Integrable time-dependent Hamiltonians

Emil Yuzbashyan

Rutgers
THE STATE UNIVERSITY OF NEW JERSEY

ICTS
International Centre for Theoretical Sciences
Tata Institute of Fundamental Research

Integrable systems in Mathematics, Condensed Matter and Statistical Physics
16 July – 10 August 2018

Is there even such a thing as integrability for a time-dependent Hamiltonian???

$$i\frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi$$

Many-body or matrix Hamiltonian with explicit (smooth) dependence on time

Q: Under what conditions on $\hat{H}(t)$ is the non-stationary Schrödinger equation exactly solvable***?***

Start with time-independent integrability

Example: 1D Hubbard model – tight-binding plus onsite Coulomb (or XXZ, BCS etc.)

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

Exact solution of the stationary Schrödinger eq. via Bethe ansatz [Lieb & Wu (1969)]

$$\hat{H}(u)\psi_n(u) = E_n(u)\psi_n(u)$$

Infinite sequence of integrals of motion polynomial in u [Shastry (1986)]

$$[\hat{H}, \hat{H}_k] = [\hat{H}_k, \hat{H}_j] = 0$$

Start with time-independent integrability

Example: 1D Hubbard model – tight-binding plus onsite Coulomb (or XXZ, BCS etc.)

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

Exact solution of the stationary Schrödinger eq. via Bethe ansatz [Lieb & Wu (1969)]

$$\hat{H}(u)\psi_n(u) = E_n(u)\psi_n(u)$$

Infinite sequence of integrals of motion polynomial in u [Shastry (1986)]

$$[\hat{H}, \hat{H}_k] = [\hat{H}_k, \hat{H}_j] = 0$$

Suppose we make u a (smooth) function of time, $u \to u(t)$

In general, this will break the integrability

$$\frac{d\hat{H}_k}{dt} = i[\hat{H}, \hat{H}_k] + \frac{\partial \hat{H}_k}{\partial t} = \frac{\partial \hat{H}_k}{\partial u}\frac{du}{dt} \neq 0$$

Commuting partners are no longer integrals of motion

Start with time-independent integrability

Example: 1D Hubbard model – tight-binding plus onsite Coulomb (or XXZ, BCS etc.)

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

Exact solution of the stationary Schrödinger eq. via Bethe ansatz [Lieb & Wu (1969)]

$$\hat{H}(u)\psi_n(u) = E_n(u)\psi_n(u) \qquad u \to u(t)$$

Instantaneous (adiabatic) eigenstates are no longer helpful due to ***Landau-Zener tunneling*** between them

$$\Psi(t) = \sum_n c_n(t) e^{-i\int dt E_n(u(t))}\psi_n(u(t))$$

$|c_n(t)| \neq \mathrm{const}$ Nonadiabatic (Landau-Zener) transitions between adiabatic states

Start with time-independent integrability

Example: 1D Hubbard model – tight-binding plus onsite Coulomb (or XXZ, BCS etc.)

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\;s} + \hat{c}^\dagger_{j+1\;s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

Q: Can we make parameters of an integrable model time-dependent without breaking the integrability, i.e. so that the non-stationary Schrödinger eq. is exactly solvable***?***

In other words, can we have integrable Landau-Zener dynamics***?***

Start with time-independent integrability

Example: 1D Hubbard model – tight-binding plus onsite Coulomb (or XXZ, BCS etc.)

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{j\,s}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{j\,s}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

Q: Can we make parameters of an integrable model time-dependent without breaking the integrability, i.e. so that the non-stationary Schrödinger eq. is exactly solvable***?***

In other words, can we have integrable Landau-Zener dynamics***?***

A: Yes, we can at least for some integrable models

Example1: Bardeen-Cooper-Schrieffer (BCS) model of superconductivity

Fermi gas plus pairing interactions between fermions

$$\hat{H}_{\mathrm{BCS}} = \sum_{k,\sigma} \varepsilon_k \hat{c}^\dagger_{k\sigma} \hat{c}_{k\sigma} - g \sum_{j,k} \hat{c}^\dagger_{j\uparrow} \hat{c}^\dagger_{j\downarrow} \hat{c}_{k\downarrow} \hat{c}_{k\uparrow}$$

Like Hubbard, there is an exact solution for the spectrum [Richardson (1964)] and nontrivial g-dependent commuting partners [Cambiaggio, Rivas, Saracena (1997)]

$g \to g(t)$

In general, this breaks the integrability

But we'll see that for certain special choices of $g(t)$ the problem remains integrable

Example1: Bardeen-Cooper-Schrieffer (BCS) model of superconductivity

Fermi gas plus pairing interactions between fermions

$$\hat{H}_{\mathrm{BCS}} = \sum_{k,\sigma} \varepsilon_k \hat{c}^\dagger_{k\sigma} \hat{c}_{k\sigma} - g \sum_{j,k} \hat{c}^\dagger_{j\uparrow} \hat{c}^\dagger_{j\downarrow} \hat{c}_{k\downarrow} \hat{c}_{k\uparrow}$$

Like Hubbard, there is an exact solution for the spectrum [Richardson (1964)] and nontrivial g-dependent commuting partners [Cambiaggio, Rivas, Saracena (1997)]

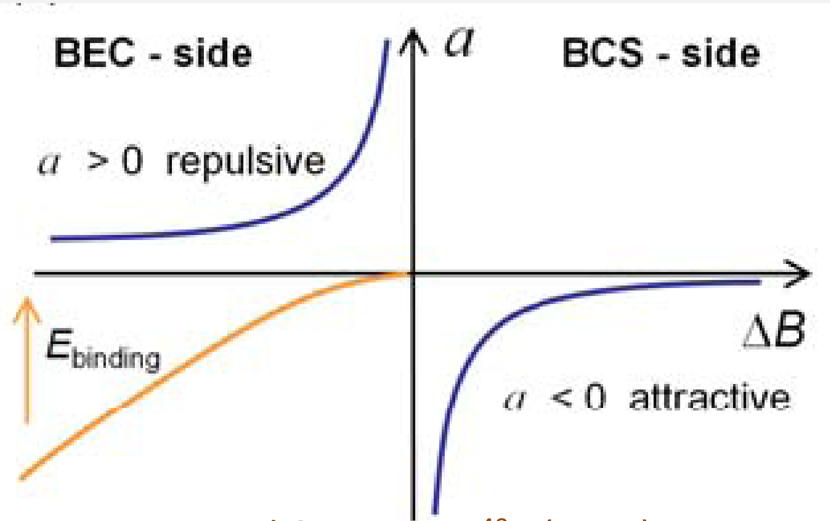
$g \to g(t)$

In general, this breaks the integrability

But we'll see that for certain special choices of $g(t)$ the problem remains integrable

In particular, we'll see that there an exact solution for $\Psi(t)$ for $g(t) = \dfrac{1}{\nu t}$

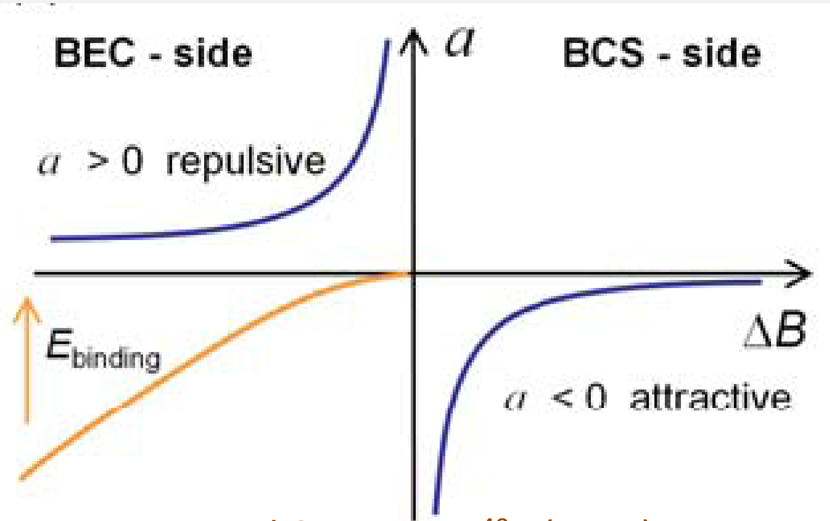
Example2: BCS-BEC Condensate of Ultracold Fermions (^{40}K, ^{6}Li)



