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Integrable time-dependent Hamiltonians, solvable Landau–Zener models and Gaudin magnets

Emil A. Yuzbashyan

Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08854, USA

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We solve the non-stationary Schrödinger equation for several time-dependent Hamiltonians, such as the BCS Hamiltonian with an interaction strength inversely proportional to time, periodically driven BCS and linearly driven inhomogeneous Dicke models as well as various multi-level Landau–Zener tunneling models. The latter are Demkov–Osherov, bow-tie, and generalized bow-tie models. We show that these Landau–Zener problems and their certain interacting many-body generalizations map to Gaudin magnets in a magnetic field. Moreover, we demonstrate that the time-dependent Schrödinger equation for the above models has a similar structure and is integrable with a similar technique as Knizhnik–Zamolodchikov equations. We also discuss applications of our results to the problem of molecular production in an atomic Fermi gas swept through a Feshbach resonance and to the evaluation of the Landau–Zener transition probabilities.

1. Introduction

Since the discovery of quantum mechanics, from the Bohr atom and the harmonic oscillator, to the present day, integrable models have played a key role in our understanding of physics at the quantum level. The field has acquired a new prominence in nonequilibrium many-body physics with direct observation of signatures of integrable dynamics in cold atom and solid state experiments [1–4] and the realization that quantum integrable systems display properties characteristic of the many-body localized phase of matter [5–9]. Nevertheless, despite great interest in nonequilibrium phenomena,

E-mail address: eyuzbash@physics.rutgers.edu.

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the vast majority of studies of integrable many-body interacting systems out of equilibrium leave out models of the Landau–Zener type where one of the couplings or external fields in the Hamiltonian varies in time in a continuous fashion.

We consider several examples of time-dependent models in this paper. The list includes Bardeen–Cooper–Schiffer (BCS) Hamiltonian with a coupling inversely proportional to time as well as periodically driven BCS models, the problem of molecular production in an atomic Fermi gas swept through a Feshbach resonance (driven inhomogeneous Dicke model), and various models of multi-level Landau–Zener tunneling. Among the latter are Demkov–Osherov [10,11], bow-tie [12], and generalized bow-tie Hamiltonians [13,14] as well as their many-body extensions. Some of these models have been analyzed by other means before, others, e.g., the time-dependent BCS model, are new. We will see that they are all closely related, construct exact general solutions of their non-stationary Schrödinger equations, and explain how they fit into the existing theory of quantum integrability. This work builds in part on Ref. [15], where nontrivial commuting partners for solvable Landau–Zener type Hamiltonians have been derived; some of our results have been briefly announced in Ref. [16].

The BCS and Dicke models are known to be related to Gaudin magnets [17–19], i.e. N commuting spin Hamiltonians of the form

$$\hat{H}_j^G = 2B\hat{s}_j^z - \sum_{k\neq j}\frac{\hat{\mathbf{s}}_j\cdot\hat{\mathbf{s}}_k}{\varepsilon_j-\varepsilon_k},\quad [\hat{H}_i^G,\hat{H}_j^G]=0,\quad i,j=1,\ldots,N, \tag{1}$$

where $\hat{\mathbf{s}}_k$ are quantum spins of arbitrary magnitude s_k, and B and ε_k are arbitrary real parameters. Amazingly, we find that *all* other models listed above also map to Gaudin magnets. Associated with Gaudin Hamiltonians are differential equations of the evolution type known as Knizhnik–Zamolodchikov (KZ) equations [20],

$$i\nu\frac{\partial\Psi_{\mathrm{KZ}}}{\partial\varepsilon_j} = \hat{H}_j^G\Psi_{\mathrm{KZ}} = \left(2B\hat{s}_j^z - \sum_{k\neq j}\frac{\hat{\mathbf{s}}_j\cdot\hat{\mathbf{s}}_k}{\varepsilon_j-\varepsilon_k}\right)\Psi_{\mathrm{KZ}}, \tag{2}$$

where ν is a real parameter and by construction $\Psi_{\mathrm{KZ}} = \Psi_{\mathrm{KZ}}(\boldsymbol{\varepsilon}, B)$ is a function of $(\varepsilon_1,\ldots,\varepsilon_N)\equiv\boldsymbol{\varepsilon}$ and B. It turns out that the non-stationary Schrödinger equation for each of our examples has the same structure as KZ equations (2) taken in a certain limit or a particular case. This allows us to transfer known results and machinery developed for the KZ equations to these and other similar time-dependent models.

