INTEGRABLE MAPPINGS FOR NONCOMMUTATIVE OBJECTS

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The method of integrable mappings is generalized to the noncommutative case. Hierarchies of integrable systems corresponding to the noncommutative Darboux–Toda substitution in the two-dimensional spaces and superspaces are constructed.

1. Introduction

Recently it has become clear [1] that the group of integrable mappings plays a key role in the theory of integrable system. But up to now only the case of commutative dependent variables has been considered. What happens when the dependent variables are noncommutative operators? The goal of the present paper is to give a particular answer to this question.

Each quantum system can be described in many different representations. The most known and used are the Schrödinger and Heisenberg representations. The first one deals with wave functions (state vectors in the Hilbert space), the second with noncommutative Heisenberg operators and equations of motion under appropriate initial conditions—commutation relations at a fixed moment of time.

The operators which we consider in the present paper are not assumed to have any concrete nature or properties. Particularly, they can be $(s \times s)$-matrix functions or any other operators defined in some representation space. Quantum Heisenberg operators are within the scope of this construction.

The method of integrable mappings seems to be the most suitable tool to solve such problems [1]. We consider two concrete examples of integrable mappings for noncommutative operators. At the moment, there is no idea about classification of all possible integrable mappings for noncommutative operators.

2. Noncommutative Darboux–Toda substitution in the two-dimensional space

2.1. Definitions

Let u and v be two operators depending on x and y coordinates of two-dimensional space.

The partial derivatives up to some sufficiently large order and the inverse operators u^{-1} and v^{-1} are assumed to exist. Consider the mapping

$$\overleftarrow{u} = v^{-1}, \qquad \overleftarrow{v} = [vu - (v_x v^{-1})_y]v \equiv v[uv - (v^{-1}v_y)_x], \tag{2.1}$$

where $\overleftarrow{u}$ and $\overleftarrow{v}$ are the final, transformed operators. The case when u and v are some $s \times s$ matrices (2.1) was considered in [2]. In the classical case u and v are commutative functions and (2.1) is the well-known Darboux–Toda substitution.

Transformation (2.1) is invertible, i.e. the initial operators u and v can be expressed through the final ones. Denoting by $\overrightarrow{u}$ and $\overrightarrow{v}$ the result of the inverse transformation we get

$$\overrightarrow{v} = u^{-1}, \qquad \overrightarrow{u} = [uv - (u_y u^{-1})_x]u \equiv u[vu - (u^{-1}u_y)_x]. \tag{2.2}$$

An s-times directly transformed operator $f(u,v)$ is denoted by $\overset{\longleftarrow s}{f} \equiv f(\overset{\longleftarrow s}{u}, \overset{\longleftarrow s}{v})$. The same s-times transformed operator with the inverse transformation is denoted by $\overset{s\longrightarrow}{f} \equiv f(\overset{s\longrightarrow}{u}, \overset{s\longrightarrow}{v})$. If

$$\begin{aligned} u_t &= F_1(u, v, u_x, v_x, u_y, v_y, \ldots), \\ v_t &= F_2(u, v, u_x, v_x, u_y, v_y, \ldots) \end{aligned} \tag{2.3}$$

is a given evolution-type system, the condition of its invariance with respect to the transformation (2.1) can be derived by differentiation of (2.1) with respect to some parameter and has the form

$$\begin{aligned} \overleftarrow{F}_1 &= \overleftarrow{u}_t = -v^{-1}v_t v^{-1} = -v^{-1}F_2 v^{-1}, \\ \overleftarrow{F}_2 &= \overleftarrow{v}_t = ([vu - (v_x v^{-1})_y]v)_t \\ &= [F_2 u + vF_1 - (F_{2x}v^{-1})_y + (v_x v^{-1}F_2 v^{-1})_y]v + [vu - (v_x v^{-1})_y]F_2. \end{aligned} \tag{2.4}$$

This is the functional symmetry equation for the substitution (2.1). If $[F_1(u,v), F_2(u,v)]$ is a solution of (2.4), the corresponding evolution-type system (2.3) is invariant with respect to substitution (2.1). The system (2.4) is linear, i.e. if $[F_1', F_2']$ and $[F_1'', F_2'']$ are solutions, then $[F_1 = aF_1' + bF_1'',\ F_2 = aF_2' + bF_2'']$ (a and b are arbitrary numbers) is also a solution. Every symmetry equation possesses a trivial solution: $[F_1 = au_x + bu_y,\ F_2 = av_x + bv_y]$. A substitution is called integrable if its symmetry equation has at least one nontrivial solution.