Detuning: $\omega_0 \approx 2\mu_B(B - B_0)$

Greiner, Regal & Jin, JILA, ^{40}K (2004)

Example2: BCS-BEC Condensate of Ultracold Fermions (^{40}K, ^{6}Li)



Greiner, Regal & Jin, JILA, ^{40}K (2004)

Detuning: $\omega_0 \approx 2\mu_B(B - B_0)$

Resonance width: $\gamma = \dfrac{g^2 \nu_F}{\varepsilon_F} \ll 1$

For a narrow resonance the BCS-BEC condensate is well described by the inhomogeneous Dicke model

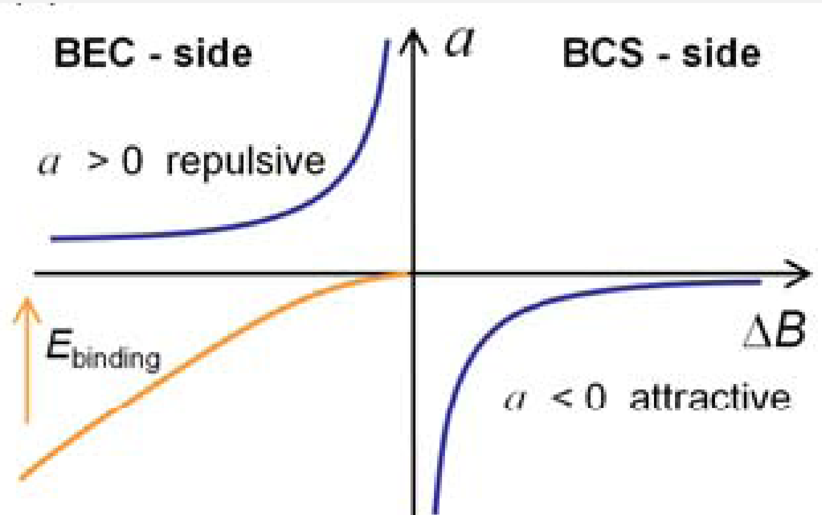
$$\hat{H}_{\mathrm{D}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \omega_0 \hat{n}_b + g \sum_{\mathbf{k}} \left(\hat{b}^{\dagger} \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} + \hat{b} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \right)$$

atoms   molecules

$$\hat{n}_b = \hat{b}^{\dagger} \hat{b}$$

Similar to BCS, this is a Bethe-ansatz-solvable model with g-dependent commuting partners [Gaudin (1983)]

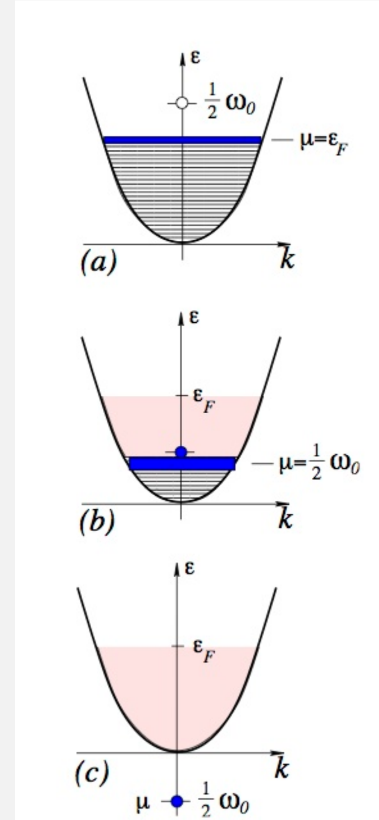
Example2: BCS-BEC Condensate of Ultracold Fermions (^{40}K, ^{6}Li)



Greiner, Regal & Jin, JILA, ^{40}K (2004)

Detuning: $\omega_0 \approx 2\mu_B(B - B_0)$

Resonance width: $\gamma = \dfrac{g^2 \nu_F}{\varepsilon_F} \ll 1$



For a narrow resonance the BCS-BEC condensate is well described by the inhomogeneous Dicke model

$$\hat{H}_{\mathrm{D}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} \underbrace{\hat{c}^{\dagger}_{\mathbf{k}\sigma}}_{\text{atoms}} \hat{c}_{\mathbf{k}\sigma} + \omega_0 \underbrace{\hat{n}_b}_{\text{molecules}} + g \sum_{\mathbf{k}} \left(\hat{b}^{\dagger} \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} + \hat{b} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \right)$$

$$\varepsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}$$

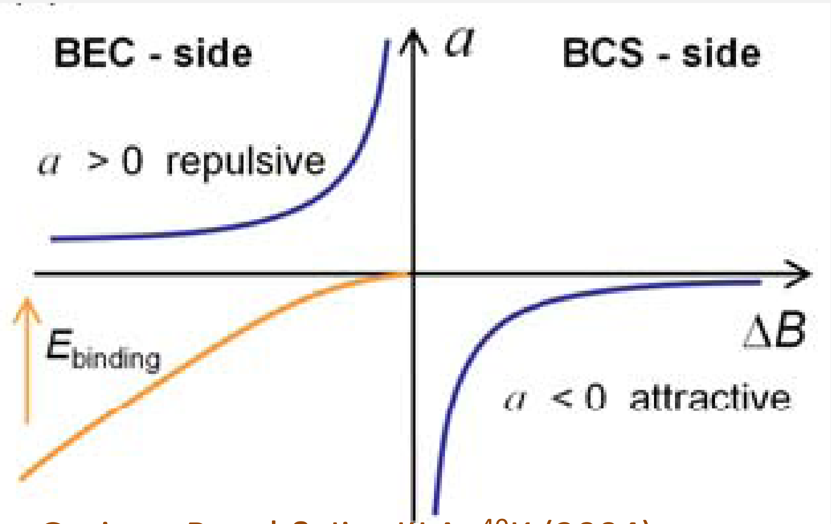
$$\hat{n}_b = \hat{b}^{\dagger} \hat{b}$$

Ground state:

$(\mathrm{a})\ \omega_0 \to +\infty$ Fermi gas

$(\mathrm{c})\ \omega_0 \to -\infty$ No atoms, everything condensed into a single mode b

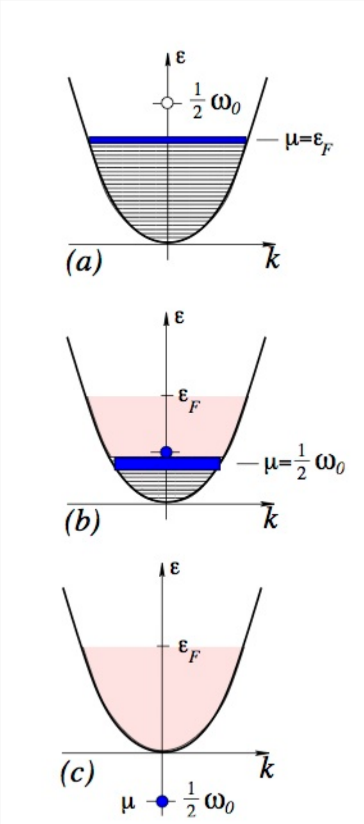
Example2: BCS-BEC Condensate of Ultracold Fermions (^{40}K, ^{6}Li)



