Relaxation and Persistent Oscillations of the Order Parameter in Fermionic Condensates

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We determine the limiting dynamics of a fermionic condensate following a sudden perturbation for various initial conditions. Possible initial states of the condensate fall into two classes. In the first case, the order parameter asymptotes to a *constant* value. The approach to a constant is oscillatory with an inverse square root decay. This happens, e.g., when the strength of pairing is abruptly changed while the system is in the paired ground state and more generally for any *nonequilibrium* state that is in the same class as the ground state. In the second case, the order parameter exhibits persistent oscillations with several frequencies. This is realized for nonequilibrium states that belong to the same class as excited stationary states.

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Response of fermionic condensates to fast perturbations is a long-standing problem [1–12]. The main difficulty is to describe the time evolution in the nonadiabatic regime when a nonequilibrium state of the condensate is created on a time scale shorter than the energy relaxation time. In this case the evolution of the system cannot be described in terms of a quasiparticle spectrum or a single time-dependent order parameter $\Delta(t)$ and particle-hole excitations [13]. One has to account for the dynamics of individual Cooper pairs, making it a complex many-body problem.

The nonadiabatic regime can be accessed experimentally in ultracold Fermi gases, where the strength of pairing between fermions can be rapidly changed [14]. Nonadiabatic measurements can be also performed in quantum circuits utilizing nanoscale superconductors where the dynamics can be initiated by fast voltage pulses [15].

Here we consider a BCS condensate that is out of equilibrium at $t = 0$ and study its time evolution for $t > 0$. Given the state of the system at $t = 0$, we predict the dynamics with no need for actually solving equations of motion. We show that possible initial states fall into two categories. In the first case, $|\Delta(t)|$ asymptotes to a constant value $\Delta_\infty < \Delta_0$. The approach to Δ_∞ is oscillatory with a $1/\sqrt{t}$ decay,

$$\frac{|\Delta(t)|}{\Delta_\infty} = 1 + a\frac{\cos(2\Delta_\infty t + \phi)}{\sqrt{\Delta_\infty t}}. \qquad (1)$$

Constants a and ϕ depend on the initial state. This is realized, e.g., when the pairing strength is abruptly changed, while the system is in the paired ground state. In the second case, $|\Delta(t)|$ oscillates persistently with several incommensurate frequencies. We propose a topological *classification* of initial states, which extends the concept of excitation spectrum to the nonlinear regime. If a state is in the same class as the paired ground state, Eq. (1) applies. Other states are topologically distinct, in which case persistent oscillations occur.

Our approach explains differences between previous studies of condensate dynamics. Linear analysis around the ground state yields [3] damped oscillations with a frequency $2\Delta_0$, where Δ_0 is the equilibrium BCS gap. Equation (1) generalizes this result to the nonlinear case and a wide range of initial conditions. An oscillatory decay following a change in the coupling strength was observed numerically [8,11]. We will see that this is due to the fact that initial states of Refs. [3,8,11] are in the same class as the BCS ground state. Undamped periodic oscillations of $|\Delta(t)|$ have been found in Refs. [4–6]. They were also seen numerically for initial states close to a normal state [7]. In contrast, Ref. [12] also starts from the normal state, but obtains a saturation to $\Delta_\infty = \Delta_0/2$. It turns out [16] that this occurs if the initial state is a *paired* state with a small seed gap $\Delta_{\rm in} \ll \Delta_0$. Quasiperiodic oscillations of the order parameter [9,10] can also be realized (see below).

In the nondissipative regime, dynamics of the condensate can be described by the BCS model. Here we are interested in the thermodynamic limit, in which case one can use the BCS mean-field approach [1]. Using Anderson's pseudospin representation [1], one can describe the mean-field evolution by a *classical* spin Hamiltonian [9] $H = \sum_j 2\epsilon_j s_j^z - g\sum_{j,k} s_j^+ s_k^-$, where ϵ_j are single-particle energies and $s_j^\pm = s_j^x \pm i s_j^y$. Dynamical variables $\mathbf{s}_j$ are vectors of fixed length, $|\mathbf{s}_j| = 1/2$. The BCS order parameter is $\Delta(t) = \Delta_x - i\Delta_y = g\sum_j s_j^-$. Equations of motion are

$$\dot{\mathbf{s}}_j = \mathbf{b}_j \times \mathbf{s}_j \qquad \mathbf{b}_j = (-2\Delta_x, -2\Delta_y, 2\epsilon_j). \qquad (2)$$