One such result is an integral representation of the general solution of KZ equations [21]. In the original KZ equations $B=0$, i.e. the first term inside the brackets in Eq. (2) is absent. However, it is not difficult to generalize the solution to the $B\neq 0$ case [22,23]. We add to this an important observation (see Section 2) that the evolution of Ψ_{KZ} with B is governed by the BCS Hamiltonian

$$i\nu\frac{\partial\Psi_{\mathrm{KZ}}}{\partial B} = \hat{H}_{\mathrm{BCS}}\Psi_{\mathrm{KZ}}, \tag{3}$$

where $\hat{H}_{\mathrm{BCS}}$ is the BCS (Richardson) Hamiltonian expressed in terms of Anderson pseudospins [24],

$$\hat{H}_{\mathrm{BCS}} = \sum_{j=1}^N 2\varepsilon_j\hat{s}_j^z - \frac{1}{2B}\sum_{j,k}\hat{s}_j^+\hat{s}_k^-, \tag{4}$$

and $(2B)^{-1}$ plays the role of the BCS coupling constant. Taking B to be a function of time, $B=B(t)$, turns Eq. (3) into the non-stationary Schrödinger equation for the Hamiltonian

$$\hat{H}_{\mathrm{BCS}}(t) = \nu^{-1}\dot{B}\sum_{j=1}^N 2\varepsilon_j\hat{s}_j^z - (2\nu B)^{-1}\dot{B}\sum_{j,k}\hat{s}_j^+\hat{s}_k^-, \tag{5}$$

whose solution is $\Psi_{\rm KZ}(\boldsymbol{\varepsilon}, B(t))$. In particular, the choice $B(t) = \nu t$ yields a BCS Hamiltonian with the coupling constant inversely proportional to time,

$$\hat{H}_{\rm BCS}(t) = \sum_{j=1}^{N} 2\varepsilon_j \hat{s}_j^z - \frac{1}{2\nu t}\sum_{j,k} \hat{s}_j^+ \hat{s}_k^-, \tag{6}$$

while a periodic $B(t)$ leads to an integrable Floquet BCS superconductor.

In the limit where the length of one of the spins, say $\mathbf{s}_N$, goes to infinity, so that it is replaced with a harmonic oscillator, $\hat{H}_N^G$ becomes the inhomogeneous Dicke (Tavis–Cummings) model [17]

$$\hat{H}_{\rm D} = \sum_{j=1}^{N-1} \xi_j \hat{s}_j^z - \omega \hat{n}_b + g\sum_{j=1}^{N-1} (\hat{b}^\dagger \hat{s}_j^- + \hat{b}\hat{s}_j^\dagger), \quad \hat{n}_b = \hat{b}^\dagger \hat{b}, \tag{7}$$

where $\hat{b}^\dagger$ and $\hat{b}$ are bosonic creation and annihilation operators. The rest of Gaudin magnets yield its commuting partners (see Section 3 for details). The corresponding ($j = N$) KZ equation (2) is replaced with

$$i\nu \frac{\partial \Psi_D}{\partial \omega} = \hat{H}_D \Psi_{\rm D}. \tag{8}$$

For $\omega = \nu t$ this is the non-stationary Schrödinger equation for the time-dependent Hamiltonian

$$\hat{H}_{\rm D}(t) = \sum_{j=1}^{N-1} \xi_j \hat{s}_j^z - \nu t \hat{n}_b + g\sum_{j=1}^{N-1} (\hat{b}^\dagger \hat{s}_j^- + \hat{b}\hat{s}_j^\dagger). \tag{9}$$

In this way we derive in Section 3 a full set of solutions of the non-stationary Schrödinger equation for $\hat{H}_{\rm D}(t)$ retracing the solution of KZ equations. This model describes the production of molecules in an atomic Fermi gas swept across an s-wave Feshbach resonance in the regime of a narrow resonance and sufficiently slow sweep rate ν [25]. Just as in the BCS Hamiltonian (4), products of fermionic creation and annihilation operators $\hat{c}_{j\sigma}^\dagger$ and $\hat{c}_{j\sigma}$, where $\sigma = \uparrow, \downarrow$, are expressed in terms of Anderson pseudospins: $2\hat{s}_j^z + 1 = \sum_\sigma \hat{c}_{j\sigma}^\dagger \hat{c}_{j\sigma}$, $\hat{s}_j^+ = \hat{c}_{j\uparrow}^\dagger \hat{c}_{j\downarrow}^\dagger$, and $\hat{s}_j^- = \hat{c}_{j\downarrow}\hat{c}_{j\uparrow}$. Suppose at $t \to -\infty$ the system is in the ground state. Since the bosonic energy $-\nu t \to +\infty$ in this limit, there are no bosons. The problem is to determine the number of bosons $\langle \hat{n}_b(t)\rangle$ as $t \to +\infty$.

More generally, in this and other problems of the Landau–Zener type, we are interested in the scattering matrix that relates the state of the system at $t = t_{\rm fin}$ to that at $t = t_{\rm in}$. For Hamiltonians linear in time, such as Eq. (9), $t_{\rm in} = -\infty$ and $t_{\rm fin} = +\infty$. In the case of the BCS Hamiltonian (6) a natural choice is $t_{\rm in} = 0^+$ and $t_{\rm fin} = +\infty$. The transition probability from one state at $t_{\rm in}$ to another at $t_{\rm fin}$ is modulus squared of the corresponding matrix element of the scattering matrix. For 2×2 Hamiltonians linear in t the problem was solved by Landau, Zener, and others back in 1932 [26–29], hence the name 'Landau–Zener'. For time-dependent Hamiltonians with larger Hilbert spaces, even for 3×3 Hermitian matrices linear in t, no general solution is available, but there is a class of models for which the multi-level version of the Landau–Zener problem is exactly solvable. For example, exact formulas for certain transition probabilities for the Dicke model (9) were conjectured empirically in Ref. [30] and later justified in Ref. [16].