2.2. Solution of the symmetry equation

First of all, let us take $F_2 = \alpha_0 v$ and $F_1 = u\beta_0$. From (2.8) we find

$$\begin{aligned} \beta_0 &= -\overrightarrow{\alpha}_0 \\ \alpha_{0xy} &= (\alpha_0 - \overleftarrow{\alpha}_0)\overleftarrow{T}_0 + T_0(\alpha_0 - \overrightarrow{\alpha}_0) + \theta\alpha_{0y}, -\alpha_{0y}\theta, \end{aligned} \tag{2.5}$$

where $T_0 = vu$ and $\theta = v_x v^{-1}$. This system has an obvious particular solution $\alpha_0{}^{(0)} = -\beta_0{}^{(0)} = 1$ which gives the first term of the hierarchy: $[F_1 = -u, \quad F_2 = v]$. The following two relations are important for farther calculations:

$$T_{0x} = \theta T_0 - T_0\overrightarrow{\theta}, \qquad \theta_y = T_0 - \overleftarrow{T}_0. \tag{2.6}$$

In fact, (2.6) is the substitution (2.1) rewritten in terms of T_0 and θ. Let us take now $\alpha_{0y} = \alpha_1\overleftarrow{T}_0 + T_0\beta_1$. This can be treated as an analog of the vector decomposition by the basic vectors. Substituting this into the second equation of the system (2.5), and expressing $\overleftarrow{T}_{0x}$ and T_{0x} from the first relation of (2.6), we find

$$(\alpha_{1x} + \overleftarrow{\alpha}_0 - \alpha_0 + \alpha_1\overleftarrow{\theta} - \theta\alpha_1)\overleftarrow{T}_0 + T_0(\beta_{1x} + \overrightarrow{\alpha}_0 - \alpha_0 + \beta_1\theta - \overrightarrow{\theta}\beta_1) = 0,$$

from which it follows that if the coefficients before $\overleftarrow{T}_0$ and after T_0 equal zero the system (2.5) is satisfied. We have

$$\begin{aligned} \alpha_{1x} &= \alpha_0 - \overleftarrow{\alpha}_0 + \theta\alpha_1 - \alpha_1\overleftarrow{\theta}, \\ \beta_{1x} &= \alpha_0 - \overrightarrow{\alpha}_0 + \overrightarrow{\theta}\beta_1 - \beta_1\theta. \end{aligned} \tag{2.7}$$

The second relation can obviously be rewritten as

$$\overleftarrow{\beta}_{1x} = -(\alpha_0 - \overleftarrow{\alpha}_0) + \theta\overleftarrow{\beta}_1 - \overleftarrow{\beta}_1\overleftarrow{\theta}.$$

We see that the system (2.7) possesses a particular solution of the form $\overleftarrow{\beta}_1 = -\alpha_1$. Differentiating the equation for α_1 with respect to y we have for that solution

$$\begin{aligned} \beta_1 &= -\overrightarrow{\alpha}_1, \\ \alpha_{1xy} &= (\alpha_1 - \overleftarrow{\alpha}_1)\overset{\leftarrow 2}{T}_0 + T_0(\alpha_1 - \overrightarrow{\alpha}_1) + \theta\alpha_{1y} - \alpha_{1y}\overleftarrow{\theta}. \end{aligned} \tag{2.8}$$