Consider the Fourier transform of the absolute value of the order parameter $|\Delta(t)|$. In the thermodynamic limit the frequency spectrum in general consists of continuum and discrete parts. Let the discrete part contain k incommensurate frequencies and its contribution to $|\Delta(t)|$ be $F_k(t)$,

$$|\Delta(t)| = F_k(t) + \int_{-D}^{D} d\omega A(\omega)\cos[\omega t + \phi(\omega)], \qquad (3)$$

where D is an ultraviolet cutoff. The integral in Eq. (3) vanishes as $t \to \infty$ for any well-behaved $A(\omega)$. Hence, $|\Delta(t)| \to F_k(t)$ for $t \to \infty$, i.e., $|\Delta(t)|$ oscillates with k frequencies for $k \geq 1$ and $|\Delta(t)| \to \Delta_\infty$ if $k = 0$.

Given the initial state, the structure of the frequency spectrum can be determined using integrability of BCS dynamics [9,10]. Frequencies of an integrable system depend only on its integrals of motion [17], which can be evaluated at $t = 0$. Frequencies can thus be determined without solving equations of motion. Here we use the method of Ref. [10]. It is convenient to introduce the following vector function (Lax vector) of an auxiliary (spectral) parameter u:

$$\mathbf{L}(u) = -\frac{\hat{\mathbf{z}}}{g} + \sum_j \frac{\mathbf{s}_j}{u - \epsilon_j}, \qquad (4)$$

where $\hat{\mathbf{z}}$ is a unit vector along the z axis. The square of the Lax vector is conserved by the evolution. The frequency spectrum is related to branch cuts of $\sqrt{\mathbf{L}^2(u)}$. Note that the numerator of $\mathbf{L}^2(u)$ is a polynomial of degree $2n$, where n is the total number of levels ϵ_j. Since $\mathbf{L}^2(u) \geq 0$, all $2n$ roots come in complex conjugate pairs. For finite n, all roots are typically distinct leading to n isolated cuts connecting pairs of conjugate roots. This situation is described by the general solution [9] for finite n. However, one can also have $2(n - m)$ roots that are real and thus double degenerate. This leaves m branch cuts corresponding to remaining m pairs of complex conjugate roots. The dynamics can now be described in terms of $m < n$ effective spins governed by the same classical Hamiltonian H but with m spins and m new effective energy levels. These *m-spin* solutions contain only m incommensurate frequencies. One frequency corresponds to a uniform rotation of all spins around the z axis. Thus, $|\Delta(t)|$ contains $m - 1$ frequencies [10].

In the thermodynamic limit some roots of $\mathbf{L}^2(u)$ merge into continuous lines and give rise to the continuum part of the spectrum, while isolated pairs of roots correspond to the discrete part. Thus, the number of discrete frequencies in $|\Delta(t)|$ is the number of isolated pairs of roots less one, $k = m - 1$. At large times $|\Delta(t)|$ exhibits persistent oscillations with k frequencies [18] and is described by an m-spin solution [19]. The number k is a topological property of the initial state. It is the number of handles on the Riemann surface of the function $\sqrt{\mathbf{L}^2(u)}$.

Discrete part of the frequency spectrum turns out to be related to discontinuities of the spin distribution $\mathbf{s}(\epsilon)$ as a function of ϵ. Indeed, consider stationary states. There are two types of such states. The BCS ground state and excited states with $\Delta \neq 0$ are obtained by aligning each spin $\mathbf{s}_j$ self-consistently along its effective magnetic field $\mathbf{b}_j$. These states can be termed *anomalous* stationary states. Choosing the x axis so that Δ is real, we obtain

$$2s_j^z = -\frac{e_j\epsilon_j}{\sqrt{\epsilon_j^2 + \Delta^2}} \qquad 2s_j^x = -\frac{e_j\Delta}{\sqrt{\epsilon_j^2 + \Delta^2}}, \qquad (5)$$

where $e_j = -1$ if the spin is parallel to the field and $e_j = 1$ otherwise. The self-consistency condition $\Delta = g\sum_j s_j^x$ yields the BCS gap equation. The state with all $e_j = 1$ is the BCS ground state. The state $e_k = -1$ and $e_{j\neq k} = 1$ has a single excited pair [1] of energy $2\sqrt{\epsilon_k^2 + \Delta_0^2}$. Using Eqs. (4) and (5) and the gap equation, we derive

$$\mathbf{L}_s(u) = -(\Delta\hat{\mathbf{x}} + u\hat{\mathbf{z}})L_s(u),$$
$$L_s(u) = \sum_j \frac{e_j}{2(u - \epsilon_j)\sqrt{\epsilon_j^2 + \Delta^2}}.$$

We see that $\mathbf{L}_s^2(u) = (u^2 + \Delta^2)L_s^2(u)$ has a pair of isolated roots at $u = \pm i\Delta$. All other roots are determined by the equation $L_s(u) = 0$ and are double degenerate.

