Multi-soliton solutions of the two-dimensional matrix Davey–Stewartson equation

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Abstract

The explicit formulae for m-soliton solutions of the $(1+2)$-dimensional matrix Davey–Stewartson equation are represented. They are found by means of the known general solution of the matrix Toda chain with fixed ends. These solutions are expressed through $m + m$ independent solutions of a pair of linear Schrödinger equations with Hermitian potentials. © 1997 Elsevier Science B.V.

1. Davey–Stewartson equation

Let u and v be two nonsingular $s \times s$ matrix functions of x and y, i.e. each matrix element is a function of the x, y coordinates of the two-dimensional space. Partial derivatives of these functions up to some sufficiently large order are assumed to exist.

We define the matrix Davey–Stewartson equation (DSE) as the following partial differential equation:

$$iu_t + au_{xx} + bu_{yy} - 2au \int dy \, (u^*u)_x - 2b \int dx \, (uu^*)_y \cdot u = 0, \qquad (1)$$

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where a and b are arbitrary real numbers and z^* is a Hermitian conjugate of a matrix z. It will be convenient to deal not with Eq. (1) but with the following expanded system which we call the matrix Davey–Stewartson system (DSS):

$$iu_t + au_{xx} + bu_{yy} - 2au \int dy\,(vu)_x - 2b \int dx\,(uv)_y \cdot u = 0\,,$$

$$-iv_t + av_{xx} + bv_{yy} - 2a \int dy\,(vu)_x \cdot v - 2bv \int dx\,(uv)_y = 0\,. \qquad (2)$$

Below, for definiteness we choose $a = b = 1$. It is easy to see that DSE is system (2) under the additional condition $v = u^*$. We call it the condition of reality.

In the case $s = 1$ (scalar case), when u and v are the scalar functions and the order of the multipliers is not essential, Eq. (1) is the usual, well-known Davey–Stewartson equation [1]. In the scalar case, soliton solutions of DSS have been obtained in [2].

2. Discrete substitution

The method we use to solve the problem is based on the discrete transformation [3] investigation. Here, we consider a concrete discrete transformation which is important for our problem.

By direct calculations it can be checked that (2) is invariant with respect to the following change of the unknown matrices u and v:

$$\tilde{u} = v^{-1}\,, \qquad \tilde{v} = \left[vu - (v_x v^{-1})_y\right] v \equiv v \left[uv - (v^{-1}v_y)_x\right]\,. \qquad (3)$$

Here $\tilde{u}$ and $\tilde{v}$ denote the "new" transformed operators. Invariance means that the matrices $\tilde{u}$ and $\tilde{v}$ satisfy the same system (2) as the matrices u and v do. Mapping (3) is an invertible one and the "old" matrices u and v can be expressed through the "new" ones

$$v = \tilde{u}^{-1}\,, \qquad u = \left[\tilde{u}\tilde{v} - (\tilde{u}_y \tilde{u}^{-1})_x\right]\tilde{u} \equiv \tilde{u}\left[\tilde{v}\tilde{u} - (\tilde{u}^{-1}\tilde{u}_x)_y\right]\,. \qquad (4)$$

Transformation (3) can be rewritten in the form of an infinite chain of equations in two equivalent ways as

$$\left((v_n)_x v_n^{-1}\right)_y = v_n v_{n-1}^{-1} - v_{n+1} v_n^{-1}\,, \qquad u_{n+1} = v_n^{-1}\,, \qquad (5)$$

or as

$$\left(v_n^{-1}(v_n)_y\right)_x = v_{n-1}^{-1} v_n - v_n^{-1} v_{n+1}\,, \qquad (6)$$

where (v_n, u_n) is the result of the n-time substitution (3) applied to some initial matrices v_0 and u_0. Sequences (5) and (6) with $v_{-1}^{-1} = v_N = 0$ boundary conditions are called the matrix Toda chain with fixed ends.