This system is completely analogous to (2.5). It also has a particular solution $\alpha_1 = -\beta_1 = 1$ which gives the next solution of the symmetry equation (2.4), $\alpha_0{}^{(1)} = \int(\overleftarrow{T}_0 - T_0)dy$. Taking $\alpha_{1y} = \alpha_2\overset{\leftarrow 2}{T}_0 + T_0\beta_2$ we can find other terms of the hierarchy using the same scheme. The system for α_2 and β_2 has the same structure as the previous systems. Its particular solution $\alpha_2 = -\beta_2 = 1$ gives

$$\alpha_0{}^{(2)} = \int dy \left[\left(\int dy \left(\overset{\leftarrow 2}{T_0} - T_0 \right) \right) \overset{\leftarrow}{T_0} - T_0 \int dy \left(\overset{\leftarrow}{T_0} - \overset{\rightarrow}{T_0} \right) \right].$$

By induction it can be proved that in the general case the equations for α_n and β_n are

$$\begin{aligned} \beta_n &= -\overset{\rightarrow}{\alpha_n}, \\ \alpha_{nxy} &= (\alpha_n - \overset{\leftarrow}{\alpha_n}) \overset{\leftarrow (n+1)}{T_0} + T_0(\alpha_n - \overset{\rightarrow}{\alpha_n}) + \theta \alpha_{ny} - \alpha_{ny} \overset{\leftarrow n}{\theta}, \end{aligned} \tag{2.9}$$

with a particular solution $\alpha_n = -\beta_n = 1$. The formula for $\alpha_0{}^{(n)}$ can be written in the symbolical form as the sum of 2^n terms

$$\begin{aligned} \alpha_0{}^{(n)} &= (-1)^n \prod_{i=1}^{n} \left(1 - L_i \exp\left[id_i + \sum_{i=k+1}^{n} d_k \right] \right) \\ &\times \int dy \left(T_0 \int dy \left(\overset{\rightarrow}{T_0} \int dy \left(\ldots \int dy \overset{(n-1)\rightarrow}{T_0} \ldots \right) \right) \right), \end{aligned} \tag{2.10}$$

where $\exp(d_p)$ shifts on unity the argument of the p-th integral

$$\ldots \int dy \overset{\leftarrow p}{T_0} \rightarrow \ldots \int dy \overset{\leftarrow (p+1)}{T_0} \ldots$$

and L_r transpose terms in the r-th brackets

$$(A_1(\ldots(A_r[\ldots])\ldots)) \rightarrow (A_1(\ldots([\ldots]A_r)\ldots)).$$

The multiplication rule for these operators is the following:

$$L_i \exp[\ldots]_1 L_j \exp[\ldots]_2 = L_i L_j \exp[[\ldots]_1 + [\ldots]_2].$$

2.3. Examples

i) $n = 0$

$$v_t = v, \qquad u_t = -u.$$

ii) $n = 1$

$$v_t = v_x, \qquad u_t = u_x.$$

iii) $n = 2$

$$v_t = v_{xx} - 2 \int (vu)_x dy \times v, \qquad u_t = -u_{xx} + 2u \int (vu)_x dy.$$

This is the matrix Davey–Stewartson system described in [3].

iv) $n = 3$

$$\begin{aligned} v_t &= v_{xxx} - 3\int (vu)_x dy \times v_x - 3\int (v_x u)_x dy \times v \\ &\quad -3\int \left[vu \int (vu)_x dy - \int (vu)_x dy \times vu \right] dy \times v, \\ u_t &= -u_{xxx} - 3u_x \int (vu)_x dy - 3u \int (v_x u)_x dy \\ &\quad -3u \int \left[vu \int dy (vu)_x - \left(\int dy (vu)_x \right) vu \right] dy. \end{aligned}$$

In the commutative case, this will be the Veselov–Novikov system.

