Superconductivity near a Quantum-Critical Point: The Special Role of the First Matsubara Frequency

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Near a quantum-critical point in a metal strong fermion-fermion interaction mediated by a soft collective boson gives rise to incoherent, non-Fermi liquid behavior. It also often gives rise to superconductivity which masks the non-Fermi liquid behavior. We analyze the interplay between the tendency to pairing and fermionic incoherence for a set of quantum-critical models with effective dynamical interaction between low-energy fermions. We argue that superconducting T_c is nonzero even for strong incoherence and/or weak interaction due to the fact that the self-energy from dynamic critical fluctuations vanishes for the two lowest fermionic Matsubara frequencies $\omega_m = \pm\pi T$. We obtain the analytic formula for T_c, which reproduces well earlier numerical results for the electron-phonon model at vanishing Debye frequency.

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Introduction.—The interplay between superconductivity and non-Fermi liquid behavior in metals is one of the most fascinating issues in the modern physics of correlated electron systems [1–18]. A generic metallic system in $D > 1$ is a Fermi liquid with coherent quasiparticles at low energies. This coherence is destroyed if the system is brought to a quantum-critical point (QCP), beyond which it develops an electronic order in the spin or charge channel. At a QCP, fluctuations of the order parameter become massless. In $D \leq 3$, the four-fermion interaction, mediated by these massless fluctuations, destroys fermionic coherence at $T = 0$, either at specific hot points on the Fermi surface [4,15,19,20], if the order has a finite momentum, or everywhere on the Fermi surface, if the order develops with $q = 0$ (Ref. [21]). The same massless fluctuations, however, also mediate the pairing interaction, and if this interaction has an attractive angular component the system can develop a superconducting instability at a finite T, before a QCP is reached. A dome of superconductivity above a QCP prevents a non-Fermi liquid, QC behavior from extending down to the lowest energies.

The existence of superconductivity near a QCP is not guaranteed, however, because strong fermionic self-energy acts against pairing. There are two effects from the self-energy. First, at $T \neq 0$ the self-energy from static (thermal) fluctuations acts as an impurity and may cause pair breaking. This is crucial for spin-triplet superconductivity, for which thermal self-energy acts as a magnetic impurity [22], but not for spin-singlet superconductivity, for which it acts as a nonmagnetic impurity and its singular contribution cancels out by Anderson theorem [23]. In this Letter, we consider spin-singlet pairing and neglect the contribution from thermal fluctuations. Second, already at $T = 0$ the self-energy produces strong upturn mass renormalization and shrinks the range of a coherent fermionic behavior. Both of these effects are detrimental to superconductivity.

The pairing amplitude and the self-energy come from the same underlying interaction mediated by a soft boson; hence, the two are generally of the same order. Zero-temperature studies of specific models in $D = 2$ and in $D = 3 - \epsilon$ have shown [3–5,8–10,14] that superconductivity does not develop at QCP; however, these studies also hinted [2–4] that the pairing at a QCP is a threshold problem and may disappear if the self-energy gets enhanced compared to the pairing amplitude. A recent study [16] made this explicit by extending a model in $D = 3 - \epsilon$ to large N in such a way that the self-energy gets enhanced, while the pairing amplitude remains intact. The authors of Ref. [16] performed $T = 0$ analysis and argued that there exists a critical N above which the pairing does not develop because decoherence, caused by strong self-energy, wins over the tendency to pairing due to an attraction.