In a (1+1)-dimensional version, mapping (3) is mentioned in [4]. In the scalar case $s = 1$, the general solution of the Toda chain with fixed ends has been found in [5] for all series of semi-simple algebras, except E_7, E_8. In [6] this result was reproduced in terms of an invariant root technique applicable to all semi-simple series.

DSS belongs to the hierarchy of integrable systems corresponding to transformation (3). This hierarchy is constructed in [7]. Supersymmetric extension is considered in [8]. And in [9] the Lax approach to the analogues hierarchies in the (1+1)-dimensional space is discussed.

The explicit general solution of the matrix Toda chain with fixed ends has been found in [10]. It was expressed through $N + N$ arbitrary independent matrix functions of a single argument $X_r(x), Y_r(y)$ as

$$v_0 = \sum_{r=1}^{N} X_r Y_r . \tag{7}$$

To (7) corresponds the following formula for u_N:

$$u_N = \sum_{r=1}^{N} \widetilde{Y}_r(x) \widetilde{X}_r(y) . \tag{8}$$

Here the matrices $\widetilde{X}$ and $\widetilde{Y}$ are not arbitrary but in some way depend on X and Y. Both these results (7) and (8) will be used in further consideration.

3. General strategy

We are going to solve DSS (2) under the condition of reality $u = v^*$. Here, we describe how the discrete transformation is used for that. A general idea is the following. At first, we take some obvious solution of DSS (2). It may not be a solution of the problem (reality condition may not be satisfied). Then, by means of the discrete transformation (3), we get from that initial, obvious solution a solution that satisfies the condition of reality.

For $u_0 = 0$ the first equation of the system (2) is satisfied identically and the second one gives

$$-iv_{0t} + v_{0xx} + v_{0yy} + V_1(t,x)v_0 + v_0 V_2(t,y) = 0 , \tag{9}$$

where V_1 and V_2 are arbitrary $s \times s$ matrix functions of their arguments (these terms arise from the undefined integrals $\int dx\,(uv)_y, \int dy\,(uv)_x$ in the system (2)). Obviously, the condition of reality is not satisfied for this solution. But after a sufficient number of discrete transformations (3), it is possible to come to the solution for which it is satisfied. To clarify this, let us consider some solution u, v satisfying the condition of reality $u = v^*$. Denoting by u_1 and v_1 and by u_{-1} and v_{-1} the results of the direct (3) and inverse (4) substitutions, respectively, one can easily check that $u_{-1} = v_1{}^*$ and $v_{-1} = u_1^*$. On the mth step, we have $u_{-m} = v_m{}^*$ and $v_{-m} = u_m^*$, where index m $(-m)$ stands for the result of the m-time direct (inverse) transformation. And vice versa, one can prove that if we begin from the solution $u_0 = 0, v_0$ and after $2m$-times discrete transformations get $u_{2m} = v_0^*, v_{2m} = 0$, the solution in the middle of the chain automatically satisfies the reality condition $u_{m+1} = v_{m+1}^*$.

The system arising from the equations $u_0 = v_{2m} = 0$ is already solved by formula (7). So it remains to solve the equation $u_{2m} = v_0^*$. It leads to the following relations between X_r and $\widetilde{X}_{\sigma[r]}$ and between Y_r and $\widetilde{Y}_{\sigma[r]}$:

$$X_r^* = \widetilde{X}_{\sigma[r]}, \qquad Y_r^* = Y_{\sigma[r]}, \tag{10}$$

where σ denotes one of the $(2m)!$ possible permutations of the $2m$ low indices. To solve (10), at first it is necessary to find the dependence of $\widetilde{X}$ and $\widetilde{Y}$ on X and Y, respectively. Finally, Eq. (9) in terms of X_r and Y_r can be rewritten as

$$-iX_{rt} + X_{rxx} + V_1(t,x)X_r = 0, \qquad -iY_{rt} + Y_{rxx} + Y_r V_2(t,y) = 0. \tag{11}$$

Thus, to find m-soliton solutions of DSE (1), it is necessary to undertake the following steps:
- find the dependencies $\widetilde{X}_i(X_1, \ldots, X_{2m})$ and $\widetilde{Y}_i(Y_1, \ldots Y_{2m})$;
- solve the system (10);
- find such a dependence of the matrix functions X_r and Y_r from the time argument that it will satisfy system (11).