First, consider the ground state, $e_j = 1$. Since $L_s(u) \to \pm\infty$ as $u \to \epsilon_j \pm 0$ for each j, all roots of $L_s(u)$ are real and located between consecutive ϵ_j. In the thermodynamic limit they merge into a line from $-D$ to D [Fig. 1(a)]. Note that the existence of a double real root between ϵ_j and ϵ_{j+1} relies on $e_j = e_{j+1}$. Further, Eq. (5) implies that when $e_j = e_{j+1}$, the components of spins $\mathbf{s}(\epsilon)$ are continuous at $\epsilon = \epsilon_j$. Now let $e_j = -e_{j+1}$. In this case the real root between ϵ_j and ϵ_{j+1} can disappear. Thus, discontinuities (jumps) in the spin distribution generate isolated complex roots [see Fig. 3(a) for an example]. Since spins far from the Fermi level are not flipped, the total number of jumps is even. In

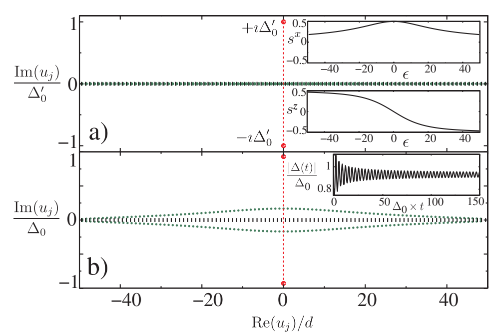

FIG. 1 (color). (a) Roots u_j of $\mathbf{L}^2(u)$ for the BCS ground state with gap Δ_0'; d is the level spacing. The absence of discontinuities in the spin distribution (insets) ($2p = 0$) implies a single pair of isolated roots at $\pm i\Delta_0'$ (red circles connected by a dashed line). Remaining roots (green circles) are real and doubly degenerate. (b) At $t = 0$ the coupling is abruptly increased so that the corresponding ground state gap is $\Delta_0 = 2.4\Delta_0'$. The line of double real roots deforms into two conjugate lines. There is $m = 2p + 1 = 1$ isolated cut of $\mathbf{L}^2(u)$ (red circles connected by a dashed line). The frequency spectrum of $|\Delta(t)|$ is thus continuous: $k = m - 1 = 0$, i.e., $|\Delta(t)|$ (inset) asymptotes to a constant Δ_∞.

general, for $2p$ jumps $L_s(u)$ can have up to p pairs of isolated roots.

One can linearize Eq. (2) around anomalous stationary states and solve for normal modes. The eigenvalues $\omega_j = 2\sqrt{u_j^2 + \Delta^2}$ are determined by the roots u_j of $\mathbf{L}_s^2(u)$. For the ground state $u_j = \epsilon_j$ up to finite size corrections [cf. Ref. [1]].

Consider few examples of nonequilibrium initial states. Let the system be in an anomalous stationary state for $t < 0$. Suppose at $t = 0$ the coupling changes abruptly from g' to g. It follows from Eq. (4) that the change in g results in a smooth deformation of the root distribution: lines of roots deform into lines. On the other hand, doubly degenerate roots become nondegenerate, e.g., p pairs of isolated double roots in addition to a pair of roots $\pm i\Delta$ become $m = 2p + 1$ pairs of single roots, i.e., $\sqrt{\mathbf{L}^2(u)}$ acquires $2p + 1$ cuts. Accordingly, $|\Delta(t)|$ exhibits persistent oscillations with $k = m - 1 = 2p$ frequencies (Fig. 3).

Let the initial state be the ground state with coupling g'. Then, the line of double real roots splits into two complex conjugate lines [Fig. 1(b)]. There is only one pair of isolated roots as in the ground state. Therefore, $k = 0$ and $|\Delta(t)|$ *asymptotes to a constant* Δ_∞ at $t \gg \tau_\Delta$, as illustrated in Fig. 1. According to Eq. (2), at large times spin $\mathbf{s}_j$ rotates in a constant magnetic field $\mathbf{b}_j = (2\Delta_\infty, 0, 2\epsilon_j)$ with a frequency $\omega(\epsilon_j) = 2\sqrt{\epsilon_j^2 + \Delta_\infty^2}$. Using this, one derives Eq. (1). The $1/\sqrt{t}$ decay law is set by the square root singularity in the spectral density [16]. Note that, although $|\Delta(t)|$ asymptotes to a constant, the final state of the system is nonstationary [12].