In this Letter, we analyze the same pairing problem, but at a nonzero T. Our result is different from Ref. [16] and earlier work by some of us (Ref. [3]). We argue that superconducting T_c is finite at arbitrary N. The reason is that the competition between the self-energy and the pairing interaction at a finite temperature is qualitatively different from that at $T = 0$. Namely, at a finite T the Matsubara self-energy $\Sigma(\omega_n)$ is a discrete variable, defined at a set of $\omega_n = \pi T(2n+1)$. It is still large for all $n \neq 0$, -1, but at the two lowest Matsubara frequencies $\omega_n = \pm\pi T$ it vanishes if we neglect the contribution from static bosonic fluctuations [24]. At the same time, the pairing interaction $\chi(\Omega_m)$, also taken without the static part (i.e., at bosonic $\Omega_m = 2\pi T m$, $m \neq 0$) is not reduced at $\Omega_m = \pi T - (-\pi T) = 2\pi T$ compared to $\chi(\Omega_m)$ at other Ω_m. As a result, the pairing interaction between fermions with $\omega_n = \pm\pi T$ is strong, while the competing contribution from the self-energy is absent. Although this holds only for the two Matsubara frequencies, we show that this is sufficient to render T_c finite. Moreover, T_c is not small and has a power-law dependence of the coupling constant, which is stronger than the logarithmical divergence in Bardeen-Cooper-Schrieffer (BCS) theory, although the latter is obtained by summing up an infinite set of Matsubara points.

In broader terms, we argue against the commonly used procedure [3,4,8–10,14,16] to obtain T_c at a QCP by computing the pairing susceptibility $\chi_{pp}(\omega)$ at $T=0$, associating the superconducting region with the range of N where $\chi_{pp}(\omega)$ becomes negative below some ω^*, and identifying T_c with $O(\omega^*)$. We argue that T_c has to be determined from the actual calculations at a finite T, and T_c generally does not scale with ω^*, except for special cases like models in $D = 3-\epsilon$ and $N = O(1)$.

To be specific, our conclusion holds for a set of QC models with dynamical interaction between fermions, for which the Eliashberg approximation [25] is valid. Within this approximation, the momentum integration in the gap equation can be carried out explicitly, and the analysis of superconductivity reduces to a set of equations for the frequency dependent pairing vertex $\Phi(\omega_m)$ and fermionic self-energy $\Sigma(\omega_m)$, both originating from the effective, momentum-averaged interaction $\chi(\omega_m - \omega'_m)$. We consider a generic case of $\chi(\Omega_m) = (g/|\Omega_m|)^\gamma$, where g is the effective fermion-boson coupling. We list specific examples of different γ below. In particular, $\gamma = 2$ corresponds to the much studied strong coupling limit of electron-phonon interaction [1,26–28]. We argue that T_c is nonzero for any γ, even if the self-energy is enhanced after a proper extension of the model to large N, as in [16]. Moreover, at large N, $T_c \approx [g/(2\pi)]/N^{1/\gamma} \approx 0.16g/N^{1/\gamma}$ is fully determined by the two lowest Matsubara frequencies. At $N=1$ this formula yields $T_c \approx 0.16g$. This value is very close to $T_c \approx 0.18g$ obtained numerically for $\gamma = 2$ (Refs. [26,29]). which implies that T_c for the QC electron-phonon problem is predominantly determined by just the two lowest Matsubara frequencies.

The model.—We consider a system of fermions at the boundary between a Fermi liquid state and a state with a long-range order in either spin or charge channel (ferromagnetism, nematic order, spin- or charge-density wave, etc.). At a QCP, the propagator of a soft boson becomes massless and mediates singular interaction between fermions. As previously stated, we treat this interaction as attractive in at least one pairing channel. This is true for QCP towards density-wave instabilities [30], but we caution that this is not always the case—e.g., for fermions at the half-filled lowest Landau level, long-range current-current interaction mediated by gapless gauge fluctuations is repulsive in all channels [14].