After this, substituting X_r and Y_r into (7), we find v_0, for which $u_{m+1} = v_{m+1}^*$ is some partial (m-soliton) solution of the Davey–Stewartson equation (1).

4. Scalar case

To gain some experience, we firstly consider the scalar case $s = 1$ for which many of the necessary calculational steps are well known and much simpler than in the general matrix case.

In this case, for the above-mentioned boundary conditions the following formulae for arbitrary k takes place [11]:

$$u_k = \frac{\mathrm{Det}_{k-1}}{\mathrm{Det}_k}, \qquad v_k = \frac{\mathrm{Det}_{k+1}}{\mathrm{Det}_k}, \qquad \mathrm{Det}_{-1} \equiv 0, \qquad \mathrm{Det}_0 \equiv 1, \tag{12}$$

where Det_k is the principle minor of the dimension k of the matrix $(v^0 \equiv v_0)$

$$\begin{pmatrix} v^0 & v_x^0 & v_{xx}^0 & \cdots \\ v_y^0 & v_{xy}^0 & v_{xxy}^0 & \cdots \\ v_{yy}^0 & v_{xyy}^0 & v_{xxyy}^0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and v^0 is determined by (7) where X_r and Y_r are arbitrary scalar functions of their arguments. Substituting (7) into the expression for u_{2m} from (12) and comparing with (8), we find

$$\widetilde{X}_r(x) = \frac{W_{2m-1}(X_1, X_2, \ldots, X_{r-1}, X_{r+1}, \ldots X_{2m})}{W_{2m}(X_1, X_2, \ldots X_{2m})}. \tag{13}$$

Here and below, W_k denotes a Wronskian of dimension k constructed from the functions in the brackets

$$W_k(g_1,\ldots,g_k) \equiv \begin{vmatrix} g_1 & g_2 & \cdots & g_k \\ g_1' & g_2' & \cdots & g_k' \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(k-1)} & g_2^{(k-1)} & \cdots & g_k^{(k-1)} \end{vmatrix}, \qquad W_0 \equiv 1. \qquad (14)$$

Expressions for $\widetilde{Y}_r$ can be got from (13) by simple exchange $X \to Y$.

In the condition of reality (10) we use the permutation $\sigma[r] = 2m - r + 1$. To solve (10) and (11), it is suitable to represent the functions X_r and Y_r in the Frobenious-like form

$$X_1 = \phi_1, \qquad X_r = \phi_1 \int dx\, \phi_2 \ldots \int dx\, \phi_r,$$

$$Y_1 = \psi_1, \qquad Y_r = \psi_1 \int dx\, \psi_2 \ldots \int dx\, \psi_r. \qquad (15)$$

From (13) we find

$$\widetilde{X}_{2m} = \left(\prod_{k=1}^{2m} \phi_k\right)^{-1}, \qquad \widetilde{X}_r = \left(\prod_{k=1}^{2m} \phi_k\right)^{-1} \int dx\, \phi_{2m} \ldots \int dx\, \phi_{2m-r}. \qquad (16)$$

Now the reality condition (10) takes the form

$$\phi_r^* = \phi_{2m-r+2} \quad (r = 2,3,\ldots,2m), \qquad \phi_{m+1} = \phi_{m+1}^* = \left(\prod_{k=1}^{m} \phi_k \phi_k^*\right)^{-1}. \qquad (17)$$

From (11) we have

$$\phi_{r_t} = \left(\phi_r \left(\ln \phi_r \prod_{k=1}^{r-1} \phi_k^2\right)'\right)'. \qquad (18)$$

