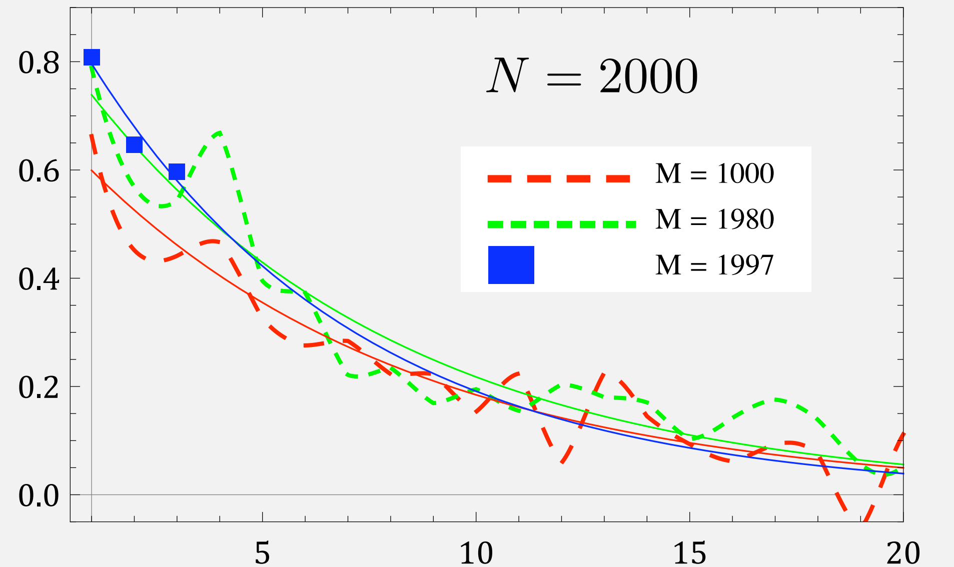
Q: How many nontrivial integrals should a system have so that its level statistics is Poisson? (numerics)

# of nontrivial integrals = Size – Type

$= N - M$

$$H(u) = \sum_{i=1}^{k} d_i H_i(u), \quad k \le N - M$$





Brody parameter ω as a function of k for $N \times N$ type M matrices.
Fit: $a \exp(-bk/\ln N)$. $b = (1.13, 1.04; 0.99, 1.03)$ for $M = (250, 480; 1000, 1980)$

$\omega = 1$ – GOE, $\omega = 0$ – Poisson

# of integrals needed $\propto \ln N$ (log of Hilbert space dim)?

Type 1 and short-range impurity problem

Every type-1 family contains a "reduced" Hamiltonian

$$\Lambda(u) = E + u|\gamma\rangle\langle\gamma|$$

$\equiv \hat{H}_{\rm BCS}$ in 1 Cooper pair sector,
GOE (exception from typical Poisson)

Type 1 *H(u)*: # of integrals =*N-1* (max # – analog of classical integrability)

Type 1 and short-range impurity problem

Every type-1 family contains a "reduced" Hamiltonian $\Lambda(u) = E + u|\gamma\rangle\langle\gamma|$

$\equiv \hat{H}_{\mathrm{BCS}}$ in 1 Cooper pair sector,
GOE (exception from typical Poisson)

Also, $\equiv \hat{H}_{\mathrm{imp}}$ short-range impurity, $u\delta(r)$, in a quantum dot

Aleiner & Matveev, PRL (1998)
Bogomolny et. al. PRL (2000)

$$\sum_i \frac{\gamma_i^2}{\lambda_m - \epsilon_i} = \frac{1}{u}$$

ε_i - eigenvalues of E
λ_m - eigenvalues of $\Lambda(u)$

$P(\{\lambda_m, \varepsilon_i\}) = \ldots$, $P(\{\lambda_m\}) = \mathrm{GOE}$? At least $P(s) \propto s^\beta$

General member of the commuting family: $H(u) = \sum_{i=1}^{N} d_i H_i(u) = T + uV$

Eigenvalues of $H(u)$: $E_m = u \sum_i \frac{d_i \gamma_i^2}{\lambda_m - \varepsilon_i}$, d_i - GOE