We assume, following earlier work [3,4,7,10,13–16, 31,32], that bosons can be treated as slow modes compared to fermions; i.e., the Eliashberg approximation is valid. Within this approximation one can explicitly integrate over the momentum component perpendicular to the Fermi surface and reduce the integral equations for the self-energy Σ and the pairing vertex Φ to the set for $\Sigma(\mathbf{k}_F, \omega_m)$ and $\Phi(\mathbf{k}_F, \omega_m)$ on the Fermi surface. We will be interested in the solution for T_c; hence, we set $\Phi(\mathbf{k}_F, \omega_m)$ to be infinitesimally small and approximate $\Sigma(\mathbf{k}_F, \omega_m)$ by its normal state value. We make one additional approximation —we assume that the dependence of $\Phi(\mathbf{k}_F, \omega_m)$ on ω_m and on the momentum direction along the Fermi surface can be factorized, i.e., that $\Phi(\mathbf{k}_F, \omega_m) = f_\Phi(\mathbf{k}_F)\Phi(\omega_m)$, where f_Φ has the symmetry of the corresponding superconducting state [4,18], and we neglect the momentum dependence of $\Sigma(\mathbf{k}_F, \omega_m)$. Under this approximation, the integration over the momentum component along the Fermi surface can be done explicitly [4,15], and the set of equations for T_c reduces to the integral equation for $\Phi(\omega_m)$ and the equation for the normal state self-energy $\Sigma(\omega_m)$:

$$\Phi(\omega_m) = \frac{g^\gamma}{N}\pi T \sum_{m' \neq m} \frac{\Phi(\omega_{m'})}{|\omega_{m'} + \Sigma(\omega_{m'})|} \frac{1}{|\omega_m - \omega_{m'}|^\gamma},$$

$$\Sigma(\omega_m) = g^\gamma \pi T \sum_{m' \neq m} \frac{\mathrm{sgn}(\omega_{m'})}{|\omega_m - \omega_{m'}|^\gamma}, \qquad (1)$$

where we incorporated the overall factors from the integration over momentum into g. As previously stated, we neglect the terms with $m = m'$ in Eq. (1) because for spin-singlet pairing such terms cancel out between $\Phi(\omega_m)$ and $\Sigma(\omega_m)$. We discuss this in more detail in Ref. [33]. The overall factor $1/N$ is the result of extending the model to an SU(N) global symmetry which involves both fermions and bosons [16]. We treat N as a parameter. Our goal is to understand whether there is a critical N above which $T_c = 0$; i.e., the normal state extends down to $T = 0$.

Models described by Eq. (1) include a model for color superconductivity [5] [$\gamma = 0_+$, $\chi(\Omega_m) \propto \log|\omega_m|$], models for spin- and charge-mediated pairing in $D = 3-\epsilon$ dimension [10,14,16] [$\gamma = O(\epsilon) \ll 1$], a 2D pairing model [35] with interaction peaked at $2k_F$ ($\gamma = 1/4$), 2D models for pairing at a nematic or Ising-ferromagnetic QCP [2,17,22] ($\gamma = 1/3$), a 2D hot-spot model for pairing at the (π, π) SDW QCP [3,4,36,37] and at a 2D CDW QCP [28,38], 2D models for pairing by undamped fermions ($\gamma = 1$), the strong coupling limit of phonon-mediated superconductivity [1,26,29], and models with parameter-dependent γ (Refs. [8,9]).

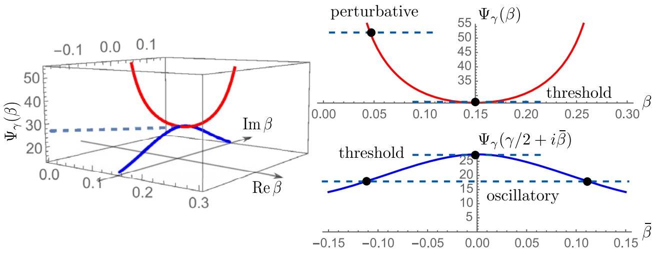

FIG. 1. Left: The plot of the function $\Psi_\gamma(\beta)$ for $\gamma = 0.3$. Right: the solution of the equation $[(1-\gamma)/2N]\Psi_\gamma(\beta) = 1$. For large N, β is real, as in perturbation theory (red line); i.e., superconductivity does not develop. For $N < N_{\rm cr}$, $\beta = \gamma/2 \pm i\bar\beta$ is complex (blue line). For a complex β, $\Phi(\omega_m)$ is oscillatory in frequency, implying that T_c is finite.