The imaginary unity i here is included into the time variable which, therefore, should be treated as a pure imaginary one from this moment. One can independently check that the systems (17) and (18) are compatible and if (18) is fulfilled for some ϕ_r, $r \leq m$, for ϕ_{2m-r+2} it also holds. Hence, it is sufficient to consider only equations with $r \leq m$ in the system (18). Now we introduce the new unknown functions $f_r^{-1} = \phi_1 \cdots \phi_r$, $r \leq m+1$. From (18) we find

$$(f_r^{-1} f_{r-1})_t = -\left(f_r^{-1} f_{r-1} (\ln f_r f_{r-1})'\right)'. \qquad (19)$$

From (17) it follows that $f_m^* = f_{m+1}^{-1}$. Substituting this in the $(m+1)$th equation of the last system, we have

$$(f_m f_m^*)_t = \left(f_m f_m^* \left(\ln f_m f_{m-1}^*\right)'\right)'. \qquad (20)$$

Eq. (20) is equivalent to the one-dimensional Schrödinger equation with arbitrary real potential

$$f_{mt} + f''_m = U f_m, \qquad U = U^*. \tag{21}$$

Now let us consider the mth equation of the system (19),

$$(f_m^{-1} f_{m-1})_t = - \left(f_m^{-1} f_{m-1} (\ln f_m f_{m-1})' \right)'. \tag{22}$$

Partially solving it as

$$f_m^{-1} f_{m-1} = z', \qquad (\ln f_m f_{m-1})' = -\frac{z_t}{z'} \tag{23}$$

and excluding the function f_{m-1}, we conclude that the function $z f_m$ satisfies the same Eq. (21) as f_m does. Denoting by u_i ($1 \leq i \leq m$) m independent solutions of (21), we find

$$f_m = u_1, \qquad z f_m = u_2 \Longrightarrow f_{m-1} = z' f_m = \frac{\begin{vmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{vmatrix}}{u_1}. \tag{24}$$

In the general case, for arbitrary i the following formula holds:

$$f_r = \frac{W_{m-r+1}}{W_{m-r}}, \qquad r \leq m, \tag{25}$$

where $W_i = W_i(u_1, \ldots, u_i)$.

To prove (25) we use the well-known Jacobi identity for determinants. Let T be some infinite in both directions matrix; $D_n(T)$ denotes the determinant of its $n \times n$ principle minor, T^s is the matrix obtained from T by deleting its sth column and T_p is the matrix obtained by deleting its pth row. In this notations the Jacobi identity takes the form

$$D_n(T) D_n(T_n^n) - D_n(T^n) D_n(T_n) = D_{n+1}(T) D_{n-1}(T). \tag{26}$$

From (26) the following identity can be derived:

$$W_i \overline{W}'_i - W'_i \overline{W}_i = W_{i-1} W_{i+1}, \tag{27}$$

where $\overline{W}_i = W_i(u_i \rightarrow u_{i+1}) = W_i(u_1, u_2, \ldots, u_{i-1}, u_{i+1})$.

Now let us partially solve Eq. (19) for arbitrary r,

$$f_{r-1} = z' f_r, \qquad (\ln f_r f_{r-1})' = -\frac{z_t}{z'}. \tag{28}$$

Excluding f_{r-1} from the last system we find that f_r and $z f_r$ are the different solutions of the same equation. And if f_r is given by formula (25), $z f_r$ can be determined as

$$z f_r = \frac{\overline{W}_{m-r+1}}{W_{m-r}}.$$

Now from (28) with the help of identity (27) we easily find

$$f_{r-1} = \frac{W_{m-r+2}}{W_{m-r+1}} \ .$$

Thus, formula (25) is proved by induction.

