Letters in Mathematical Physics **35**: 345–349, 1995.

© 1995 *Kluwer Academic Publishers. Printed in the Netherlands.*

The General Solution of Two-Dimensional Matrix Toda Chain Equations with Fixed Ends

A. N. LEZNOV and E. A. YUZBASHJAN
Institute for High Energy Physics, 142284 Protvino, Moscow Region, Russia

(Received: 28 November 1994; in final form: 4 March 1995)

Abstract. It is shown that the two-dimensional matrix Toda chain determines the group of discrete symmetries of the two-dimensional matrix nonlinear Schrödinger equation (the matrix generalization of the Davey–Stewartson system). The general solution of this chain with definite boundary conditions is obtained in explicit form.

Mathematics Subject Classifications (1991). 35Q51, 35Q55.

1. Introduction

We shall understand by the matrix Davey–Stewartson equation the following system of two equations for two unknown $s \times s-$ matrix functions u, v:

$$\begin{aligned} &u_t + au_{xx} + bu_{yy} - 2au \int \mathrm{d}y(uv)_x - 2b \int \mathrm{d}x(uv)_y u = 0, \\ &-v_t + av_{xx} + bv_{yy} - 2a \int \mathrm{d}y(uv)_x v - 2bv \int \mathrm{d}x(uv)_y = 0, \end{aligned} \tag{1.1}$$

where a, b are arbitrary numerical parameters and x, y are the coordinates of two-dimensional space. In the particular case $s = 1$, when the order of multipliers is not essential, (1.1) is the usual Davey–Stewartson equation in its original form [1].

2. Discrete Substitution

By direct but tedious computations, one can become convinced that the system (1.1) is invariant with respect to the following transformation of the dependent variables;

$$\tilde{u} = \frac{1}{v}, \qquad \tilde{v} = [vu - (v_x v^{-1})_y]v \equiv v[uv - (v^{-1} v_y)_x]. \tag{2.1}$$

The substitution (2.1) is the discrete transformation [2] with respect to which all the equations of the matrix Davey–Stewartson hierarchy are invariant. In the case of a one-dimensional space, this substitution was mentioned in [3].

The substitution (2.1) is invertible and the 'old' functions u, v may be represented in terms of the new ones as

$$v = \frac{1}{\tilde{u}}, \qquad u = [\tilde{u}\tilde{v} - (\tilde{u}_y u^{-1})_x]\tilde{u} \equiv \tilde{u}[\tilde{v}\tilde{u} - (u^{-1}\tilde{u}_x)_y]. \tag{2.2}$$

The substitution (2.1) may be rewritten in the form of an infinite chain of equations

$$((v_n)_x v_n^{-1})_y = v_n v_{n-1}^{-1} - v_{n+1} v_n^{-1}, \qquad (u_{n+1} = v_n^{-1}) \tag{2.3}$$

where (v_{n-1}, u_{n-1}) is to be understood as the result of the n-fold application of the substitution (2.1) to some given matrix-functions (v_0, u_0).

Generally, the chain (2.3) is infinite in both directions, but it may be interrupted by appropriate boundary conditions. The case in which the boundary conditions $v_{-1}^{-1} = v_{N+1} = 0$ are imposed will be called the matrix Toda chain with fixed ends.

In the scalar case $s = 1$; Equation (2.3) is equivalent to the Toda lattice in its original form (describing points on the real line with an exponential interaction between nearest neighbours), when this set of equations is written in terms of the variables x_n defined by $v_n = \exp x_n$. After another change of variables, $\rho_n = x_n - x_{n-1}$, this system takes the familiar form

$$(\rho_n)_{xy} = \exp \rho_{n-1} - 2\exp \rho_n + \exp \rho_{n+1}.$$

These are exactly the equations of the two-dimensional Toda lattice, the general solution of which with fixed ends was found in [4] for all series of semisimple Lie algebras except for E_7, E_8. In [5], this result was reproduced in terms of invariant root techniques applicable to all semisimple series.

The goal of this Letter is to obtain the general solution of the matrix Toda chain with fixed ends in explicit form.

3. General Solution

First let us observe that from (2.1) it follows that

$$v_{n+1} u_{n+1} - v_n u_n = -((v_n)_x v_n^{-1})_y.$$

Keeping in mind that $u_0 = v_{-1}^{-1} = 0$, we immediately obtain

$$v_{n+1} = -\left[\sum_{t=0}^{n} ((v_t)_x v_t^{-1})_y\right] v_n = -v_n \left[\sum_{t=0}^{n} (v_t^{-1}(v_t)_y)_x\right]. \tag{3.1}$$

The single equation to determine one unknown function v_0 ($v_{N+1} = 0$) takes the form

$$\sum_{t=0}^{N} (v_t)_x v_t^{-1} = A_N(x), \qquad \sum_{t=0}^{N} v_t^{-1}(v_t)_y = B_N(y), \tag{3.2}$$

where $A_N(x)$, $B_N(y)$ are arbitrary $s \times s$ matrix functions of the corresponding arguments.

We shall employ the following notation:

$$X^r = (v_r)_x v_r^{-1}, \qquad S^n = \sum_{t=0}^{n} X^t,$$

and the corresponding expressions with respect to the coordinate y.

In this notation, (3.1) may be rewritten as

$$v_{n+1} = -(S^n)_y v_n = (S^n)_y (S^{n-1})_y v_{n-1} = -\dots \tag{3.3}$$

From (3.3), the following recurrence relation for the determination of X^n, S^n ensues:

$$X^n = (S^{n-1}_{xy} + S^{n-1}_y X^{n-1})(S^{n-1}_y)^{-1},$$
$$S^n = \left(\sum_{t=0}^{n-1} [S^t_x + S^t X^t] \right)_y (S^{n-1}_y)^{-1}. \tag{3.4}$$

Let us first consider solutions of (3.2) for the starting values $N = 0, 1, 2, \dots$ which allow us to obtain the solution for the general case of arbitrary N by induction.