Greiner, Regal & Jin, JILA, ^{40}K (2004)

Detuning: $\omega_0 \approx 2\mu_B(B - B_0)$

Resonance width: $\gamma = \dfrac{g^2\nu_F}{\varepsilon_F} \ll 1$



For a narrow resonance the BCS-BEC condensate is well described by the inhomogeneous Dicke model

$$\hat{H}_{\mathrm{D}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \omega_0 \hat{n}_b + g\sum_{\mathbf{k}} \left(\hat{b}^{\dagger} \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} + \hat{b} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \right)$$

atoms molecules

Linear sweep across the Feshbach resonance: $\omega_0 = -\nu t$ $\qquad i\dfrac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi$

At $t \to -\infty$: $\langle \hat{n}_b \rangle = 0,\ \langle \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} \rangle = \theta(k - k_F)$

At $t \to +\infty$: $\langle \hat{n}_b \rangle = ?,\ \langle \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} \rangle = ?$ $\qquad \hat{n}_b = \hat{b}^{\dagger}\hat{b}$

Multi-level Landau-Zener problem

$$H(t) = A + Bt \qquad i\frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi$$

A, B – $N \times N$ time-independent Hermitian matrices

$$\Psi(t \to -\infty) = |\mathrm{in}\rangle, \quad \Psi(t \to +\infty) = S|\mathrm{in}\rangle$$

S – scattering matrix = ? Transition probabilities: $p_{i\to k} = |S_{ik}|^2$

B – diagonal (diabatic basis)

Multi-level Landau-Zener problem

$$H(t) = A + Bt \qquad i\frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi$$

A, B – $N \times N$ time-independent Hermitian matrices

$$\Psi(t \to -\infty) = |\text{in}\rangle, \quad \Psi(t \to +\infty) = S|\text{in}\rangle$$

S – scattering matrix = ? Transition probabilities: $p_{i \to k} = |S_{ik}|^2$

B – diagonal (diabatic basis)

$N = 2$ Landau, Zener, Majorana, Stuckelberg (1932)

$$H(t) = \begin{pmatrix} 0 & g/2 \\ g/2 & 0 \end{pmatrix} + \begin{pmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{pmatrix} t$$

$\Psi(t)$ – solution in terms of parabolic cylinder functions

Survival probability
(Landau-Zener formula)

$$p_{0 \to 0} = 1 - e^{-\frac{\pi g^2}{\lambda}} \to 1 \text{ as } \lambda \to 0$$

(adiabaticity)

Multi-level Landau-Zener problem

$$H(t) = A + Bt \qquad i\frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi$$

A, B – $N \times N$ time-independent Hermitian matrices

$$\Psi(t \to -\infty) = |\text{in}\rangle, \quad \Psi(t \to +\infty) = S|\text{in}\rangle$$

S – scattering matrix = ? Transition probabilities: $p_{i\to k} = |S_{ik}|^2$

B – diagonal (diabatic basis)

$N > 2$ No general solution, only certain special cases

Q: Under what conditions on $H(t) = A + Bt$, i.e. for which A and B is the multi-level Landau-Zener problem exactly solvable? What is the solution?

By definition solvable iff: $p_{i\to k} = f_{\text{elem}}(A_{ij}, B_{ij})$

Exactly solvable multi-level Landau-Zener problems

A. Trivial/reducible MLZ problems

arbitrary spin in linear in time magnetic field

$$H(t) = \begin{pmatrix} 0 & g/2 \\ g/2 & 0 \end{pmatrix} + \begin{pmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{pmatrix} t = g\frac{\sigma_x}{2} + \lambda t \frac{\sigma_z}{2} \longrightarrow g S_x + \lambda t S_z$$

arbitrary rep of $\mathrm{su}(2)$

Exactly solvable multi-level Landau-Zener problems

A. Trivial/reducible MLZ problems

arbitrary spin in linear in time magnetic field

$$H(t) = \begin{pmatrix} 0 & g/2 \\ g/2 & 0 \end{pmatrix} + \begin{pmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{pmatrix} t = g\frac{\sigma_x}{2} + \lambda t \frac{\sigma_z}{2} \longrightarrow gS_x + \lambda t S_z$$

arbitrary rep of $\mathrm{su}(2)$

The time evolution operator belongs to the SU(2) group (rotation)

$$U(t) = e^{-i\alpha(t)\hat{S}_z} e^{-i\beta(t)\hat{S}_y} e^{-i\gamma(t)\hat{S}_z} \equiv \hat{R}(\alpha, \beta, \gamma)$$

Euler angles $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are the same as in the 2×2 LZ problem

Transition probabilities are modulus squared of the elements of the Wigner D-matrix

$$p_{m \to m'} = |\langle m|\hat{R}(\alpha, \beta, \gamma)|m'\rangle|^2$$

Hioe, J. Opt. Soc. Am. B 4, 1327 (1987)

Exactly solvable multi-level Landau-Zener problems

A. Trivial/reducible MLZ problems

Driven Quantum Ising Model: $$H = -J\sum_{n=1}^{N}\left[h(t)\sigma_n^x + \sigma_n^z\sigma_{n+1}^z\right], \quad h(t) = -\lambda t$$

After Jordan-Wigner followed by Fourier this reduces to the 2×2 LZ problem

Dziarmaga, PRL 95, 245701 (2005)

$$H = J\sum_k \left\{2[h-\cos(ka)]c_k^\dagger c_k + \sin(ka)[c_k^\dagger c_{-k}^\dagger + c_{-k}c_k] - h\right\}$$

Exactly solvable multi-level Landau-Zener problems

A. Trivial/reducible MLZ problems

Driven Quantum Ising Model: $$H = -J\sum_{n=1}^{N}\left[h(t)\sigma_n^x + \sigma_n^z\sigma_{n+1}^z\right], \quad h(t) = -\lambda t$$

After Jordan-Wigner followed by Fourier this reduces to the 2×2 LZ problem

Dziarmaga, PRL 95, 245701 (2005)

$$H = J\sum_k \left\{2[h-\cos(ka)]c_k^\dagger c_k + \sin(ka)[c_k^\dagger c_{-k}^\dagger + c_{-k}c_k] - h\right\}$$

Density of kinks in $N\to\infty$ limit for a sweep across QPT from paramagnet $(h \gg 1)$ to ferromagnet at $h=0$

$$n = \frac{1}{2\pi}\left(\frac{\hbar\lambda}{2J}\right)^{1/2}$$

Scaling with the rate λ agrees with Kibble-Zurek mechanism

And many more trivial/reducible MLZ problems…

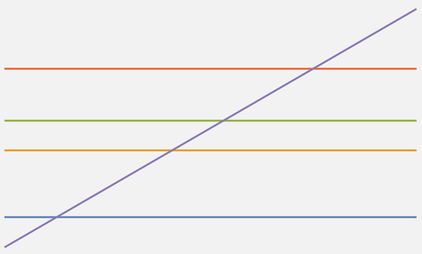
Exactly solvable multi-level Landau-Zener problems

B. Three irreducible exactly solvable MLZ problems since 1932

1. Demkov-Osherov model

Soviet Phys. JETP (1968)

$$H_{\mathrm{DO}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



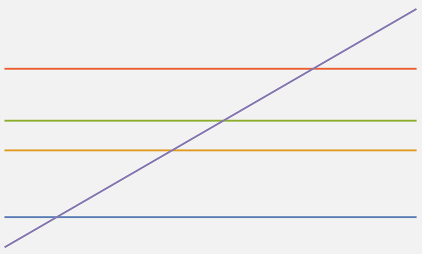
Exactly solvable multi-level Landau-Zener problems

B. Three irreducible exactly solvable MLZ problems since 1932

1. Demkov-Osherov model

Soviet Phys. JETP (1968)