***Q:* Can we determine the statistics of eigenvalues of *H(u)* analytically?**

Type 1: Second "Hamiltoniazation" & Localization

Every type-1 family contains a "reduced" Hamiltonian

$$\Lambda(u) = E + u|\gamma\rangle\langle\gamma|$$

All members of a commuting family have the same eigenstates – can consider any one of them

$$\Lambda(u) \to \hat{H}(\Lambda) = \sum_{ij} \Lambda_{ij}(u) c_i^\dagger c_j$$

$$\Lambda(u) \to \hat{H}(u) = \sum_i \varepsilon_i \hat{n}_i + u \sum_{ij} \gamma_i \gamma_j c_i^\dagger c_j$$

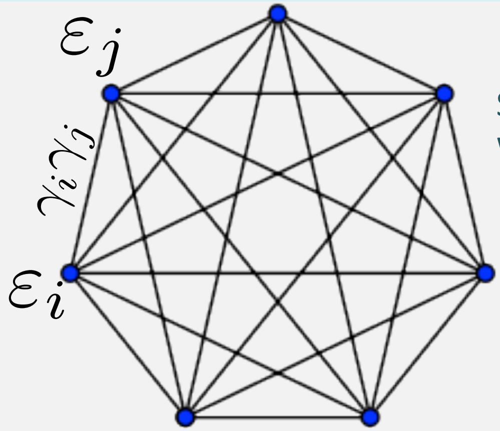
Infinite range hopping in the Hilbert space between the eigenstates of $u=0$ or generally $u=u_0$ Hamiltonian

$$H_i(u) \to \hat{H}_i(u) = \hat{n}_i + u \sum_{j \neq i} \frac{\gamma_i \gamma_j (c_i^\dagger c_j + c_j^\dagger c_i) - \gamma_i^2 \hat{n}_j - \gamma_j^2 \hat{n}_i}{\varepsilon_i - \varepsilon_j}$$

$$[\hat{H}_i(u), \hat{H}_j(u)] = 0, \quad \hat{H}(u) = \sum_i \varepsilon_i \hat{H}_i(u) + \text{const}$$

Type 1: Second "Hamiltoniazation" & Localization

$$\hat{H}(u) = \sum_i \varepsilon_i \hat{n}_i + u \sum_{ij} \gamma_i \gamma_j c_i^\dagger c_j \quad u < 0$$



Source: Wikipedia

ε_i, γ_i - random (arbitrary)

Complete graph, (N-1)-simplex

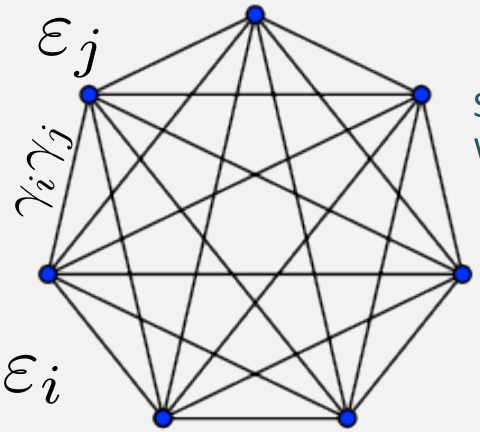
Exact solution: $\displaystyle\sum_{i=1}^{N} \frac{\gamma_i^2}{\lambda_m - \epsilon_i} = \frac{1}{u}, \quad |\lambda_m\rangle = \sum_{i=1}^{N} \frac{\gamma_i c_i^\dagger}{\lambda_m - \epsilon_i}|0\rangle$

Participation ratio: $\displaystyle \mathrm{PR}_{\lambda_m} = \frac{\left[\sum_i \frac{\gamma_i^2}{(\lambda_m - \varepsilon_i)^2}\right]^2}{\sum_i \frac{\gamma_i^4}{(\lambda_m - \varepsilon_i)^4}}$

All states are localized except the ground state. Ground state delocalizes at $|u_c|/\delta \sim 1/\log(N)$

δ – average level spacing between ε_i

$$\hat{H}(u) = \sum_i \varepsilon_i \hat{n}_i + u \sum_{ij} \gamma_i \gamma_j c_i^\dagger c_j \quad u < 0$$



Source: Wikipedia

ε_i, γ_i - random (arbitrary)

Complete graph, (N-1)-simplex

Excited states localized at any *u* [see also Ossipov (2013)]

Ground state extended for $|u| >> 1/\log(N)$. Delocalization of the ground state at $|u_c|/\delta \sim 1/\log(N)$ corresponds to the superconducting transition in H_{BCS}