3.1. $N = 0$

In the sum (3.2), we have only one term and for the v_0 we obtain the obvious solution

$$v_0 = \phi_0(x)\bar{\phi}_0(y),$$

where ϕ_0, $\bar{\phi}_0$ are arbitrary $s \times s$ matrix functions of their arguments.

3.2. $N = 1$

Equation (3.2) may be rewritten with the help of (3.4) as ($S^0 \equiv X^0$)

$$(X^0_x + X^0 X^0)_y = A_1(x) X^0_y, \qquad X^0_x + X^0 X^0 = A_1(x) X^0 + A_0(x).$$

Keeping in mind the determination of X^0 (Y^0), we obtain

$$v^0_{xx} = A_1(x) v^0_x + A_0(x) v^0, \qquad v^0_{yy} = v^0_y B_1(y) + v^0 B_0(y). \tag{3.5}$$

The system (3.5) has obviously the following general solution:

$$v^0 = \phi_0(x)\bar{\phi}_0(y) + \phi_1(x)\bar{\phi}_1(y),$$

where, as in the previous example, $\phi_p(x)\bar{\phi}_p(y)$ are arbitrary matrix functions of their arguments.

3.3. $N = 2$

We reproduce the corresponding steps of the computations in this case without detailed comments.

$$S^2 = A_2(x), \qquad S^0_x + S^0 X^0 + S^1_x + S^1 X^1 = A_2(x) S^1 + A_1(x).$$

From the last equality, with the help of (3.4), we have

$$X^0_{xx} + 2X^0_x X^0 + X^0 X^0_x + (X^0)^3 = A_2(X^0_x + (X^0)^2) + A_1 X^0 + A_0.$$

Keeping in mind the definition of X^0, we finally obtain two equations

$$v^0_{xxx} = A_2 v^0_{xx} + A_1 v^0_x + A_0 v^0, \qquad v^0_{yyy} = v^0_{yy} B_2 + v^0_y B_1 + v^0 B_0$$

with the general solution

$$v^0 = \sum_{t=0}^{2} \phi_t(x)\bar{\phi}_t(y).$$

3.4. THE CASE OF ARBITRARY N

Let us define by induction the values of S^n_p as follows:

$$S^n_p = \sum_{q=0}^{n} [(S^q_{p-1})_x + S^q_{p-1} X^q], \tag{3.6}$$

with boundary condition $S^q_0 = 1$. Comparing (3.4) with (3.6) we notice that S^n from (3.4) coincides with S^n_1 from (3.6). Bearing in mind (3.4) and the definition (3.6), we immediately obtain the recurrence relation

$$S^n_p = (S^{n-1}_{p+1})_y[(S^{n-1}_1)_y]^{-1}. \tag{3.7}$$

Using (3.7), it is possible to represent all unknown matrix-valued functions of the chain v_n in terms of the function v_0 alone. The system of two equations which determine v_0 (3.2) takes the form

$$\begin{aligned} v^0_{\underbrace{x\ldots x}_{N+1}} &= A_N(x) v^0_{\underbrace{x\ldots x}_{N}} + \ldots + A_0(x) v^0, \\ v^0_{\underbrace{y\ldots y}_{N+1}} &= v^0_{\underbrace{y\ldots y}_{N}} B_n(y) + \ldots + v^0 B_0(y), \end{aligned} \tag{3.8}$$

from which we obtain

$$v^0 = \sum_{t=0}^{N} \phi_t(x)\bar{\phi}_t(y). \tag{3.9}$$

We have shown (and from the examples for $N = 0, 1, 2, \ldots$ it is absolutely clear) that v_0 as a function of a single argument $x(y)$ satisfies the linear equation with constant coefficients and, for this reason, the form (3.9) for its solution is the only possible one.

4. Conclusion

The main result of this Letter is contained in the previously unknown (3.9) general solution of the matrix Toda chain (2.2) with fixed ends. In the scalar case ($s = 1$), this solution, as is well known, is closely connected with the theory of representations of semisimple Lie algebras [6]. In the text of the Letter, we have obtained an expression for v_n. But we are sure that this expression may be rewritten in 'determinant' (i.e. more explicit) form. For us, it is not clear what is the connection of the proposed solution (3.9) with the theory of group representations (if there is any). We hope to come back to these interesting questions in further publications.

Acknowledgement

For one of the authors (A.N.L), the research of this Letter was made possible in part by Grant NRMMOOO of the International Scientific Foundation.

References

1. Davey, A. and Stewartson, K., *Proc. Roy. Soc A* **338**, 101–110 (1974).
2. Leznov, A. N., Nonlinear symmetries of integrable systems, *J. Soviet Lazer Res.* **3-4**, 278–288 (1992); Backlund transformation for integrable systems, Preprint IHEP-92-112 DTP (1992).
3. Leznov, A. N., Shabat, A. B., and Yamilov, R. I., *Phys. Lett. A* **174**, 397–402 (1993).
4. Leznov, A. N., *Teor. Mat. Fiz.* **42**, 343 (1980).
5. Leznov, A. N. and Saveliev, M. V., *Lett. Math. Phys.* **3**, 489 (1979).
6. Leznov, A. N. and Saveliev, M. V., *Progress in Physics 15*, Birkhauser, 1982, p. 290.