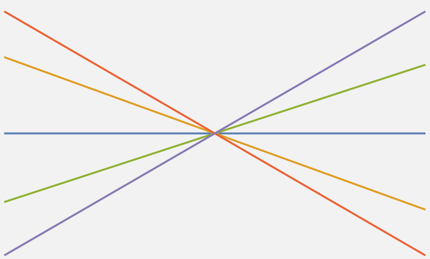
$$H_{\mathrm{DO}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



2. Bow-tie model

Ostrovsky & Nakamura, J. Phys. A (1997)

$$H_{\mathrm{bt}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$



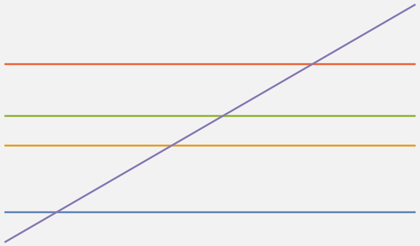
Exactly solvable multi-level Landau-Zener problems

B. Three irreducible exactly solvable MLZ problems since 1932

1. Demkov-Osherov model

Soviet Phys. JETP (1968)

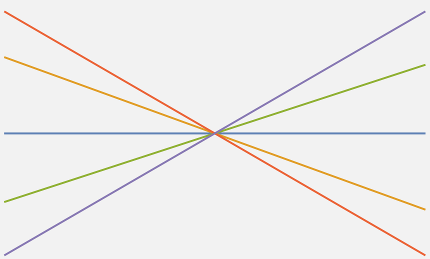
$$H_{\mathrm{DO}}(t)=\begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix}+t\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



2. Bow-tie model

Ostrovsky & Nakamura, J. Phys. A (1997)

$$H_{\mathrm{bt}}(t)=\begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & 0 \end{pmatrix}+t\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$



3. Inhomogeneous Dicke model

Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, PRL (2018)

$$\hat{H}_{\mathrm{D}}=\sum_{\mathbf{k},\sigma}\varepsilon_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}\sigma}\hat{c}_{\mathbf{k}\sigma}-(\nu t)\hat{n}_b+g\sum_{\mathbf{k}}\left(\hat{b}^{\dagger}\hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow}+\hat{b}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow}\right)$$

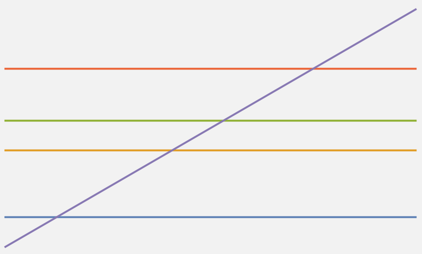
Exactly solvable multi-level Landau-Zener problems

B. Three irreducible exactly solvable MLZ problems since 1932

1. Demkov-Osherov model

Soviet Phys. JETP (1968)

$$H_{\rm DO}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



$$S = \begin{pmatrix} p_2\cdots p_N & q_2p_3\cdots p_N & q_3p_4\ldots p_N & q_4p_5\cdots p_N & \cdots & q_N \\ q_2 & p_2 & 0 & 0 & \cdots & 0 \\ p_2q_3 & q_2q_3 & p_3 & 0 & \cdots & 0 \\ p_2p_3q_4 & q_2p_3q_4 & q_3q_4 & p_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_2\cdots p_{N-1}q_N & q_2p_3\cdots p_{N-1}q_N & q_3p_4\cdots p_{N-1}q_N & q_4p_5\cdots p_{N-1}q_N & \cdots & q_N \end{pmatrix}$$

$$p_k = e^{-\frac{\pi g_k^2}{\lambda}}, \quad q_k = \sqrt{1-p_k^2}$$

$$S = S_{\rm LZ}^{1N}\ldots S_{\rm LZ}^{13}S_{\rm LZ}^{12}$$

Exactly solvable multi-level Landau-Zener problems

Q: What is special about these models? What sets them apart from any other Hamiltonian linear in time?

$$H(t) = A + Bt \qquad A, B - N \times N \text{ time-independent Hermitian matrices}$$

Insight from Integrable Matrix Theory (counterpart of Random Matrix Theory for quantum regular as opposed to chaotic systems)

Owusu &Yuzbashyan, J. Phys. A (2011)
Yuzbashyan & Shastry, J. Stat. Phys. (2013)
Yuzbashyan, Shastry, Scaramazza, PRE (2016)

First, consider an abstract N x N Hermitian matrix M

$$M = \begin{pmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{pmatrix}$$

Makes no sense to talk about its integrability

First, consider an abstract N x N Hermitian matrix M

$$M = \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times \end{pmatrix}$$

Makes no sense to talk about its integrability

For example, there is no natural notion of a nontrivial integral of motion

For any M there is a full set of $\mathrm{M_k}$ such that $[M_k, M_j] = [M_k, M] = 0$

And any integral of motion $M_k = \sum_{n=1}^{N} a_n M^n$

All Hermitian matrices look the same from this point of view

The situation changes if we introduce & fix parameter dependence

Let $H(t) = A + Bt$, t – real parameter and A, B – Hermitian matrices

Suppose we require a commuting partner also linear in t: $\widetilde{H}(t) = \widetilde{A} + \widetilde{B}t$

$$\left[\widetilde{H}(t), H(t)\right] = 0, \ \text{for all } t$$

$$\Updownarrow$$

$$[\widetilde{B}, B] = 0, \quad [\widetilde{A}, B] = [A, \widetilde{B}], \quad [\widetilde{A}, A] = 0$$

These commutation relations severely constraint matrix elements of $H(t)$

For a generic/typical $H(t)$ – no commuting partners except itself and identity

Now can separate generic matrices (no commuting partners) from special (integrable matrices)

N x N Hamiltonians linear in a parameter separate into two distinct classes

$$H(t) = A + Bt$$

No commuting partners linear in t other than itself and identity (typical) – nonintegrable, need $N^2/2$ real parameters to specify $H(t)$

Nontrivial commuting partners $H_k(u)=A_k+B_k t$ exist – integrable, turns out need less than $4N$ parameters – measure zero in the space of linear Hamiltonians

1. Bethe-ansatz-like exact solution for the spectrum

2. Level crossings (typically $N^2/2$ crossings)

3. Can generate basis-independent ensembles of integrable matrices. Level statistics are typically Poissonian

Exactly solvable multi-level Landau-Zener problems

Q: What is special about these models? What sets them apart from any other Hamiltonian linear in time?

$$H(t) = A + Bt \qquad A, B - N \times N \text{ time-independent Hermitian matrices}$$

A: They are integrable matrices as defined above!

Patra & Yuzbashyan, J. Phys. A (2015)

Exactly solvable multi-level Landau-Zener problems

Q: What is special about these models? What sets them apart from any other Hamiltonian linear in time?

$H(t) = A + Bt \qquad A, B - N \times N$ time-independent Hermitian matrices

A: They are integrable matrices as defined above!

Patra & Yuzbashyan, J. Phys. A (2015)

Example 1: Demkov-Osherov model

$$H_{\mathrm{DO}} = \lambda t|1\rangle\langle 1| + \sum_{k=2}^{N} \left(g_k|1\rangle\langle k| + g_k|k\rangle\langle 1| + a_k|k\rangle\langle k|\right)$$

Has N independent nontrivial commuting partners linear in t

$$H_j = (t - a_j)|j\rangle\langle j| - g_j|1\rangle\langle j| - g_j|j\rangle\langle 1| + \sum_{k\neq j} \frac{g_j g_k|j\rangle\langle k| + g_j g_k|k\rangle\langle j| - g_k^2|j\rangle\langle j| - g_j^2|k\rangle\langle k|}{a_k - a_j}$$

$$[H_j, H_k] = [H_j, H_{\mathrm{DO}}] = 0$$

Exactly solvable multi-level Landau-Zener problems

Q: What is special about these models? What sets them apart from any other Hamiltonian linear in time?