The argument for the threshold.—To set the stage for our analysis, we briefly display the argument for the existence of a threshold in N for T_c. The argument is based on the analysis of the pairing susceptibility at $T = 0$ for $0 < \gamma < 1$ (Refs. [3,16]). At $T = 0$ the self-energy has a non-Fermi liquid form: $\Sigma(\omega_m) = |\omega_m|^{1-\gamma}\omega_0^\gamma {\rm sgn}(\omega_m)$, where $\omega_0 = g[2/(1-\gamma)]^{1/\gamma}$. Substituting this $\Sigma(\omega_m)$ into the equation for $\Phi(\omega)$ and adding up a bare Φ_0, one can compute the $T = 0$ pairing susceptibility $\chi_{pp}(\omega_m) = \Phi(\Omega_m)/\Phi_0$ at $\omega_m < \omega_0$ order by order in $1/N$. The building block for series for $\chi_{pp}(\omega_m)$ is $\int d\omega_{m'} 1/(|\omega_{m'} - \omega_m|^\gamma |\omega_{m'}|^{1-\gamma})$, where the first term comes from the interaction and the second from the self-energy. The integrand scales as $1/|\omega_{m'}|$ at $\omega_{m'} > \omega_m$; hence, the series for $\chi_{pp}(\omega_m)$ is logarithmic. At $N \gg 1$ the coupling is weak and one can just sum up the series of leading logarithms, as in BCS theory. However, this analogy does not go further because in our case, each logarithm is cut by ω_m rather than by T and the summation of the logarithms yields $\chi_{pp}(\omega_m) = 1 + \alpha \log(\omega_0/|\omega_m|) + \alpha^2/2[\log(\omega_0/|\omega_m|)]^2 + \alpha^3/6[\log(\omega_0/|\omega_m|)]^3 + \cdots = (\omega_0/|\omega_m|)^\alpha$, where $\alpha = (1-\gamma)/N$. This susceptibility is positive; i.e., the summation of the leading logarithms does not give rise to pairing.

This line of reasoning is developed further by solving for the susceptibility beyond the logarithmical approximation. The $1/\omega_{m'}$ scaling of the kernel suggests a power-law form $\Phi(\omega_m) \propto (\omega_0/|\omega_m|)^\beta$ at $\omega_m < \omega_0$. Substituting this into (1) and evaluating the integrals, we obtain an equation on β of the form $(\alpha/2)\Psi_\gamma(\beta) = 1$, where $\Psi_\gamma(\beta) = \Gamma(\beta)\Gamma(\gamma-\beta)/\Gamma(\gamma) + \Gamma(1-\gamma)\{[\Gamma(\beta)/\Gamma(1-\gamma+\beta)] + [\Gamma(\gamma-\beta)/\Gamma(1-\beta)]\}$. We plot $\Psi_\gamma(\beta)$ in Fig. 1. Solving for β as a function of α and γ and choosing the branch which gives $\beta \approx \alpha$ at small α, consistent with logarithmical perturbation theory, we find that β increases with α, reaches the value $\gamma/2$ at a critical $\alpha_{\rm cr} = (1-\gamma)/N_{\rm cr}$, and at larger α (i.e., smaller N) becomes complex: $\beta = \gamma/2 \pm i\bar\beta$, where $\bar\beta \propto (\alpha - \alpha_{\rm cr})^{1/2} \sim (N_{\rm cr} - N)^{1/2}$. As the consequence, χ_{pp} becomes an oscillating function of ω_m: $\chi_{pp}(\omega_m) \propto (\omega_0/|\omega_m|^{\gamma/2}) \cos[\bar\beta \log(\omega_0/|\omega_m|) + \psi_0]$, where ψ_0 is an arbitrary phase. Oscillations of the pairing susceptibility cannot be obtained within a perturbation theory and their presence was interpreted as the sign that the system has already undergone a pairing instability at some finite T_c. To obtain T_c, earlier works used the $T = 0$ form of $\chi_{pp}(\omega_m)$ and identified T_c with the largest ω_m at which $\chi_{pp}(\omega_m)$ first becomes negative. At $\alpha \geq \alpha_{\rm cr}$, when β is small, this yields [3,16,39] $T_c \sim \omega_0 e^{-a/(N_{\rm cr}-N)^{1/2}}$, where $a = O(1)$.