The connection between solvable time-dependent models and KZ equations we establish in this paper should be especially useful for evaluating the scattering matrix and transition probabilities. In the theory of KZ equations, elements of the scattering matrix are known as transition functions between asymptotic solutions of these equations. There is a well-developed technique based on the general solution of KZ equations to evaluate these transition functions explicitly in terms of quantum 6j-symbols [31]. Therefore, it should be possible to similarly determine scattering matrices as well other quantities of interest for models we analyze in this paper. We do not dwell on it further here leaving this calculation for the future.

Well-known examples of nontrivial[1] solvable multi-level Landau–Zener models are Demkov–Osherov, bow-tie, and generalized bow-tie models. The first two are the following time-dependent $N \times N$ matrix Hamiltonians:

$$H_{\rm DO} = t|1\rangle\langle 1| + \sum_{k=2}^{N} \left(p_k|1\rangle\langle k| + p_k|k\rangle\langle 1| + a_k|k\rangle\langle k|\right), \tag{10}$$

$$H_{\rm bt} = \sum_{k=2}^{N} \left(p_k|1\rangle\langle k| + p_k|k\rangle\langle 1| + r_k t|k\rangle\langle k|\right), \tag{11}$$

where, p_k, a_k, and r_k are real parameters. Both describe a single level $|1\rangle$ coupled to $N-1$ other levels that are not directly coupled to each other. Diagonal matrix elements (diabatic energy levels) are linear functions of time. In the Demkov–Osherov model, also known as the equal slope model, all diabatic energies except the first one have the same slope, which has been set to zero in Eq. (10) without loss of generality. In the bow-tie model the slopes are distinct and the diabatic level diagram resembles a bow-tie.

We show in Section 4 that both these models map to one of the Gaudin magnets, say $\hat{H}_1^G$ in Eq. (1), restricted to the sector where the total spin polarization S_z differs by one from its minimum, $S_z = S_z^{\rm min} + 1 = -\sum_k s_k + 1$. Note that since $\hat{S}_z = \sum_k \hat{s}_k^z$ commutes with all $\hat{H}_j^G$, they break down into blocks corresponding to different eigenvalues S_z of $\hat{S}_z$. The $N \times N$ block of $\hat{H}_1^G$ that corresponds to $S_z = S_z^{\rm min} + 1$ (as well as the one for $S_z = S_z^{\rm max} - 1$) maps to the Demkov–Osherov or bow-tie Hamiltonians via a change of variables, while similar blocks of the remaining Gaudin magnets become their commuting partners. This also allows us to construct general solutions of the non-stationary Schrödinger equations for these models following the same procedure as that for the KZ equations. As we will also see in Section 4, the generalized bow-tie model obtains from the bow-tie Hamiltonian (11) via a simple time-independent unitary transformation, so it is not an essentially independent model and the same results as for the bow-tie model apply in this case.

Let us also mention a generalization of the Demkov–Osherov model that describes a system of spinless fermions interacting with a time-dependent impurity level [32],

$$\hat{H}_f = t\hat{n}_1 + \sum_{k=2}^{N} p_k\hat{c}_1^\dagger\hat{c}_k + p_k c_k^\dagger\hat{c}_1 + a_k(1-u\hat{n}_1)\hat{n}_k, \quad \hat{n}_j \equiv \hat{c}_j^\dagger\hat{c}_j. \tag{12}$$

Here $\hat{c}_j^\dagger$ and $\hat{c}_j$ are fermionic creation and annihilation operators and u is a real parameter. It has been conjectured that the problem of determining the scattering matrix relating states at $t = \pm\infty$ is exactly solvable for this model [32]. We will show in Section 5 that this model too stems from Gaudin magnets (1) and thus determine its commuting partners.

At this point it is useful to switch gears and consider instead of Eqs. (2) and (3) an abstract set of multi-time Schrödinger equations

$$i\nu\frac{\partial\Psi(\boldsymbol{x})}{\partial x_j} = \hat{H}_j\Psi(\boldsymbol{x}), \qquad j = 0, 1, \ldots, n; \tag{13}$$

where $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ are n real parameters on which the Hamiltonians $\hat{H}_j$ depend. Compatibility of differential equations (13) imposes severe restrictions on the choice of $\hat{H}_j$. Indeed, equating mixed derivatives, $\partial_k\partial_j\Psi(\boldsymbol{x}) = \partial_j\partial_k\Psi(\boldsymbol{x})$, we obtain

$$\partial_j\hat{H}_k - \partial_k\hat{H}_j - i[\hat{H}_k, \hat{H}_j] = 0, \quad k, j = 0, \ldots, n; \tag{14}$$

where $\partial_j \equiv \partial/\partial x_j$. These consistency conditions guarantee the existence of a joint solution $\Psi(\boldsymbol{x})$ of Eq. (13) for any initial condition, see, e.g., Ref. [33]. Let us also introduce a nonabelian gauge field

[1] 'Nontrivial' in this context excludes reducible Landau–Zener models. For example, the Landau–Zener problem $g\hat{J}_x + \nu t\hat{J}_z$ for an arbitrary spin $\hat{\mathbf{J}}$ reduces to that for spin-1/2, i.e. the original 2×2 Landau–Zener problem etc., see, e.g., Ref. [15] for more details.