$H(t) = A + Bt \qquad A, B - N \times N$ time-independent Hermitian matrices

A: They are integrable matrices as defined above!

Example 2: inhomogeneous Dicke model

$$\hat{H}_{\mathrm{D}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} - (\nu t)\hat{n}_b + g \sum_{\mathbf{k}} \left(\hat{b}^{\dagger} \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} + \hat{b} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \right)$$

Anderson pseudospins: $s^z_{\mathbf{k}} \equiv \frac{1}{2}\left[c^{\dagger}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} + c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow} - 1 \right], \quad s^-_{\mathbf{k}} \equiv c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}, \quad s^+_{\mathbf{k}} \equiv c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow}$

$$\hat{H}_{\mathrm{D}}(t) = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} s^z_{\mathbf{k}} - (\nu t)\hat{n}_b + g \sum_{\mathbf{k}} \left(\hat{b}^{\dagger} s^-_{\mathbf{k}} + \hat{b} s^+_{\mathbf{k}} \right)$$

$$\hat{H}_{\mathbf{k}}(t) = (\varepsilon_{\mathbf{k}} + \nu t) s^z_{\mathbf{k}} + g(\hat{b}^{\dagger} s^-_{\mathbf{k}} + \hat{b} s^+_{\mathbf{k}}) + 2g^2 \sum_{p \neq k} \frac{\vec{s}_{\mathbf{k}} \cdot \vec{s}_{\mathbf{p}}}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{p}}}$$

$$[\hat{H}(t), \hat{H}_{\mathbf{k}}(t)] = [\hat{H}_{\mathbf{k}}(t), \hat{H}_{\mathbf{p}}(t)] = 0, \quad \forall t, \mathbf{k}, \mathbf{p}$$

Exactly solvable multi-level Landau-Zener problems

Q: What is special about these models? What sets them apart from any other Hamiltonian linear in time?

$$H(t) = A + Bt \qquad A, B - N \times N \text{ time-independent Hermitian matrices}$$

A: They are integrable matrices as defined above!

Patra & Yuzbashyan, J. Phys. A (2015)

$$\exists \quad H_k(t) = A_k + B_k t : \quad [H_k(t), H(t)] = 0 \quad \forall t$$

Q: What is the role of these commuting partners? How do they help us solve for the dynamics of the system?

$$i\frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi, \qquad \Psi(t) = ?$$

They aren't conserved: $\dfrac{dH_k}{dt} = i[H, H_k] + \dfrac{\partial H_k}{\partial t} = B_k \neq 0$

Exactly solvable multi-level Landau-Zener problems

Q: What is special about these models? What sets them apart from any other Hamiltonian linear in time?

$H(t) = A + Bt \qquad A, B - N \times N$ time-independent Hermitian matrices

A: They are integrable matrices as defined above!

Patra & Yuzbashyan, J. Phys. A (2015)

$$\exists \quad H_k(t) = A_k + B_k t : \quad [H_k(t), H(t)] = 0 \quad \forall t$$

Q: What is the role of these commuting partners? How do they help us solve for the dynamics of the system?

$$i\frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi, \quad \Psi(t) = ?$$

A: They determine the evolution of the system with respect to parameters other than time!

$$i\frac{\partial \Psi}{\partial x_k} = \hat{H}_k \Psi$$

Idea: The non-stationary Schrödinger equation can be consistently embedded into a set of multi-time Schrödinger equations

Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, PRL (2018)

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} = \hat{H}\Psi \\ \\ i\nu\dfrac{\partial \Psi}{\partial x_k} = \hat{H}_k\Psi, \quad k = 1, \dots, n-1 \end{cases}$$

$$x_0 \equiv \nu t, \quad \hat{H}_0 \equiv \hat{H}, \quad \partial_k = \frac{\partial}{\partial x_k}, \quad \boldsymbol{x} = (x_0, \dots, x_{n-1})$$

$$i\nu\partial_k\Psi(\boldsymbol{x}) = \hat{H}_k\Psi(\boldsymbol{x})$$

$$\Updownarrow$$

Consistency: $\partial_j \hat{H}_k - \partial_k \hat{H}_j - i[\hat{H}_k, \hat{H}_j] = 0$

Idea: The non-stationary Schrödinger equation can be consistently embedded into a set of multi-time Schrödinger equations

Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, PRL (2018)

$$\begin{cases} i\dfrac{\partial\Psi}{\partial t} = \hat{H}\Psi \\[2ex] i\nu\dfrac{\partial\Psi}{\partial x_k} = \hat{H}_k\Psi, \quad k = 1,\ldots,n-1 \end{cases}$$

$$x_0 \equiv \nu t, \quad \hat{H}_0 \equiv \hat{H}, \quad \partial_k = \frac{\partial}{\partial x_k}, \quad \boldsymbol{x} = (x_0,\ldots,x_{n-1})$$

$$i\nu\partial_k\Psi(\boldsymbol{x}) = \hat{H}_k\Psi(\boldsymbol{x})$$

$$\Updownarrow$$

Consistency: $\underbrace{\partial_j\hat{H}_k - \partial_k\hat{H}_j}_{\text{real}} - \underbrace{i[\hat{H}_k,\hat{H}_j]}_{\text{imaginary}} = 0$

$$\Updownarrow$$

$$\begin{cases} \partial_j\hat{H}_k = \partial_k\hat{H}_j & \leftarrow \text{Additional constraint} \\[2ex] [\hat{H}_k,\hat{H}_j] = 0 & \leftarrow \text{Integrability of the underlying model} \end{cases}$$

Idea: The non-stationary Schrödinger equation can be consistently embedded into a set of multi-time Schrödinger equations

Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, PRL (2018)

Example: inhomogeneous Dicke model

$$\hat{H}_{\mathrm{D}}(t) = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} s^z_{\mathbf{k}} + \omega_0 \hat{n}_b + g \sum_{\mathbf{k}} \left(\hat{b}^\dagger s^-_{\mathbf{k}} + \hat{b} s^+_{\mathbf{k}} \right)$$

$$\hat{H}_{\mathbf{k}}(t) = (\varepsilon_{\mathbf{k}} - \omega_0) s^z_{\mathbf{k}} + g(\hat{b}^\dagger s^-_{\mathbf{k}} + \hat{b} s^+_{\mathbf{k}}) + 2g^2 \sum_{p\neq k} \frac{\vec{s}_{\mathbf{k}} \cdot \vec{s}_{\mathbf{p}}}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{p}}}$$

$$x_0 = -\omega_0 = \nu t, \quad x_k = \varepsilon_{\mathbf{k}}$$

$$\frac{\partial H_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{p}}} = 2g^2 \frac{\vec{s}_{\mathbf{k}} \cdot \vec{s}_{\mathbf{p}}}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{p}})^2} = \frac{\partial H_{\mathbf{p}}}{\partial \varepsilon_{\mathbf{k}}} \qquad \frac{\partial H_{\mathrm{D}}}{\partial \varepsilon_{\mathbf{k}}} = s^z_{\mathbf{k}} = \frac{\partial H_{\mathbf{k}}}{\partial(-\omega_0)}$$

$$\begin{cases} \partial_j \hat{H}_k = \partial_k \hat{H}_j & \leftarrow \text{Additional constraint} \\ [\hat{H}_k, \hat{H}_j] = 0 & \leftarrow \text{Integrability of the underlying model} \end{cases}$$

Multi-level Landau-Zener problem

$$
\begin{cases}
i\dfrac{\partial \Psi}{\partial t} = \hat{H}\Psi \\[2em]
i\nu\dfrac{\partial \Psi}{\partial x_k} = \hat{H}_k\Psi
\end{cases}
$$