Finite T analysis.—We now perform the actual analysis at a finite T and argue that it yields a result different from the one at $T = 0$. Namely, we argue that T_c is nonzero for any N and only tends to zero when N tends to infinity. We show that this result originates from the vanishing of the self-energy at Matsubara frequencies $\omega_m = \pm\pi T$. The special role of the lowest Matsubara frequencies cannot be detected in the $T = 0$ analysis in which Matsubara frequency is a continuous variable.

Vanishing of the self-energy $\Sigma(\omega_m = \pm\pi T)$ can be readily seen from Eq. (1). We have $\Sigma(\pi T) = [g/(2\pi T)]^\gamma \pi T \sum_{m'\neq 0} {\rm sgn}(2m'+1)/|m'|^\gamma$, and the sums over positive and negative m' cancel each other. The same holds for $\omega_m = -\pi T$. For any other $m \geq 1$, $\Sigma(\omega_m > 0) \sim \omega_m [g/(2\pi T)]^\gamma \gg \omega_m$; i.e., at low T the self-energy at $|\omega_m| \neq \pi T$ well exceeds the bare ω_m term in the fermionic propagator. Note in passing that the vanishing of $\Sigma(\omega_m = \pm\pi T)$ in our analysis does not actually imply that at this frequency a fermion is a free quasiparticle, because we eliminated from $\Sigma(\omega_m)$ the contribution from static critical fluctuations [the $m = m'$ term in Eq. (1)]. Such a contribution is irrelevant for the pairing, but it is parametrically larger than T near a QCP; hence, the full self-energy has a non-Fermi liquid form even at $\omega_m = \pm\pi T$.

To make our point about T_c, we consider large N and small T. Neglecting ω_m compared to the self-energy for all m except $m = 0$ and $m = -1$, using the symmetry conditions $\Phi(\omega_m) \equiv \Phi_m = \Phi_{-m-1}$ and $\Sigma(\omega_m) \equiv \Sigma_m = -\Sigma_{-m-1}$, and introducing $\bar\Phi_m \equiv \Phi_m/(\pi T K_T)$, $\bar\Sigma_m \equiv \Sigma_m/(\pi T K_T)$, where $K_T = [g/(2\pi T)]^\gamma \gg 1$, we rewrite the gap equation in (1) as a set of coupled equations for $\bar\Phi_{m=0,-1}$ and $\bar\Phi_{m>0}$:

$$\bar\Phi_0 = \frac{K_T}{N}\bar\Phi_{-1} + \frac{1}{N}\sum_{m>0}\frac{\bar\Phi_m}{\bar\Sigma_m}\left[\frac{1}{m^\gamma} + \frac{1}{(m+1)^\gamma}\right],$$

$$\bar\Phi_{m>0} = \frac{K_T}{N}\left[\frac{\bar\Phi_0}{m^\gamma} + \frac{\bar\Phi_{-1}}{(m+1)^\gamma}\right] + \frac{1}{N}\sum_{m'>0,m'\neq m}\frac{\bar\Phi_m}{\bar\Sigma_m}\left[\frac{1}{|m-m'|^\gamma} + \frac{1}{(m+m'+1)^\gamma}\right]. \tag{2}$$

We distinguish $\bar\Phi_0$ and $\bar\Phi_{-1}$ in (2) only for illustrative purposes. In fact, the two are equal, $\bar\Phi_0 = \bar\Phi_{-1}$.