$\mathcal{A}(\boldsymbol{x})$, $\mathcal{A}_j = -i\hat{H}_j$. Eq. (14) is then the zero curvature condition: $\mathcal{F}_{jk} \equiv \partial_j\mathcal{A}_k - \partial_k\mathcal{A}_j - [\mathcal{A}_j, \mathcal{A}_k] = 0$, meaning that the formal solution of Eq. (13) in terms of an ordered exponent is independent of the path connecting two fixed points in the space of real parameters $\boldsymbol{x}$.

Suppose the Hamiltonians $\hat{H}_j$ are real symmetric. Then, the imaginary and real parts of Eq. (14) yield

$$\left[\hat{H}_j(\boldsymbol{x}), \hat{H}_k(\boldsymbol{x})\right] = 0, \tag{15}$$

$$\partial_j\hat{H}_k(\boldsymbol{x}) = \partial_k\hat{H}_j(\boldsymbol{x}), \quad j, k = 0, 1, \dots, n. \tag{16}$$

These equations are useful for identifying potentially solvable time-dependent models [16]. For example, it is straightforward to verify that the choice $x_j = \varepsilon_j$, $\hat{H}_j = \hat{H}_j^G$ for $j \geq 1$, $x_0 = B$, $n = N$, and $\hat{H}_0 = \hat{H}_{\text{BCS}}$ satisfies Eqs. (15) and (16). The condition $[\hat{H}_{\text{BCS}}, \hat{H}_j^G] = 0$ follows from the identities

$$\sum_j 2\varepsilon_j\hat{H}_j^G = 2B\hat{H}_{\text{BCS}} - S_z^2 + S_z + \sum_j \mathbf{s}_j^2, \quad \sum_j \hat{H}_j^G = 2B\hat{S}_z, \tag{17}$$

where $\hat{\mathbf{S}} = \sum_j \hat{\mathbf{s}}_j$ is the total spin. As noted above, this ensures a joint solution of Eqs. (2) and (3) for any initial condition. Therefore, any solution of KZ equations (2) *must* also be a solution of Eq. (3) up to a multiplicative factor $C(B)$ independent of ε_j. We will show below that in fact $C(B)$ is also independent of B, i.e. is a constant, so that the solution of Eq. (3) coincides with the known solution of KZ equations.

In principle, zero curvature conditions (15) and (16) seem to provide a framework for constructing integrable time-dependent Hamiltonians that do not necessarily reduce to Gaudin magnets and associated KZ equations. However, so far we have not encountered a single nontrivial example, solvable multi-level Landau–Zener problems included, for which this is the case, i.e. which is not in the Gaudin–KZ class of models. Let us emphasize that the restriction to real symmetric $\hat{H}_j(\boldsymbol{x})$ is important here.[2] Otherwise, there are of course numerous other realizations of Eq. (14), usually for $n = 1$, among integrable nonlinear partial differential equations [34,35]. Naturally, by definition we include in the Gaudin–KZ class other integrable versions of Gaudin magnets and corresponding KZ equations, such as trigonometric, hyperbolic, generalized to other Lie algebras instead of spin $su(2)$ etc. [17,19,36–39].

Note also that making ε_j functions of time, $\varepsilon_j = \varepsilon_j(t)$, we obtain [23] the non-stationary Schrödinger equation for the Hamiltonian $\nu^{-1}\dot{\varepsilon}_j\hat{H}_j^G$ directly from KZ equations (2). The solution of this Schrödinger equation is $\Psi_{\text{KZ}}(\boldsymbol{\varepsilon}(t), B)$. What we do in this paper to derive various models listed above and solutions of their non-stationary Schrödinger equations from the Gaudin–KZ system is much more general. First, we map $\boldsymbol{\varepsilon}$, B, spin magnitudes s_j, and associated Gaudin magnets $\hat{H}_j^G$ to a new set of variables $\boldsymbol{x}$ and associated Hamiltonians $\hat{H}_j$, so that the new system satisfies the zero curvature condition (16). In general, the solution $\Psi(\boldsymbol{x})$ of the so constructed new set of multi-time Schrödinger equations (13) *cannot* be obtained by performing the above mapping directly in $\Psi_{\text{KZ}}(\boldsymbol{\varepsilon}, B)$. Instead, we have to derive $\Psi(\boldsymbol{x})$ anew following the same overall approach as in the case of KZ equations,[3] . That the mapping is zero-curvature-preserving ensures that this approach works for the new system. It is often highly nontrivial to find the proper set of variables $\boldsymbol{x}$ and, especially, an appropriate construction for the solution, see, e.g., the bow-tie example in Section 4.2. Having done so, we make x_j time-dependent to obtain the solution of the non-stationary Schrödinger equation for the Hamiltonian $\nu^{-1}\dot{x}_j\hat{H}_j$ from $\Psi(\boldsymbol{x})$. For example, for the Demkov–Osherov model (10), $x_0 = B - \sum_{k=2}^N s_k\varepsilon_k^{-1}$, $p_k = \sqrt{s_k}\varepsilon_k^{-1}$, and $x_k = a_k = -\varepsilon_k^{-1}$ for $k \geq 2$, while $\hat{H}_1 = H_{\text{DO}}$. Having derived $\Psi(\boldsymbol{x})$ for this case, we set $x_0 = t$ to get the solution of the corresponding non-stationary Schrödinger equation.