Formal solution: $\Psi(\boldsymbol{x}) = T\exp\left(-i\int_{\mathcal{P}} \hat{H}_k dx_k\right)\Psi(\boldsymbol{x}_0)$

Multi-level Landau-Zener problem

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} = \hat{H}\Psi \\ \\ i\nu\dfrac{\partial \Psi}{\partial x_k} = \hat{H}_k\Psi \end{cases}$$

Formal solution: $\Psi(\boldsymbol{x}) = T\exp\left(-i\int_{\mathcal{P}} \hat{H}_k dx_k\right)\Psi(\boldsymbol{x}_0)$

Path-independent

Consistency: $\partial_j \hat{H}_k - \partial_k \hat{H}_j - i[\hat{H}_k, \hat{H}_j] = 0$

$$\Downarrow$$

Non-abelian gauge field $\mathcal{A}_k = -i\hat{H}_k$ has zero curvature

[Not to be confused with zero curvature representation of nonlinear PDEs]

Multi-level Landau-Zener problem

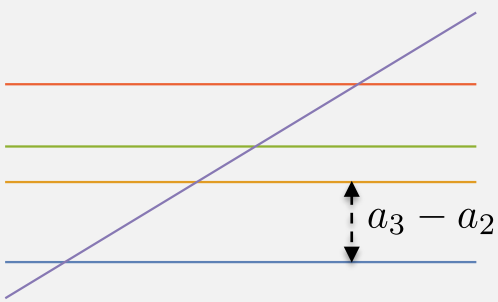
$$
\begin{cases}
i\dfrac{\partial \Psi}{\partial t} = \hat{H}\Psi \\[2em]
i\nu\dfrac{\partial \Psi}{\partial x_k} = \hat{H}_k\Psi
\end{cases}
$$

Formal solution: $\Psi(\boldsymbol{x}) = T\exp\left(-i\int_{\mathcal{P}} \hat{H}_k dx_k\right)\Psi(\boldsymbol{x}_0)$

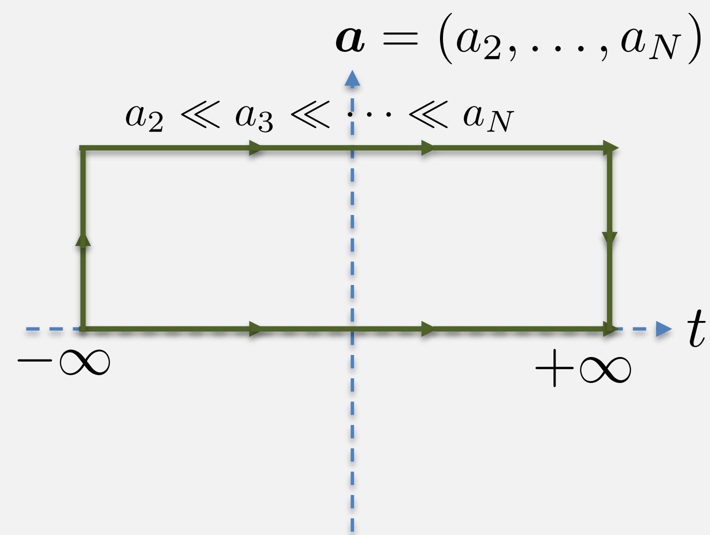
Path-independent

Example: Demkov-Osherov model

$$
H_{\mathrm{DO}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}
$$



$$x_k = a_k$$



Multi-level Landau-Zener problem

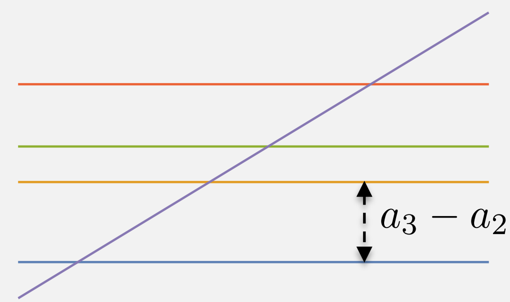
$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} = \hat{H}\Psi \\[2em] i\nu\dfrac{\partial \Psi}{\partial x_k} = \hat{H}_k\Psi \end{cases}$$

Formal solution: $\Psi(\boldsymbol{x}) = T\exp\left(-i\int_{\mathcal{P}} \hat{H}_k dx_k\right)\Psi(\boldsymbol{x}_0)$

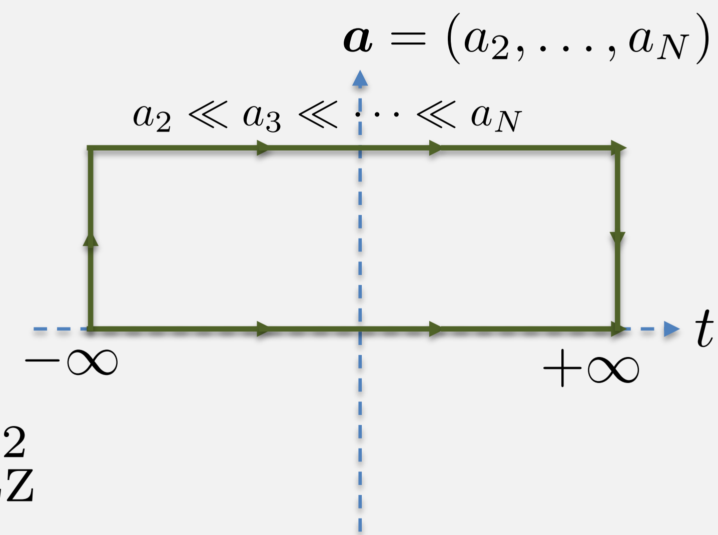
Path-independent

Example: Demkov-Osherov model

$$H_{\mathrm{DO}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



$$x_k = a_k$$



Energy levels well separated (i.e. evolution is adiabatic and no transitions occur) everywhere along the contour except near crossings where 2×2 LZ scattering events take place

$$\Rightarrow S = S_{\mathrm{LZ}}^{1N} \ldots S_{\mathrm{LZ}}^{13} S_{\mathrm{LZ}}^{12}$$

Knizhnik-Zamolodchikov equations

$$i\nu \frac{\partial \Psi}{\partial \varepsilon_j} = \hat{H}_j \Psi$$

$$\hat{H}_j = -\sum_{k \neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k}$$ – Gaudin magnets

$$[\hat{H}_j, \hat{H}_k] = 0$$

Q: Is their any relationship between the multi-time Schrödinger equations we derived for solvable Landau-Zener models and Knizhnik-Zamolodchikov equations?

Generalized Knizhnik-Zamolodchikov equations

$$i\nu \frac{\partial \Psi}{\partial \varepsilon_j} = \hat{H}_j \Psi \qquad \hat{H}_j = \boxed{2Bs_j^z} - \sum_{k\neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k}$$

– Gaudin magnets

$$[\hat{H}_j, \hat{H}_k] = 0$$

$$\sum_k 2\varepsilon_k \hat{H}_k \propto \hat{H}_{\mathrm{BCS}} = \sum_k 2\varepsilon_k s_k^z - \frac{1}{2B} \sum_{j,k} s_j^+ s_k^- \qquad [\hat{H}_{\mathrm{BCS}}, \hat{H}_k] = 0$$

BCS model of superconductivity in Anderson pseudospin representation

$$s_k^z = \frac{\hat{n}_k - 1}{2}, \quad s_k^- = c_{k\downarrow} c_{k\uparrow}, \quad s_k^+ = c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger; \qquad g = \frac{1}{2B}$$

$$\hat{H}_{\mathrm{BCS}} = \sum_{k,\sigma} \varepsilon_k \hat{c}_{k\sigma}^\dagger \hat{c}_{k\sigma} - g \sum_{j,k} \hat{c}_{j\uparrow}^\dagger \hat{c}_{j\downarrow}^\dagger \hat{c}_{k\downarrow} \hat{c}_{k\uparrow}$$