3. Noncommutative Darboux–Toda substitution in the two-dimensional superspace

3.1. Definitions

Now let us analyse a situation when the operators under consideration depend on the Grassmann variables θ_+ and θ_- in addition to the spatial coordinates x and y. Consider the mapping

$$\overleftarrow{u} = v^{-1}, \quad \overleftarrow{v} = -[D_-(D_+ v \times v^{-1}) + vu]v \equiv v[D_+(v^{-1}D_- v) - uv], \tag{3.1}$$

where

$$D_+ = \frac{\partial}{\partial \theta_+} + \theta_+ \frac{\partial}{\partial x}, \quad D_- = \frac{\partial}{\partial \theta_-} + \theta_- \frac{\partial}{\partial y}, \quad D_+^2 = \frac{\partial}{\partial x}, \quad D_-^2 = \frac{\partial}{\partial y}.$$

The notation in this section is the same as in the previous one. The inverse transformation has the form

$$\overrightarrow{v} = u^{-1}, \quad \overrightarrow{u} = -[D_+(D_- u \times u^{-1}) + uv]u \equiv u[D_-(u^{-1}D_+ u) - vu]. \tag{3.2}$$

The symmetry equation for (3.1) is the following:

$$\begin{aligned} \overleftarrow{F_1} &= -v^{-1} F_2 v^{-1}, \\ \overleftarrow{F_2} &= F_2[D_+(v^{-1}D_- v) - uv] \\ &\quad + v[D_+(-v^{-1}F_2 v^{-1} D_- v) + D_+(v^{-1}D_- F_2) - F_1 v - uF_2]. \end{aligned} \tag{3.3}$$

3.2. Solution of the symmetry equation

Here we get the hierarchy of solutions of the symmetry equation (3.3). For this we use the same general method as in the previous section. But there is some difference. As we will see below, particular solutions of (3.3) can be found only at even steps when the unknown operators are the bosonic-like whereas at odd steps they are fermionic-like.

Substituting $F_1 = u\beta_0$ and $F_2 = \alpha_0 v$ into (3.3) we find

$$\begin{aligned} \beta_0 &= -\overrightarrow{\alpha_0}, \\ D_+D_-\alpha_0 &= (\overleftarrow{\alpha}_0 - \alpha_0)\overleftarrow{T_0} + T_0(\alpha_0 - \overrightarrow{\alpha_0}) + \theta D_-\alpha_0 + D_-\alpha_0\theta, \end{aligned} \tag{3.4}$$

where $T_0 = vu$ and $\theta = D_+v \times v^{-1}$. This system has the particular solution $\alpha_0 = -\beta_0 = 1$ which gives $[F_1 = -u, F_2 = v]$.

The transformation (3.1) can be rewritten in terms of T_0 and θ,

$$D_+T_0 = \theta T_0 - T_0\overrightarrow{\theta}, \qquad D_-\theta = -T_0 - \overleftarrow{T_0}. \tag{3.5}$$

Taking now $D_-\alpha_0 = \alpha_1\overleftarrow{T_0} + T_0\beta_1$, for α_1 and β_1 we have

$$\begin{aligned} D_+\alpha_1 &= \overleftarrow{\alpha}_0 - \alpha_0 + \theta\alpha_1 + \alpha_1\overleftarrow{\theta}, \\ D_+\beta_1 &= \alpha_0 - \overrightarrow{\alpha}_0 + \overrightarrow{\theta}\beta_1 + \beta_1\theta. \end{aligned}$$

For $\overleftarrow{\beta}_1 = \alpha_1$, the second equation directly follows from the first one. Acting on the equation for α_1 with D_-, we get

$$\begin{aligned} \beta_1 &= \overrightarrow{\alpha_1}, \\ -D_+D_-\alpha_1 &= (\alpha_1 + \overleftarrow{\alpha}_1)\overleftarrow{T_0}^{2} - T_0(\alpha_1 + \overrightarrow{\alpha_1}) + D_-\alpha_1\overleftarrow{\theta} - \theta D_-\alpha_1. \end{aligned} \tag{3.6}$$

This is a typical system for odd steps. Comparing it with (2.8) we notice that the difference between these systems is that (3.6) has no numerical particular solutions (α_1 and β_1 are fermionic-like). However, it is possible to continue the reduction by using a decomposition: $D_-\alpha_1 = \alpha_2\overleftarrow{T_0}^{2} + T_0\beta_2$. We have

$$\begin{aligned} -D_+\alpha_2 &= \overleftarrow{\alpha}_1 + \alpha_1 - \theta\alpha_2 + \alpha_2\overleftarrow{\theta}^{2}, \\ -D_+\beta_2 &= -(\alpha_1 + \overrightarrow{\alpha_1}) - \overrightarrow{\theta}\beta_2 + \beta_2\overleftarrow{\theta}. \end{aligned}$$