2. Time-dependent BCS Hamiltonians

Here we write down a complete set of solutions of the evolution equation (3) for the BCS Hamiltonian (4) or, equivalently, the solution of the non-stationary Schrödinger equation for time-dependent BCS Hamiltonians (5) and (6). As discussed above, this is also the solution of the KZ

2 See also the footnote 5.

3 Except in the case of the BCS Hamiltonian (4) where we simply show that the existing solution $\Psi_{\text{KZ}}(\boldsymbol{\varepsilon}, B)$ satisfies Eq. (3).

equations (2) up to a multiplicative factor $C(B)$. Since the former is known, all we need to do is to determine $C(B)$. We will see that $C(B)$ is a B-independent constant and, therefore, the solution of Eq. (3) is just the known solution of the KZ equations. Note however that various modifications of the BCS Hamiltonian (4), e.g., adding a term $-2\mu(B)\hat{S}_z$ or replacing $\sum_{j,k}\hat{s}_j^+\hat{s}_k^-$ with $\hat{\mathbf{S}}^2$, do not affect the zero curvature conditions (15) and (16) while resulting in a non-constant $C(B)$.

We start with a review of the solution of KZ equations [21–23]. The problem of determining exact eigenvalues and eigenstates of Gaudin magnets (1) is solvable. The wavefunctions are of the form

$$\Phi \equiv |\Phi(\boldsymbol{\lambda},\boldsymbol{\varepsilon})\rangle = \prod_{\alpha=1}^{M}\hat{L}^+(\lambda_\alpha)|0\rangle, \quad \hat{L}^+(\lambda) = \sum_{j=1}^{N}\frac{\hat{s}_j^+}{\lambda-\varepsilon_j}, \tag{18}$$

where $|0\rangle$ is the minimal weight state with all spins pointing in the negative z-direction, $\hat{s}_j^z|0\rangle = -s_j|0\rangle$, $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_M)$, and $\boldsymbol{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_N)$. The state Φ is a simultaneous eigenstate of $\hat{H}_j^G$ when λ_α satisfy a set of algebraic equations (Bethe equations). The idea is to work with the states Φ with unconstrained λ_α (off-shell Bethe Ansatz). The only additional ingredient we need is the action of $\hat{H}_j^G$ on these states

$$\hat{H}_j^G\Phi = h_j\Phi + \sum_\alpha \frac{f_\alpha \hat{s}_j^+\Phi_{\not\alpha}}{\lambda_\alpha-\varepsilon_j}, \quad \Phi_{\not\alpha} = \prod_{\beta\ne\alpha}\hat{L}^+(\lambda_\beta), \tag{19}$$

where $\Phi_{\not\alpha}$ is the state Φ with the term $\hat{L}^+(\lambda_\alpha)$ deleted,

$$h_j = -2Bs_j - \sum_{k\ne j}\frac{s_j s_k}{\varepsilon_j-\varepsilon_k} + \sum_\alpha \frac{s_j}{\varepsilon_j-\lambda_\alpha}, \tag{20}$$

and

$$f_\alpha = 2B - \sum_j \frac{s_j}{\varepsilon_j-\lambda_\alpha} - \sum_{\beta\ne\alpha}\frac{1}{\lambda_\beta-\lambda_\alpha}. \tag{21}$$

Note that if we were to set $f_\alpha = 0$, Φ and h_j would become eigenstates and eigenvalues of $\hat{H}_j^G$, respectively.