Sierra, Nucl. Phys. B (2000); Amico, Falci, Fazio, J. Phys. A (2001); Sedrakyan & Galitskii, PRB (2010); Fioretto, Caux, Gritsev, New J. Phys. (2014)

Generalized Knizhnik-Zamolodchikov equations

$$i\nu \frac{\partial \Psi}{\partial \varepsilon_j} = \hat{H}_j \Psi \qquad \hat{H}_j = \boxed{2Bs_j^z} - \sum_{k\neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k}$$ – Gaudin magnets

$$[\hat{H}_j, \hat{H}_k] = 0$$

$$\sum_k 2\varepsilon_k \hat{H}_k \propto \hat{H}_{\mathrm{BCS}} = \sum_k 2\varepsilon_k s_k^z - \frac{1}{2B}\sum_{j,k} s_j^+ s_k^- \qquad [\hat{H}_{\mathrm{BCS}}, \hat{H}_k] = 0$$

Observation: The evolution of the system with magnetic field B is governed by the BCS Hamiltonian [Yuzbashyan, Ann. Phys. (2018)]

$$i\nu \frac{\partial \Psi}{\partial B} = \hat{H}_{\mathrm{BCS}} \Psi$$

This equation is consistent with the generalized KZ equations, because the BCS Hamiltonians satisfies the zero curvature conditions:

$$\frac{\partial \hat{H}_k}{\partial B} = 2s_k^z = \frac{\partial \hat{H}_{\mathrm{BCS}}}{\partial \varepsilon_k}$$

KZ-BCS equations

$$\begin{cases} i\nu \dfrac{\partial \Psi}{\partial \varepsilon_j} = \hat{H}_j \Psi \\[2em] i\nu \dfrac{\partial \Psi}{\partial B} = \hat{H}_{\mathrm{BCS}} \Psi \end{cases}$$

$$\hat{H}_j = 2Bs_j^z - \sum_{k \neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k}$$ – Gaudin magnets

$$\hat{H}_{\mathrm{BCS}} = \sum_k 2\varepsilon_k s_k^z - \frac{1}{2B} \sum_{j,k} s_j^+ s_k^-$$

KZ-BCS equations

$$\begin{cases} i\nu \dfrac{\partial \Psi}{\partial \varepsilon_j} = \hat{H}_j \Psi & \hat{H}_j = 2Bs_j^z - \displaystyle\sum_{k\neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k} \text{ – Gaudin magnets} \\ i\nu \dfrac{\partial \Psi}{\partial B} = \hat{H}_{\mathrm{BCS}} \Psi & \hat{H}_{\mathrm{BCS}} = \displaystyle\sum_k 2\varepsilon_k s_k^z - \frac{1}{2B} \sum_{j,k} s_j^+ s_k^- \end{cases}$$

Integrable time-dependent BCS Hamiltonians: let $B = B(t)$

$$B(t) = \nu t \implies \quad \hat{H}_{\mathrm{BCS}}(t) = \sum_j 2\varepsilon_j \hat{s}_j^z - \frac{1}{\nu t} \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-$$

$$B(t) = \sin(\nu t) \implies \quad \hat{H}_{\mathrm{BCS}}(t) = \cos(\nu t) \sum_j 2\varepsilon_j \hat{s}_j^z - \cot(\nu t) \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-$$

Solution of the non-stationary Schrödinger eq: $\Psi(t) = \Psi_{\mathrm{KZ}}[B(t)]$

Yuzbashyan, Ann. Phys. (2018)

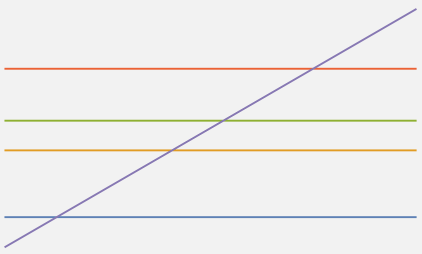
What about exactly solvable multi-level Landau-Zener problems?

Three irreducible exactly solvable MLZ problems since 1932

1. Demkov-Osherov model

Soviet Phys. JETP (1968)

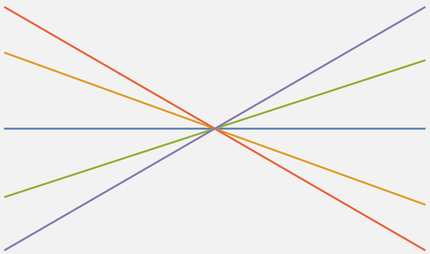
$$H_{\mathrm{DO}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



2. Bow-tie model

Ostrovsky & Nakamura, J. Phys. A (1997)

$$H_{\mathrm{bt}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$



3. Inhomogeneous Dicke model

Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, PRL (2018)

$$\hat{H}_{\mathrm{D}}(t) = \sum_k \varepsilon_k s_k^z - (\nu t)\hat{n}_b + g \sum_k \left(\hat{b}^\dagger s_k^- + \hat{b} s_k^+ \right)$$

There is a mapping from Gaudin magnets to each of these models!

Gaudin magnets

$$\hat{H}_j = 2Bs_j^z - \sum_{k\neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k}$$

Demkov-Osherov model

Bow-tie model

$$s_0 \to \infty, \quad \hat{s}_0^z \to \hat{n}_b - s_0$$

$$\hat{s}_0^- \to \sqrt{2s}\hat{b}, \quad \hat{s}_0^+ \to \sqrt{2s}\hat{b}^\dagger$$

$$\text{Then,} \quad \hat{H}_0 \to \hat{H}_{\mathrm{D}}(t)$$

Inhomogeneous Dicke model

$$\hat{H}_{\mathrm{D}}(t) = \sum_k \varepsilon_k s_k^z - (\nu t)\hat{n}_b + g\sum_k \left(\hat{b}^\dagger s_k^- + \hat{b} s_k^+\right)$$

Plus various new integrable time-dependent Hamiltonians result if we replace spin SU(2) with other Lie algebras or consider hyperbolic or trigonometric Gaudin magnets

Gaudin magnets

$$\hat{H}_j = 2Bs_j^z - \sum_{k\neq j} \frac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k} \longrightarrow$$

Demkov-Osherov model

$$H_{\mathrm{DO}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$[S^z_{\mathrm{tot}}, \hat{H}_j] = 0 \Rightarrow \quad H_j = \begin{pmatrix} \boxed{1 \times 1} & S^z_{\mathrm{tot}} = \min & \\ & \boxed{\begin{matrix} S^z_{\mathrm{tot}} = \min + 1 \\ N \times N \end{matrix}} & \\ & & \ddots \end{pmatrix}$$

Block-diagonal

$$s_1 = 1, \quad \varepsilon_1 = 0, \quad s_k = \frac{g_k^2}{a_k^2}, \quad \varepsilon_k = -\frac{1}{a_k}, \quad 2B = t - \sum_{k=2}^{N} \frac{g_k^2}{a_k}$$

$N \times N$ block of $\hat{H}_1 \longrightarrow H_{\mathrm{DO}}(t)$

$N \times N$ blocks of $\hat{H}_j \longrightarrow$ commuting partners H_j of $H_{\mathrm{DO}}(t)$

Crucially, the new system satisfies the zero curvature condition

$$H_{\mathrm{DO}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Demkov-Osherov model

$$\boxed{\begin{gathered} [H_j, H_{\mathrm{DO}}] = [H_j, H_k] = 0 \\ \frac{\partial H_j}{\partial a_k} = \frac{\partial H_k}{\partial a_j}, \quad \frac{\partial H_{\mathrm{DO}}}{\partial a_k} = \frac{\partial H_k}{\partial t} \end{gathered}}$$

Guaranteed by the mapping from Gaudins

Unrelated to the mapping, but holds

$$\Updownarrow$$

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} = H_{\mathrm{DO}}(t)\Psi \\[2ex] i\dfrac{\partial \Psi}{\partial a_k} = H_k \Psi \end{cases}$$

The non-stationary Schrödinger eq. can be consistently embedded into a set of multi-time Schrödinger eqs.