Can explicitly determine exact PR in $N \to \infty$ limit when ε_i, γ_i are distributed with a smooth density, i.e. neglecting mesoscopic fluctuations in the DoS

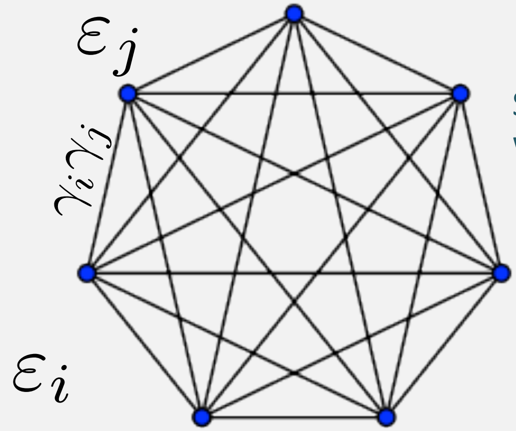
e.g. for $\varepsilon_i \in [-W/2, W/2]$ with $\rho(\varepsilon_i) = \text{const}$ and $\gamma_i = 1$

Excited states: $$\text{PR}_{\lambda_m} = \frac{3 + 3f^2(\varepsilon_m)}{1 + 3f^2(\varepsilon_m)}, \quad f(x) = -\frac{\delta}{\pi u} + \frac{1}{\pi} \ln \frac{2x + W}{W - 2x}, \quad 1 \le \text{PR}_{\lambda_m} \le 3$$

Ground state: $$\text{PR}_{g.s.} = \frac{3N}{1 + 2\cosh(\delta/u)} \propto N$$

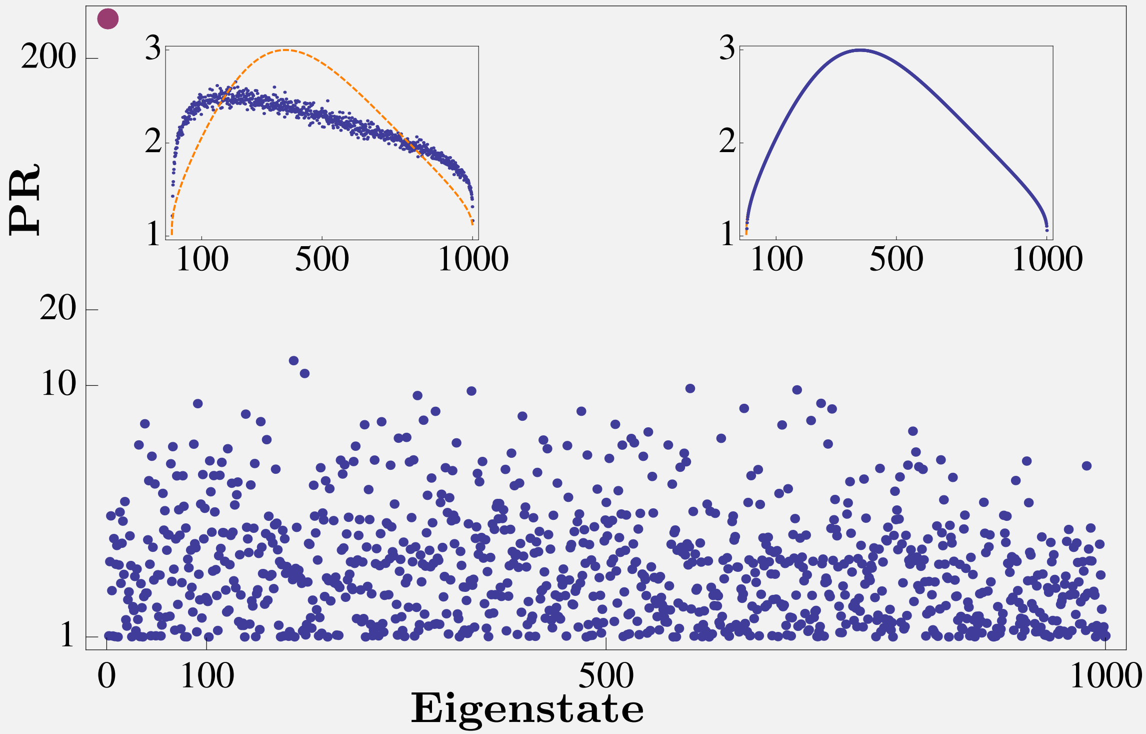
$$\hat{H}(u) = \sum_i \varepsilon_i \hat{n}_i + u \sum_{ij} \gamma_i \gamma_j c_i^\dagger c_j \quad u < 0$$