The next step is to introduce a function $\mathcal{S}$, known as the Yang–Yang action, defined through equations

$$\frac{\partial\mathcal{S}}{\partial\varepsilon_j} = h_j, \quad \frac{\partial\mathcal{S}}{\partial\lambda_\alpha} = f_\alpha. \tag{22}$$

Explicitly we obtain

$$\begin{aligned}\mathcal{S}(\boldsymbol{\lambda},\boldsymbol{\varepsilon}) = -2B\sum_j \varepsilon_j s_j + 2B\sum_\alpha \lambda_\alpha - \frac{1}{2}\sum_j\sum_{k\ne j} s_j s_k \ln(\varepsilon_j-\varepsilon_k) + \\ \sum_j\sum_\alpha s_j\ln(\varepsilon_j-\lambda_\alpha) - \frac{1}{2}\sum_\alpha\sum_{\beta\ne\alpha}\ln(\lambda_\beta-\lambda_\alpha).\end{aligned} \tag{23}$$

The solution of the KZ equations is

$$\Psi_{\mathrm{KZ}}(B,\boldsymbol{\varepsilon}) = \oint_\gamma d\boldsymbol{\lambda}\exp\left[-\frac{i\mathcal{S}(\boldsymbol{\lambda},\boldsymbol{\varepsilon})}{\nu}\right]|\Phi(\boldsymbol{\lambda},\boldsymbol{\varepsilon})\rangle, \quad d\boldsymbol{\lambda} = \prod_{\alpha=1}^{M} d\lambda_\alpha, \tag{24}$$

where the closed contour γ is such that the integrand comes back to its initial value after λ_α has described it. We verify that this is indeed a solution by substituting $\Psi_{\mathrm{KZ}}(B,\boldsymbol{\varepsilon})$ into KZ equations (2) and using Eqs. (18), (19) and (22). The only 'trick' involved is to notice that the integral of a complete derivative with respect to any λ_α is zero due to the above property of the contour γ. Moreover, it has been shown that any solution of KZ equations (2) is a linear combination of solutions of the form (24) with different choices of γ, see Ref. [31] and references therein.

To demonstrate that $\Psi_{\rm KZ}(B, \boldsymbol{\varepsilon})$ is also the solution of the BCS evolution equation (3) (thus also proving $C(B) = \text{const}$), we apply both sides of Eq. (17) to $\Psi_{\rm KZ}$

$$2i\nu \sum_j \varepsilon_j \frac{\partial \Psi_{\rm KZ}}{\partial \varepsilon_j} = 2i\nu B \frac{\partial \Psi_{\rm KZ}}{\partial B} + A\Psi_{\rm KZ}, \tag{25}$$

where

$$A = \sum_j s_j(s_j+1) + M - \sum_j s_j - \Big(M - \sum_j s_j\Big)^2, \tag{26}$$

and we used the fact that $\Psi_{\rm KZ}$ is an eigenstate of $\hat{S}_z$ with the eigenvalue $M - \sum_j s_j$. There are several ways to prove Eq. (25). A simple way is to consider the scaling

$$\varepsilon_j \to a\varepsilon_j, \quad \lambda_\alpha \to a\lambda_\alpha, \quad B \to a^{-1}B. \tag{27}$$

Eqs. (18), (23), and (24) imply

$$\Phi \to a^{-M}\Phi, \quad \mathcal{S} \to \mathcal{S} + \frac{A}{2}\ln a, \quad d\boldsymbol{\lambda} \to a^M \lambda, \tag{28}$$

and therefore

$$\Psi_{\rm KZ}(a^{-1}B, a\boldsymbol{\varepsilon}) = \exp\left(\frac{A\ln a}{2i\nu}\right)\Psi_{\rm KZ}(B, \boldsymbol{\varepsilon}). \tag{29}$$

Differentiating this equation with respect to a and then setting $a = 1$, we derive Eq. (25). We have also verified Eq. (25) directly by substituting Eq. (24) into Eq. (25) and using Eqs. (19), (20), and (21). Thus, $\Psi_{\rm KZ}(B, \boldsymbol{\varepsilon})$ is the solution of the BCS evolution equation (3), while $\Psi_{\rm KZ}(B(t), \boldsymbol{\varepsilon})$ and $\Psi_{\rm KZ}((\nu t)^{-1}, \boldsymbol{\varepsilon})$ solve the non-stationary Schrödinger equations for time-dependent BCS Hamiltonians (5) and (6), respectively.

3. Driven inhomogeneous Dicke model

In this section, we first describe the mapping from Gaudin magnets (1) to the inhomogeneous Dicke model

$$\hat{H}_{\rm D} = \sum_{j=1}^{N-1} \xi_j \hat{s}_j^z - \omega\hat{n}_b + g\sum_{j=1}^{N-1}(\hat{b}^\dagger \hat{s}_j^- + \hat{b}\hat{s}_j^\dagger), \quad \hat{n}_b = \hat{b}^\dagger\hat{b}, \tag{30}$$

and then use it to derive the general solution of its non-stationary Schrödinger equation for $\omega = \nu t$ from the solution of KZ equations (24).