At vanishing $1/N$ Eq. (2) has a solution at $K_T = N$, i.e., at $T = T_c = (g/2\pi)/N^{1/\gamma}$. Indeed, the first equation in (2)

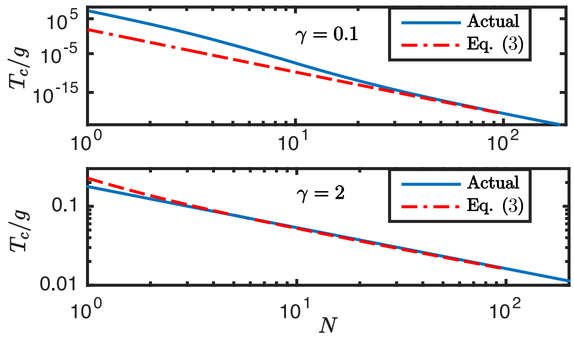

FIG. 2. Superconducting T_c, obtained by solving the gap Eq. (2) numerically (labeled as "actual"), vs the analytical result from Eq. (3). Upper panel: $\gamma = 0.1$; lower panel: $\gamma = 2$. In both cases the analytical T_c perfectly matches the numerical one at large N. For $N = O(1)$, the numerical solution yields much larger T_c than Eq. (3) for $\gamma = 0.1$, but for $\gamma = 2$ numerical and analytical results remain close even for $N = 1$.

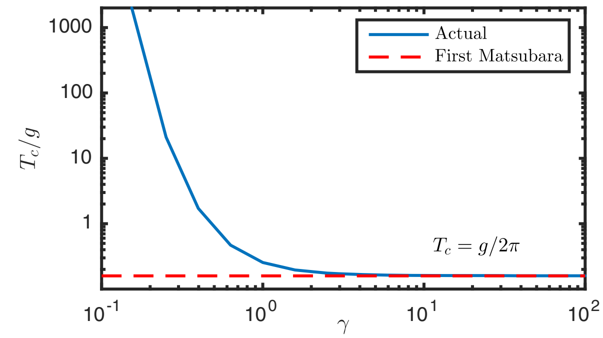

FIG. 3. The numerical result for T_c at $N = 1$ as a function of γ. At small γ, T_c is determined by all Matsubara frequencies and increases exponentially with decreasing γ (see the text). At $\gamma > 1$ it rapidly approaches $T_c = g/2\pi$, which we obtained analytically from the two lowest Matsubara frequencies.

is satisfied, while the second one determines $\overline{\Phi}_m$ for all $m > 0$ in terms of $\overline{\Phi}_0$: $\overline{\Phi}_{m>0} = \overline{\Phi}_0[1/m^\gamma + 1/(m+1)^\gamma]$. Plugging this $\overline{\Phi}_{m>0}$ into the first equation in (2), we obtain T_c with $1/N$ correction (see Ref. [33] for details):

$$T_c \approx \frac{g}{2\pi}\frac{1}{N^{1/\gamma}}\left(1 + \frac{\delta_\gamma}{N\gamma}\right), \quad (3)$$

where $\delta_\gamma = \sum_{m>0}[1/m^\gamma + 1/(m+1)^\gamma]^2/\overline{\Sigma}_m$ is a number of order one. We see that T_c is nonzero for *any* N, i.e., no matter how strong is the self-energy at Matsubara frequencies ω_m with $m \neq 0, -1$. We also see that superconducting T_c is predominantly determined by the two lowest Matsubara frequencies, for which the pairing interaction is strong, but the self-energy vanishes. This new understanding is very different from the previous one that superconductivity at a QCP originated from the pairing of incoherent fermions at $T = 0$.