Solution of the generalized KZ eqs. via off-shell Bethe ansatz

Off-shell Bethe states: $\Phi(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}) = \prod_{\alpha=1}^{M} \hat{L}^{+}(\lambda_\alpha)|0\rangle, \quad \hat{L}^{+}(\lambda) = \sum_{j=1}^{N} \frac{\hat{s}_j^{+}}{\lambda - \varepsilon_j}$

Yang-Yang action:

$$\mathcal{S}(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}) = -2B \sum_j \varepsilon_j s_j + 2B \sum_\alpha \lambda_\alpha - \frac{1}{2} \sum_j \sum_{k \neq j} s_j s_k \ln(\varepsilon_j - \varepsilon_k) + \sum_j \sum_\alpha s_j \ln(\varepsilon_j - \lambda_\alpha) - \frac{1}{2} \sum_\alpha \sum_{\beta \neq \alpha} \ln(\lambda_\beta - \lambda_\alpha)$$

Solution of KZ eqs:

$$\Psi_{\mathrm{KZ}}(B, \boldsymbol{\varepsilon}) = \oint_\gamma d\boldsymbol{\lambda} \exp\left[-\frac{i\mathcal{S}(\boldsymbol{\lambda}, \boldsymbol{\varepsilon})}{\nu}\right] \Phi(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}), \quad d\boldsymbol{\lambda} = \prod_{\alpha=1}^{M} d\lambda_\alpha$$

Babujian, J. Phys. A (1993); Fioretto, Caux, Gritsev, New J. Phys. (2014)

Solution of the generalized KZ eqs. via off-shell Bethe ansatz

Off-shell Bethe states: $\Phi(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}) = \prod_{\alpha=1}^{M} \hat{L}^+(\lambda_\alpha)|0\rangle, \quad \hat{L}^+(\lambda) = \sum_{j=1}^{N} \frac{\hat{s}_j^+}{\lambda - \varepsilon_j}$

Yang-Yang action:

$$\mathcal{S}(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}) = -2B \sum_j \varepsilon_j s_j + 2B \sum_\alpha \lambda_\alpha - \frac{1}{2} \sum_j \sum_{k \neq j} s_j s_k \ln(\varepsilon_j - \varepsilon_k) + \sum_j \sum_\alpha s_j \ln(\varepsilon_j - \lambda_\alpha) - \frac{1}{2} \sum_\alpha \sum_{\beta \neq \alpha} \ln(\lambda_\beta - \lambda_\alpha)$$

Solution of KZ eqs:

$$\Psi_{\mathrm{KZ}}(B, \boldsymbol{\varepsilon}) = \oint_\gamma d\boldsymbol{\lambda} \exp\left[-\frac{i\mathcal{S}(\boldsymbol{\lambda}, \boldsymbol{\varepsilon})}{\nu}\right] \Phi(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}), \quad d\boldsymbol{\lambda} = \prod_{\alpha=1}^{M} d\lambda_\alpha$$

Babujian, J. Phys. A (1993); Fioretto, Caux, Gritsev, New J. Phys. (2014)

Solution of the non-stationary Schrödinger eq. for the time-dependent BCS Hamiltonians: $\Psi(t) = \Psi_{\mathrm{KZ}}[B(t), \boldsymbol{\varepsilon}]$

A similar technique solves the non-stationary Schrödinger eq. for Demkov-Osherov, bow-tie & driven inhomogeneous Dicke models

Example: Demkov-Osherov model

$$H_{\mathrm{DO}} = \lambda t|1\rangle\langle 1| + \sum_{k=2}^{N} \left(g_k|1\rangle\langle k| + g_k|k\rangle\langle 1| + a_k|k\rangle\langle k|\right)$$

Off-shell Bethe states: $\Phi_{\mathrm{DO}}(\eta, \boldsymbol{a}) = |1\rangle - \sum_{j=2}^{N} \frac{g_j|j\rangle}{a_j - \eta}$

Yang-Yang action: $\mathcal{S}_{\mathrm{DO}}(\eta, \boldsymbol{a}, t) = \eta t - \frac{\eta^2}{2} + \sum_{j=2}^{N} p_j^2 \ln\left(\frac{a_j}{a_j - \eta}\right)$

Solution of the non-stationary Schrödinger eq:

$$\Psi_{\mathrm{DO}}(t, \boldsymbol{a}) = \oint_{\gamma} d\eta e^{-i\mathcal{S}_{\mathrm{DO}}(\eta, \boldsymbol{a}, t)} \Phi_{\mathrm{DO}}(\eta, \boldsymbol{a})$$

Summary

- Formulated a set of conditions under which the non-stationary Schrödinger eq. for a time-dependent quantum Hamiltonian is integrable – embedding into a system of consistent multi-time Schrödinger eqs.

- New intergrable $H(t)$, e.g., the BCS model with coupling $\propto 1/t$, a Floquet BCS model and linearly driven inhomogeneous Dicke model

- Exactly solvable multi-level Landau-Zener problems fit into this construction

- All nontrivial integrable $H(t)$ to date map to Gaudin magnets. Their non-stationary Schrödinger eq. is solvable via off-shell Bethe ansatz

- This theory explains why the scattering matrix factorizes for integrable $H(t)$

Open Questions

- Formulated a set of conditions under which the non-stationary Schrödinger eq. for a time-dependent quantum Hamiltonian is integrable – embedding into a system of consistent multi-time Schrödinger eqs.

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} = \hat{H}\Psi \\ \\ i\nu\dfrac{\partial \Psi}{\partial x_k} = \hat{H}_k\Psi \end{cases} \Longrightarrow \begin{cases} \partial_j \hat{H}_k = \partial_k \hat{H}_j \\ \\ [\hat{H}_k, \hat{H}_j] = 0 \end{cases}$$

- All nontrivial integrable $H(t)$ to date map to Gaudin magnets. Their non-stationary Schrödinger eq. is solvable via off-shell Bethe ansatz

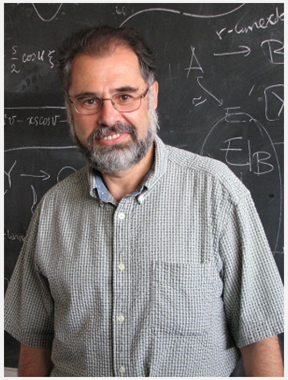
  Q Are there integrable $H(t)$ that do not map to Gaudin magnets? If not, then why? Any integrable $H(t)$ not listed in this talk?

  Q Can we introduce time dependence into, e.g., XXZ or Hubbard Hamiltonian without breaking integrability?

Gaudin magnets: $\hat{H}_j = 2Bs_j^z - \sum_{k\neq j} \dfrac{\vec{s}_j \cdot \vec{s}_k}{\varepsilon_j - \varepsilon_k}$

Aniket Patra
Rutgers

Nikolai Sinitsyn
LANL



Vladimir Chernyak
Wayne State Univ.

Chen Sun
Texas A&M Univ.

- Patra & Yuzbashyan, J. Phys. A 48, 245303 (2015).
- Sinitsyn, Yuzbashyan, Chernyak, Patra & Sun, Phys. Rev. Lett 120, 190402 (2018).
- Yuzbashyan, Ann. Phys. 392, 323 (2018).

National Science Foundation
WHERE DISCOVERIES BEGIN