Finally, for functions ϕ_r from the definition of f_r and formulae (17), we have

$$\phi_{m+1} = v_1 v_1^* \, , \qquad \phi_r = \frac{W_{m-r+2} W_{m-r}}{W_{m-r+1}^2} \, ,$$

$$\phi_1 = \frac{W_{m-1}}{W_m} \, , \qquad \phi_r^* = \phi_{2m-r+2} \, , \quad r \le m \, . \tag{29}$$

Analogues expressions take place for functions ψ_k,

$$\psi_{m+1} = v_1 v_1^* \, , \qquad \psi_r = \frac{W_{m-r+2} W_{m-r}}{W_{m-r+1}^2} \, ,$$

$$\psi_1 = \frac{W_{m-1}}{W_m} \, , \qquad \psi_r^* = \psi_{2m-r+2} \, , \quad r \le m \, . \tag{30}$$

In (30), $W_i = W_i(v_1, \ldots, v_i)$ and $v_i \equiv v_i(y)$, $1 \le i \le m$, are m independent solutions of the $(1 + 1)$-dimensional linear Schrödinger equation with arbitrary real potential V,

$$v_{it} + v_i'' = V v_i \, , \qquad V = V^* \, . \tag{31}$$

5. Matrix case

Here, we consider a general problem as it has been formulated in Sections 1 and 3. We find m-soliton solutions of DSE for arbitrary dimension of the unknown matrix u. We therefore find matrix generalizations of all formulae of the previous section. It turns out that quasi-determinants of matrices with noncommutative entries play the role of usual determinants. Conception of a quasi-determinant has been recently introduced by Gelfand and Retarh [12]. We use an independent technique, more appropriate for our particular case, but quasi-determinants can be used as well.

With the chain (5,6) under the above-mentioned boundary conditions we connect the following recurrent relations:

$$R_n \equiv v_n^{-1} v_{ny} \, , \qquad S_n^q \equiv \sum_{k=0}^{n-1} (S_{k_y}^{q-1} + R_k S_k^{q-1}) \, , \tag{32}$$

with the boundary conditions $S_i^0 \equiv 1$ for arbitrary i. From definitions (32) and Eqs. (5) and (6) we easily find

$$S_n^1 = \sum_{k=0}^{n} R_n \, , \qquad S_0^q = v_0^{-1} v_0 \underbrace{y \cdots y}_{q} \, , \tag{33}$$

$$v_{n+1} = -v_n (S_{n+1}^1)_x = (-1)^{n+1} v_0 (S_1^1)_x (S_2^1)_x \cdots (S_{n+1}^1)_x \, . \tag{34}$$

For matrix functions S_n^q the following relation is true:

$$S_n^q = \left[(S_{n-1}^1)_x\right]^{-1} (S_{n-1}^{q+1})_x .$$ (35)

Now let us find the dependence of $\widetilde{X}$ on X. For this, we use the fact that each matrix function X_i is determined only by matrices $X_1, \ldots, X_{2m}$, and, therefore, we can choose the matrices $Y_1, \ldots, Y_{2m}$ in an arbitrary way. It is convenient to choose

$$Y_i = \frac{y^{i-1}}{(i-1)!} E , \qquad v_0 = X_1 + \frac{y}{1!} X_2 + \cdots + \frac{y^{2m-1}}{(2m-1)!} X_{2m} ,$$ (36)

where E is the unity $s \times s$ matrix. Substituting (36) into the expression for v_{2m-1} from (34), we find

$$v_{2m-1}^{-1}\big|_{y=0} = - \left[X_1 (T_0^1)_x (T_1^1)_x \cdots (T_{2m-2}^1)_x\right]^{-1} ,$$ (37)

where matrices T_q^n are determined by the following relations:

$$T_n^q = \left[(T_{n-1}^1)_x\right]^{-1} (T_{n-1}^{q+1})_x ,$$ (38)

with the boundary conditions

$$T_0^q = S_0^q\big|_{y=0} = X_1^{-1} X_{q+1} .$$ (39)

Expression (37) corresponds to the one of the functions $\widetilde{X}_i$. And since these functions can be enumerated in various ways, we can choose

$$(\widetilde{X}_1)^{-1} = X_1 (T_0^1)_x (T_1^1)_x \cdots (T_{2m-2}^1)_x \equiv F(x_1, \ldots, x_{2m}) .$$ (40)

A formula for an arbitrary i can be derived from (40) by the cycled permutation of the indices

$$\widetilde{X}_i^{-1} = (-1)^{i-1} F(\sigma_i[x_1, \ldots, x_{2m}]) .$$ (41)