ε_i, γ_i - random (arbitrary)



Source: Wikipedia

Mesoscopic fluctuations:



Excited states:

$\mathrm{PR}^{\max}_{\lambda_m} \approx \alpha \ln N$

due to clustering in ε_i

PR for $u = -.004$, $N = 10^3$. ε_i, γ_i are independent random numbers uniformly distributed in interval $(-1, 1)$

What can we achieve with this notion of quantum integrability? – quite a lot!!

Definition: Quantum Hamiltonian H_0 is integrable if…

Consequences:

1. Exact Solution
2. Generate (ensembles of) integrable models
3. Commuting integrals $[H_i, H_j]=0;\ i, j=0,1\ldots$
4. Energy level crossings?
5. Poisson level statistics *and exceptions*
6. Generalized Gibbs Ensemble for dynamics?

Proof of Generalized Gibbs Ensemble for Type 1

$$\rho = Z^{-1} e^{-\sum_i \beta_i H_i} \qquad \langle O(t)\rangle_{t\to\infty} = \mathrm{Tr}\, \rho O\,?$$

$$\langle \mathrm{in}|H_i|\mathrm{in}\rangle = \mathrm{Tr}\, \rho H_i$$

Type 1 *H(u)*: # of integrals =*N-1* (max # – analog of classical integrability)

$$\langle O(t)\rangle_{t\to\infty} = \sum_{m=1}^{N} |c_m|^2 O_{mm} \qquad |\mathrm{in}\rangle = \sum_m c_m |\lambda_m\rangle$$

(diagonal ensemble)

\# of integrals $= N-1 =$ # of parameters $\beta_i =$ # of independent $|c_m|$,
i.e. enough integrals to reproduce all $|c_m|$

Can determine β_i such that $\langle O(t)\rangle_{t\to\infty} = \mathrm{Tr}\, \rho O$

Specifically, $\beta_i = \dfrac{1}{u} \displaystyle\sum_m \frac{\ln |c_m|^2}{\mathcal{N}_m^2 (\lambda_m - \varepsilon_i)}$

As in Classical Mechanics integrals fully constrain the motion apart from linear in time phases (angle variables) that cancel out upon time-averaging. In both cases integrals completely fix infinite time averages.

Proof of Generalized Gibbs Ensemble for Type 1

$$\rho = Z^{-1} e^{-\sum_i \beta_i H_i}$$

$$\langle O(t) \rangle_{t\to\infty} = \mathrm{Tr}\,\rho O\,?$$

$$\langle \mathrm{in}|H_i|\mathrm{in}\rangle = \mathrm{Tr}\,\rho H_i$$

$H_{\mathrm{eff}}(u)$ – a member of the commuting family

General member of the commuting family: $H(u) = \sum_{i=1}^{N} d_i H_i(u) = T + uV$

For quantum quenches, $u_i \to u_f$, in type 1 $H_{\mathrm{eff}}(u) \neq \beta H(u)$

The system effectively thermalizes with a different Hamiltonian (related to the localization of eigenstates $H(u_f)$ in the eigenspace of $H(u_i)$ seen above)

In a nonintegrable system expect $H_{\mathrm{eff}} = \beta H(u)$,
e.g. if we take T and V to be random matrices, $H_{\mathrm{eff}} = 0 \times H(u)$

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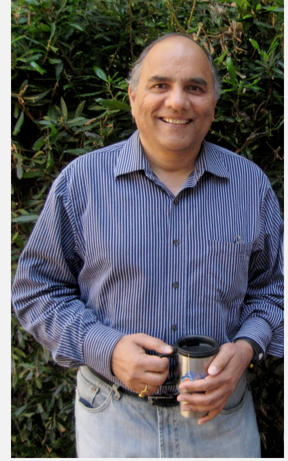

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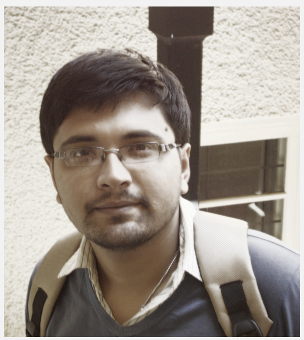

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