There is another type of stationary states—*normal* states. In these states each spin is aligned along the z axis, $s_j^z = z_j/2 = \pm 1/2$. The Fermi state is $z_j = -\mathrm{sgn}\,\epsilon_j$ (levels below the Fermi energy are occupied, above − empty). States with other z_j correspond to particle-hole excitations of the Fermi gas. For example, a state $z_k = \mathrm{sgn}\,\epsilon_k < 0$ has a pair of fermions removed from the level ϵ_k. The Lax vector for normal states is

$$\mathbf{L}_n(u) = L_n(u)\hat{\mathbf{z}} \qquad L_n(u) = -\frac{1}{g} + \sum_j \frac{z_j}{2(u - \epsilon_j)}. \quad (6)$$

All roots of $\mathbf{L}^2(u) = L_n^2(u)$ are thus doubly degenerate. Note the absence of a branch cut of $\sqrt{\mathbf{L}^2(u)}$ connecting the points $u = \pm i\Delta$. Further analysis is similar to that for anomalous states. The Fermi state has a single jump in the $s_z(\epsilon)$ at the Fermi level. This results in a pair of complex conjugate isolated roots [Fig. 2(a)], which can be determined from the equation $L_n(u) = 0$. In the thermodynamic limit, $u = \pm i\Delta_0/2$. The rest of the roots are real and form a continuous line. A normal state with $2p + 1$ jumps in $s_z(\epsilon)$ can have up to $p + 1$ pairs of isolated roots. Each root is doubly degenerate. Linearizing Eq. (2) around a normal state, we obtain normal frequencies $\omega_j = 2u_j$, where $L_n(u_j) = 0$. In particular, the Fermi state has a

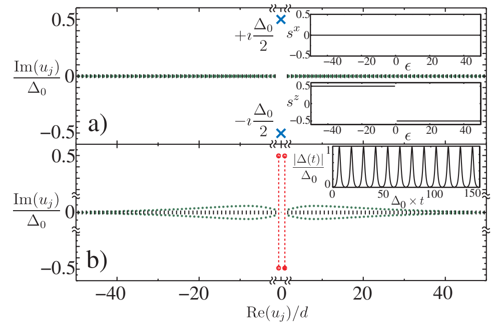

FIG. 2 (color). (a) Roots of $\mathbf{L}^2(u)$ for the Fermi state. There is $2p + 1 = 1$ discontinuity in the spin distribution (insets) resulting in $p + 1 = 1$ pair of imaginary double roots $u_j = \pm i\Delta_0/2$ (blue crosses). Remaining roots (green circles) are real and doubly degenerate. (b) Initially the system is in the Fermi state. A small "external field" $-2\Delta_{\mathrm{QF}}\hat{\mathbf{x}} = -5.4 \times 10^{-2}\Delta_0\hat{\mathbf{x}}$ is added to Eq. (2) for a time $t^* = 1/\Delta_0$. The $p + 1 = 1$ pair of double roots $u_j = \pm i\Delta_0/2$ splits into $m = 2p + 2 = 2$ isolated cuts (red circles connected by dashed lines); the line of real roots splits into two conjugate lines. The frequency spectrum of $|\Delta(t)|$ has $k = m - 1 = 1$ discrete frequency, i.e., $|\Delta(t)|$ (inset) exhibits periodic oscillations.

single unstable mode that corresponds to $u_j = \pm i\Delta_0/2$ and grows as $e^{\Delta_0 t}$ indicating the pairing instability of the Fermi state [20]. Remaining frequencies are real and correspond to the precession of spins at their natural frequencies, $\omega_k = 2\epsilon_k$ up to finite size corrections.

Let the system be in a normal state at $t = 0$. Since, within mean field, normal states are unstable equilibria, a small perturbation is needed to start off the dynamics. A typical deviation splits all double degenerate roots as illustrated in Fig. 2(b). Real roots split into two complex conjugate lines. Degenerate roots at $u = \pm i\Delta_0/2$ split into $m = 2$ isolated cuts. Since $k = m - 1 = 1$ in Eq. (3), $|\Delta(t)|$ will exhibit undamped periodic oscillations (Fig. 2). At large times it is described by a 2-spin solution [19].