Suppose the magnitude s of one of the spins, e.g., $\hat{\mathbf{s}}_N$ diverges. We use a slightly modified Holstein–Primakoff transformation from spin to bosonic creation and annihilation operators,

$$\hat{s}_N^- = \sqrt{2s}\left(1 - \frac{\hat{n}_b}{2s}\right)^{1/2}\hat{b}, \quad \hat{s}_N^+ = \sqrt{2s}\hat{b}^\dagger\left(1 - \frac{\hat{n}_b}{2s}\right)^{1/2}, \quad \hat{s}_N^z = \hat{n}_b - s. \tag{31}$$

We only need the terms that do not vanish in the limit $s \to \infty$,

$$\hat{s}_N^- = \sqrt{2s}\hat{b} + O(s^{-1/2}), \quad \hat{s}_N^+ = \sqrt{2s}\hat{b}^\dagger + O(s^{-1/2}), \quad \hat{s}_N^z = \hat{n}_b - s. \tag{32}$$

Gaudin magnets (1) now become

$$\begin{aligned} \hat{H}_j^G &= 2B\hat{s}_j^z - \sum_{k\neq j}\frac{\hat{\mathbf{s}}_j\cdot\hat{\mathbf{s}}_k}{\varepsilon_j - \varepsilon_k} - \frac{\sqrt{2s}(\hat{s}_j^-\hat{b}^\dagger + \hat{s}_j^+\hat{b}) + 2(\hat{n}_b - s)\hat{s}_j^z}{2(\varepsilon_j - \varepsilon_N)}, \\ \hat{H}_N^G &= 2B(\hat{n}_b - s) - \sum_j \frac{\sqrt{2s}(\hat{s}_j^-\hat{b}^\dagger + \hat{s}_j^+\hat{b}) + 2(\hat{n}_b - s)\hat{s}_j^z}{2(\varepsilon_j - \varepsilon_N)}, \end{aligned} \tag{33}$$

where $j = 1, \ldots, N-1$ and the summation over k in $\hat{H}_j^G$ is from $k = 1$ to $k = N-1$.

Let us make the following replacements:

$$\varepsilon_j = -\frac{\xi_j}{2g^2}, \quad \varepsilon_N = \frac{\sqrt{2s}}{2g}, \quad 2B - \frac{s}{\varepsilon_N} = \omega; \quad j \le N-1. \tag{34}$$

Note that $\omega = -2g^2\varepsilon_N + 2B$, so effectively ω replaces ε_N. Performing this variable change in Eq. (33), expanding in $\sqrt{2s}$ and keeping only the non-vanishing terms, we obtain

$$\hat{H}_j^G \to \hat{H}_j^D = (\omega + \xi_j)\hat{s}_j^z + g(\hat{s}_j^- \hat{b}^\dagger + \hat{s}_j^+ \hat{b}) + 2g^2 \sum_{k\ne j} \frac{\hat{\mathbf{s}}_j \cdot \hat{\mathbf{s}}_k}{\xi_j - \xi_k}, \tag{35}$$

$$\hat{H}_N^G \to -2Bs + g\sqrt{2s}(\hat{n}_b + \hat{S}_z) - \hat{H}_D, \quad \hat{S}_z = \sum_{j=1}^{N-1} \hat{s}_j^z. \tag{36}$$

Since the quantity $\hat{n}_b + \hat{S}_z$ commutes with $\hat{H}_j^D$ and the Dicke Hamiltonian $\hat{H}_D$, all Hamiltonians $\hat{H}_j^D$ and $\hat{H}_D$ mutually commute. Moreover, the second zero curvature condition (16) holds for $x_j = \xi_j, \hat{H}_j = \hat{H}_j^D$ for $N-1 \ge j \ge 1, x_0 = \omega$, and $\hat{H}_0 = \hat{H}_D$.

To construct the solution of the non-stationary Schrödinger equation for the driven inhomogeneous Dicke model $\hat{H}_D(t)$,

$$i\frac{\partial \Psi_D}{\partial t} = \hat{H}_D(t)\Psi_D, \quad \hat{H}_D(t) = \sum_{j=1}^{N-1} \xi_j \hat{s}_j^z - \nu t \hat{n}_b + g\sum_{j=1}^{N-1} (\hat{b}^\dagger \hat{s}_j^- + \hat{b}\hat{s}_j^\dagger), \tag{37}$$

all we need to do is to apply transformations (32) and (34) to the formulas leading to the solution of the KZ equations in the previous section and take the limit $s \to \infty$. The off-shell Bethe states (18) become

$$\Phi \equiv |\Phi_D(\boldsymbol{\lambda}, \boldsymbol{\xi})\rangle = \prod_{\alpha=1}^{M} \hat{L}^+(\lambda_\alpha)|0\rangle, \quad \hat{L}^+(\lambda) = \hat{b}^\dagger + g\sum_{j=1}^{N-1} \frac{\hat{s}_j^+}{\lambda - \xi_j}. \tag{38}$$