An arbitrary multiplier can be added into formula (41). It will be counted in the expression for $\widetilde{Y}_i$. We added $(-1)^{i-1}$ to do further calculations more convenient. Using (41) and (34), we find

$$\widetilde{Y}_1^{-1} = -(Q_{2m-2}^1)_y \cdots (Q_0^1)_y , \qquad Y_1 \equiv G(Y_1, \ldots, Y_{2m}) ,$$ (42)

$$\widetilde{Y}_i^{-1} = (-1)^i G(\sigma_i[Y_1, \ldots, Y_{2m}]) ,$$ (43)

where

$$Q_n^s = (Q_{n-1}^{s+1})_y \left[(Q_{n-1}^1)_y\right]^{-1} , \qquad Q_0^s = Y_{s+1} Y_1^{-1} .$$ (44)

Now as in the previous section, we represent the initial functions X_r, Y_r in the Frobenious-like form,

$$X_1 = \phi_1, \qquad X_2 = \phi_1 \int dx\, \phi_2, \qquad X_3 = \phi_1 \int dx\, \phi_2 \int dx\, \phi_3, \quad \dots,$$

$$Y_1 = \psi_1, \qquad Y_2 = \int dx\, \psi_2 \cdot \psi_1, \qquad Y_3 = \int dx \left(\int dx\, \psi_3 \cdot \psi_2 \right) \cdot \psi_1, \quad \dots \quad (45)$$

After the permutation of the indices, the formula for $\widetilde{X}_r$ coincides with (16). The only difference is that in the matrix case the order of the multipliers must be taken into account

$$\widetilde{X}_1 = p, \qquad \widetilde{X}_2 = \int dx\, \phi_{2m} \cdot p, \qquad \widetilde{X}_3 = \int dx \left(\int dx\, \phi_{2m-1} \cdot \phi_{2m} \right) \cdot p, \quad \dots,$$

$$\widetilde{Y}_1 = -s, \qquad \widetilde{Y}_2 = -s \int dx\, \psi_{2m}, \qquad \widetilde{Y}_3 = -s \int dx\, \psi_{2m} \int dx\, \psi_{2m-1}, \quad \dots, \quad (46)$$

where $p = (\phi_1 \cdots \phi_{2m})^{-1}$ and $s = (\psi_{2m} \cdots \psi_1)^{-1}$. The condition of reality taken in the form $\widetilde{X}_r = X_r^*$, $\widetilde{Y}_r = Y_r^*$ reads

$$\phi_r^* = \phi_{2m-r+2}, \quad 2 \leq r \leq m,$$
$$\phi_{m+1}^{-1} = (\phi_{m+1}^*)^{-1} = (\phi_1 \phi_2 \cdots \phi_m)^* (\phi_1 \phi_2 \cdots \phi_m),$$
$$\psi_r^* = \psi_{2m-r+2}, \quad 2 \leq r \leq m,$$
$$\psi_{m+1}^{-1} = (\psi_{m+1}^*)^{-1} = -(\psi_m \psi_{m-1} \cdots \psi_1)^* (\psi_m \psi_{m-1} \dots \psi_1). \quad (47)$$

The fact that all functions X_r are solutions of the same equation (11) leads to the following system:

$$-(\phi_s)_t + (2(\phi_1 \phi_2 \dots \phi_{s-1})^{-1} (\phi_1 \phi_2 \dots \phi_{s-1})' \phi_s + \phi_s')' = 0. \quad (48)$$

Introducing now the functions $f_r^{-1} = \phi_1 \phi_2 \dots \phi_r$ from (48), we find

$$-(f_{r-1} f_r^{-1})_t = \left[f_{r-1}' f_r^{-1} - f_{r-1} (f_r^{-1})' \right]'. \quad (49)$$