Taking $\overleftarrow{\beta}_2 = -\alpha_2$ after simple calculations we find

$$\begin{aligned} \beta_2 &= -\overrightarrow{\alpha_2}, \\ D_+D_-\alpha_2 &= (\overleftarrow{\alpha}_2 - \alpha_2)\overleftarrow{T_0}^{3} + T_0(\alpha_2 - \overrightarrow{\alpha_2}) + D_-\alpha_1\overleftarrow{\theta}^{2} + \theta D_-\alpha_2. \end{aligned} \tag{3.7}$$

The particular solution of this system $\alpha_2 = -\beta_2 = 1$ corresponds to the trivial system $[F_1 = au_x + bv_y, F_2 = av_x + bv_y]$. All systems derived at even steps will be similar to (3.7). The particular solution of each system of that kind gives a nontrivial and nonlinear evolution-type system invariant with respect to the transformation (3.1) (see the $k = 2$ example). By induction one can easily prove that for arbitrary $n = 2k + 1$ we have

$$\begin{aligned} D_-\alpha_{n-1} &= \alpha_n \overset{\leftarrow n}{T_0} + T_0 \overrightarrow{\alpha}_n, \\ D_-\alpha_n &= \alpha_{n+1} \overset{\leftarrow n+1}{T_0} - T_0 \overrightarrow{\alpha}_{n+1}, \\ D_+D_-\alpha_{n-1} &= (\overleftarrow{\alpha}_{n-1} - \alpha_{n-1})\overset{\leftarrow n}{T_0} - T_0(\alpha_{n-1} - \overrightarrow{\alpha}_{n-1}) + D_-\alpha_{n-1} \overset{\leftarrow n-1}{\theta} + \theta D_-\alpha_{n-1}. \end{aligned} \tag{3.8}$$

Using the $\alpha_{2k} = 1$ particular solution of system (3.8), we construct the k-th term of the hierarchy. The final result can be represented in the following form:

$$\begin{aligned} \alpha_0{}^{(k)} = (-1)^k \prod_{i=1}^{2k} \left(1 - (-1)^i L_i \exp\left[\mathrm{id}_i + \sum_{i=k+1}^{2k} d_k\right]\right) \\ \times D_-^{-1}(T_0 D_-^{-1}(\overrightarrow{T}_0 D_-^{-1}(\ldots D_- -1 \overset{(n-1)\rightarrow}{T_0} \ldots))). \end{aligned} \tag{3.9}$$

The notation here is the same as in (2.10).

3.3. Examples

i) $k = 0$

$$v_t = v, \qquad u_t = -u.$$

ii) $k = 1$

$$v_t = v_x, \qquad u_t = u_x.$$

iii) $k = 2$

$$\begin{aligned} v_t &= v_{xx} - 2D_-^{-1}(vu)_x D_+ v - 2D_-^{-1}(vD_+u) \times v + 2D_-^{-1}\left[vuD_-^{-1}(vu)_x + D_-^{-1}(vu)_x \times vu\right], \\ u_t &= -u_{xx} + D_+uD_-^{-1}(vu)_x - 2uD_-^{-1}(D_+vu)_x - 2uD_-^{-1}\left[vuD_-^{-1}(vu)_x + D_-^{-1}(vu)_x \times vu\right]. \end{aligned}$$

4. Conclusion

The main concrete results of the paper are the constructed hierarchies of integrable systems (2.10) and (3.9) invariant with respect to the corresponding noncommutative Darboux–Toda transformation.

Note that the scheme of our calculations is similar to a computer program algorithm—there are many identical operations which can be interrupted at an arbitrary step. The structure of the group of integrable mappings is coded in this scheme. If it were possible to translate it into the group-theoretical language, we would be closer to the classification theorem for integrable mappings.

It is well known that quantum integrable systems are closely connected with the quantum algebras [4]. We hope that future investigations will allow to find a connection between the method described in the present paper and the sufficiently developed formalism of quantum algebras.

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