The value of T_c.—In Fig. 2 we show T_c given by Eq. (3), together with the numerical solution of the gap equation. We see that at large N the actual solution and the one from Eq. (3) agree quite well, as expected. The agreement does not extend to $N \sim 1$ at small γ, but gets progressively better for larger γ, for which T_c is predominately determined by the first two Matsubara frequencies even for $N = 1$, i.e., $T_c \approx g/(2\pi)$. Other Matsubara frequencies account only for a small correction to $T_c = g/(2\pi)$. To verify this, we computed the leading correction in $1/\gamma$ for an arbitrary N and obtained $T_c = g/2\pi(s/N)^{1/\gamma}$, where $s = s(N)$ is determined from $J_{3/2+N/s}(1/s)/J_{1/2+N/s}(1/s) = s - 1$, where J is a Bessel function (see Ref. [33] for details). At $N = 1$, $s = 1.1843$, at $N \gg 1$, $s = 1 + 1/(2N)$, in agreement with Eq. (3) [in Eq. (3), $\delta_\gamma \to 1/2$ at $\gamma \to \infty$]. For the strong coupling limit of electron-phonon superconductivity ($\gamma = 2$, $N = 1$), $T_c \approx 0.17g$, which is very close to $0.18g$, obtained in extensive numerical studies [26,29] on a large mesh of Matsubata frequencies. This has been noticed in Ref. [29] but not related to the absence of the self-energy at $\omega_m = \pm\pi T$.

For completeness, we also computed T_c at small γ and $N = O(1)$. In this regime $T_c \gg \omega_0$ (see Fig. 2) and the self-energy is again irrelevant, but now simply because at $T = T_c$, $\omega_m \gg \Sigma(\omega_m)$ for all m. Neglecting $\Sigma(\omega_{m'})$ in Eq. (1), we obtain (see Ref. [33] for details)

$$T_c \sim \omega_0(\gamma N)^{-1/\gamma} \sim \frac{g}{2\pi N^{1/\gamma}} e^{\log(b/\gamma)/\gamma} \gg \frac{g}{2\pi N^{1/\gamma}}, \quad (4)$$

where $b = O(1)$. A similar result for the pairing scale has been obtained in Refs. [14,40] using the RG procedure. Note in passing that the divergence of T_c at $\gamma \to 0$ is the consequence of the fact that in this limit the effective interaction $\chi(\Omega_m) = (g/|\Omega_m|)^\gamma$ tends to a constant, while there is no upper cutoff in the theory. If we add a cutoff, we indeed obtain that T_c saturates.

In Fig. 3 we plot T_c at $N = 1$ obtained numerically from the Eliashberg equation (1). We see that at $\gamma > 1$, T_c rapidly approaches $g/2\pi$—the result which we obtained analytically from the two lowest Matsubara frequencies. We emphasize that at both small and large γ the fermionic self-energy is irrelevant for T_c. At $\gamma \sim 1$, it does affect the value of T_c, but is not crucial in the sense that a comparable T_c is obtained without including the self-energy.

Conclusion.—In this Letter, we computed superconducting T_c for a set of quantum-critical models with Eliashberg-type effective dynamical interaction between low-energy fermions. We found that superconductivity always develops above a quantum-critical point, no matter what the interplay between the pairing interaction and the fermionic incoherence at $T = 0$. We argued that the proper calculation of T_c should be done directly at a finite temperature, and T_c is nonzero due to the fact that at a finite T the self-energy vanishes at the two lowest fermionic Matsubara frequencies $\omega_m = \pm\pi T$. This implies that fermionic incoherence at a QCP is not an obstacle for superconductivity. We caution, however, that this is true for the Eliashberg T_c, which does not include fluctuations of the pairing gap. The analysis of the gap fluctuations requires separate consideration.

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