Then, from (47) and (49) we conclude that f_m is a solution of the $(1+1)$-dimensional linear Schrödinger equation with Hermitian potential

$$f_{m\,t} + f_m'' = W f_m, \qquad W = W^*. \quad (50)$$

A solution of system (49) can be found in the same way as in the previous section. The matrix case does not require the use of the Jacobi identity because recurrent definitions are used,

$$f_1 = u_1, \qquad f_{m-r} = (U_r^1)' \cdots (U_0^1)' u_1. \quad (51)$$

Matrix functions U_n^q are determined by the following recurrent relations:

$$U_n^q = (U_{n-1}^{q+1})' \left[(U_{n-1}^1)' \right]^{-1},$$

with the boundary conditions

$$U_0^r = u_{r+1} u_1^{-1} \,,$$

where matrices u_r are different solutions of Eq. (50). Finally, for ϕ_{m-r} we have

$$\phi_1 = f_1^{-1} \,, \qquad \phi_{m-r} = (U_r^1)' \,, \quad 0 \le r \le m - 2 \,,$$
$$\phi_{m+1} = u_1 u_1^* \,, \qquad \phi_r^* = \phi_{2m-r+2} \,, \quad 2 \le r \le m \,. \tag{52}$$

For ψ_i, we derive

$$\psi_1^{-1} = v_1 (V_{m-2}^1)' \cdots (V_0^1)' v_1 \,,$$
$$\psi_{m-r} = (U_r^1)' \,, \quad 0 \le r \le m - 2 \,,$$
$$\psi_{m+1} = -v_1 v_1^* \,, \qquad \psi_r^* = \psi_{2m-r+2} \,, \quad 2 \le r \le m \,, \tag{53}$$

where

$$V_n^q = \left[(V_{n-1}^1)' \right]^{-1} (V_{n-1}^{q+1})' \,, \qquad V_0^r = v_1^{-1} v_{r+1}$$

and matrices $v_i(y)$ are different solutions of the $(1+1)$-dimensional linear Schrödinger equation with an arbitrary Hermitian potential

$$v_{it} + v_i'' = v_i M \,, \qquad M = M^* \,.$$

Now substituting (52) and (53) directly into (45) and (7) and then into the formula for v_{m+1} from (34), we find the m-soliton solution of the matrix DSE. We do not write down the corresponding expression, because it can easily be derived, but is too large to represent it here.

6. The simplest example of the one-soliton solution

Substituting $m = 1$ into the formulae of the last section, we find

$$v_0 = X_1 Y_1 + X_2 Y_2 \,, \qquad X_1 = \phi_1 \,, \qquad X_2 = \phi_1 \int dx \, \phi_2 \,,$$

$$Y_1 = \psi_1 \,, \qquad Y_2 = \int dx \, \psi_2 \cdot \psi_1 \,.$$

After this, we find the following expression for the one-soliton solution of the matrix DSE:

$$u_1 = \psi_1^{-1} \left(1 + \int dx \, \phi_2 \int dy \, \psi_2 \right)^{-1} \phi_1^{-1} \,. \tag{54}$$

The matrix functions $\phi(t, x)$ and $\psi(t, y)$ are determined by u and v solutions of the one-dimensional linear Schrödinger equations,

$$\phi_1 = u^{-1} \,, \qquad \phi_2 = u u^* \,,$$
$$\psi_1 = v^{-1} \,, \qquad \psi_2 = -v^* v \,,$$

$$u_t + u_{xx} + uM_1(t, x) = 0\,,$$

$$v_t + v_{yy} + M_2(t, y)v = 0\,, \qquad M_{1,2} = M_{1,2}^*\,.$$

7. Conclusion

The main result of the paper are the explicit expressions for the m-soliton solutions of the $(1+2)$-dimensional matrix Davey–Stewartson equation. By means of the corresponding formulae of Sections 4 and 5 these solutions are expressed through the $m + m$ independent solutions of a pair of linear $(1+1)$-dimensional Schrödinger equations.

From the group-theoretical point of view it means that we have realized the finite-dimensional representation of the group of integrable mappings. This viewpoint remained beyond our concrete calculations.

Note that restriction with the finite-dimensional matrices is absolutely nonessential. We had never used this restriction and, moreover, the dimension (s) was not included in any expression.

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