The dynamics in normal states can be triggered by quantum fluctuations. In Ref. [12] this is modeled by adding a small *external* field $-2\Delta_{\mathrm{QF}}\hat{\mathbf{x}}$ to Eq. (2). This makes $\mathbf{L}^2(u)$ time dependent: $d\mathbf{L}^2(u)/dt = 2\Delta_{\mathrm{QF}}L_y/g$. Being applied for a short time t^*, the external field drives the system out of the normal state. Treating the evolution of $\mathbf{L}^2(u)$ perturbatively, we find that the new positions of the roots are determined by the equation $L_n(u) = \pm 2i\Delta_{\mathrm{QF}}t^*/g$. Degenerate complex roots split along the real axis into two cuts [Fig. 2(b)] separated by $2\Delta_{\mathrm{QF}}^2 t^* d/g$, where $d = \langle \epsilon_{j+1} - \epsilon_j \rangle$ is the level spacing. Thus, $k = 1$ and $|\Delta(t)|$ exhibits periodic oscillations (Fig. 2). The frequency spectrum has a single discrete frequency corresponding to the unstable mode rather than being continuous as general arguments of Ref. [12] assume.

We see that for initial conditions close to the Fermi state a periodic solution is "dynamically selected" [7], while

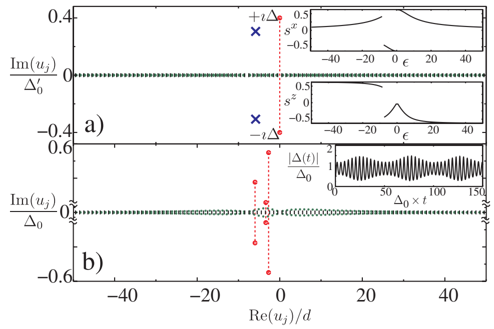

FIG. 3 (color). (a) Roots of $\mathbf{L}^2(u)$ for an *anomalous* stationary state (see the text) that has been obtained from the ground state with gap Δ_0' by flipping spins in energy interval $(-0.37\Delta_0', 0)$. It has a gap $\Delta = 0.4\Delta_0'$. Spin flips result in $2p = 2$ discontinuities in the spin distribution (insets). There is a $p = 1$ pair of isolated double roots (blue crosses) in addition to a pair of single roots $u_j = \pm i\Delta$ (red circles connected by a dashed line). The remaining roots (green circles) are real and doubly degenerate. (b) At $t = 0$ the coupling constant is increased so that the corresponding ground state gap is $\Delta_0 = 1.55\Delta_0'$. The $p = 1$ pair of isolated double roots splits into two cuts, resulting in $m = 2p + 1 = 3$ isolated cuts (red circles connected by dashed lines). The frequency spectrum of $|\Delta(t)|$ has $k = m - 1 = 2$ discrete frequencies; i.e., $|\Delta(t)|$ (inset) displays oscillations with two basic frequencies.

other initial conditions ''select'' damped or multifrequency undamped oscillations. In more conventional terms, this corresponds to a basic fact that the evolution of an integrable system depends on initial conditions. All different behaviors are captured by the general solution [9]. Here we systematically classified possible initial states and specialized the general solution to each type of initial conditions; i.e., we developed ''selection rules'' for the BCS dynamics. Which behavior is realized in a particular experimental setup depends on the initial state of the condensate. In this respect, the periodic solution is somewhat special: if we start from the ground state with a small nonzero $\Delta_{\mathrm{in}} \ll \Delta_0$, the order parameter $|\Delta(t)|$ asymptotes to a constant value Δ_∞.

Excited normal states have several jumps in the spin distribution and can therefore display oscillations with more than one frequency. Consider, e.g., a state where pairs in energy interval from $-\epsilon_a$ to $-\epsilon_b$ have been removed. The initial spin distribution is $s_z(\epsilon) = -\mathrm{sgn}[\epsilon(\epsilon + \epsilon_a)(\epsilon + \epsilon_b)]$. The $2p + 1 = 3$ jumps result in $p + 1 = 2$ pairs of isolated double degenerate complex roots. As a small perturbation splits these roots into $m = 2p + 2 = 4$ cuts, $|\Delta(t)|$ oscillates with $k = m - 1 = 3$ frequencies.

In conclusion, we have shown how to predict the BCS dynamics from the initial state of the condensate. We classified initial states by their integrals of motion—roots of $\mathbf{L}^2(u)$. For states with a root diagram as in the paired ground state, the order parameter $|\Delta(t)|$ displays damped oscillations described by Eq. (1). Other states are of the same type as excited stationary states of the BCS Hamiltonian. In these cases $|\Delta(t)|$ oscillates persistently with few incommensurate frequencies. The number of frequencies is related to the number of jumps in the pseudospin distribution of the corresponding stationary state. For the situation most relevant to the experiments on cold fermions—an abrupt change of the coupling—we predict damped oscillations with $1/\sqrt{t}$ decay.

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