The action of $\hat{H}_j^D$ and $\hat{H}_D$ on these states is

$$\hat{H}_j^D \Phi = h_j \Phi + g\sum_\alpha \frac{f_\alpha \hat{s}_j^+ \Phi_{\not\alpha}}{\lambda_\alpha - \xi_j}, \quad \hat{H}_D \Phi = h_D \Phi + \sum_\alpha f_\alpha \hat{b}^\dagger \Phi_{\not\alpha} \tag{39}$$

where $\Phi_{\not\alpha}$ is defined as before in Eq. (19) and

$$h_j = -\omega s_j + 2g^2 \sum_{k\ne j} \frac{s_j s_k}{\xi_j - \xi_k} - 2g^2 \sum_\alpha \frac{s_j}{\xi_j - \lambda_\alpha}, \quad h_D = \sum_\alpha \lambda_\alpha - \sum_k s_k \xi_k, \tag{40}$$

$$f_\alpha = \omega - 2g^2\lambda_\alpha + 2g^2 \sum_j \frac{s_j}{\xi_j - \lambda_\alpha} + 2g^2 \sum_{\beta\ne\alpha} \frac{1}{\lambda_\beta - \lambda_\alpha}, \tag{41}$$

where $\omega = \nu t$. We define the Yang–Yang action $\mathcal{S}_D$ through

$$\frac{\partial \mathcal{S}_D}{\partial \xi_j} = h_j, \quad \frac{\partial \mathcal{S}_D}{\partial \omega} = h_D, \quad \frac{\partial \mathcal{S}_D}{\partial \lambda_\alpha} = f_\alpha, \tag{42}$$

which upon integration yield

$$\begin{aligned}\mathcal{S}_D(\boldsymbol{\lambda}, \boldsymbol{\xi}, t) = -\nu t \sum_k s_k \xi_k + \nu t \sum_\alpha \lambda_\alpha + g^2 \sum_j \sum_{k\ne j} s_j s_k \ln(\xi_j - \xi_k) - \\ 2g^2 \sum_j \sum_\alpha s_j \ln(\xi_j - \lambda_\alpha) + g^2 \sum_\alpha \sum_{\beta\ne\alpha} \ln(\lambda_\beta - \lambda_\alpha).\end{aligned} \tag{43}$$

Finally, the solution of the non-stationary Schrödinger equation (37) has the same form as the solution of KZ equation (24), i.e.

$$\Psi_D(t,\boldsymbol{\xi}) = \oint_\gamma d\boldsymbol{\lambda} \exp\left[-\frac{iS_D(\boldsymbol{\lambda},\boldsymbol{\xi},t)}{\nu}\right]|\Phi_D(\boldsymbol{\lambda},\boldsymbol{\xi})\rangle, \quad d\boldsymbol{\lambda} = \prod_{\alpha=1}^{M} d\lambda_\alpha. \tag{44}$$

Indeed, we verify directly

$$\begin{aligned} &i\frac{\partial \Psi_D}{\partial t} - \hat{H}_D(t)\Psi_D = -\oint_\gamma d\boldsymbol{\lambda} e^{\frac{-iS_D}{\nu}} \sum_\alpha f_\alpha \hat{b}^\dagger \Phi_{\not\alpha} = \\ &-i\nu \sum_\alpha \oint_\gamma \left(\prod_{\beta\neq\alpha} d\lambda_\beta\right) d\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha}\left(\hat{b}^\dagger \Phi_{\not\alpha} e^{\frac{-iS_D}{\nu}}\right) = 0, \end{aligned} \tag{45}$$

where we used the second equation in (40) and the last two equations in (42). By construction $\Psi_D(t,\boldsymbol{\xi})$ also satisfies the remaining ($j = 1, \ldots, N-1$) Eqs (13) with $x_j = \xi_j$ and $\hat{H}_j = \hat{H}_j^D$. Let us also mention an interesting application [40] of the off-shell Bethe Ansatz to the homogeneous, $\xi_j = \xi$, Dicke model with non-integrable time-dependence of the detuning ω.

4. Multi-level Landau–Zener models

Here we describe the mapping from a sector of Gaudin magnets (1) to Demkov–Osherov [10], bow-tie [12], and generalized bow-tie [13,14] models. We then derive solutions of their non-stationary Schrödinger equations with the same approach as for KZ equations in Section 2. We note that certain integral representations of these solutions have been constructed before [10–13]. Our main point in this Section are not the solutions themselves, but to show that these models belong to the Gaudin–KZ class and to derive their commuting partners.

The mapping proceeds in two steps. First, we note that Gaudin magnets (1) commute with the z-projection of the total spin $\hat{S}_z$ and consider their next to minimum weight sector where $S_z = S_z^{\min}+1 = -\sum_j s_j + 1$. An orthonormal basis in this subspace is

$$|k\rangle = \frac{\hat{s}_k^+|0\rangle}{\sqrt{2s_k}}, \quad k = 1,\ldots,N; \tag{46}$$

where $|0\rangle$ is the lowest weight state $S^z = S^z_{\min}$ as in Section 2. In this basis Gaudin Hamiltonians (1) are N commuting $N \times N$ matrices

$$H_j^G = \tilde{H}_j - \left(2B\, s_j + \sum_{k\neq j}\frac{s_j s_k}{\varepsilon_j - \varepsilon_k}\right)\mathbb{1}, \quad j = 1,\ldots,N; \tag{47}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix,

$$\tilde{H}_j = 2B|j\rangle\langle j| - \sum_{k\neq j}\frac{\gamma_j\gamma_k|j\rangle\langle k| + \gamma_j\gamma_k|k\rangle\langle j| - \gamma_k^2|j\rangle\langle j| - \gamma_j^2|k\rangle\langle k|}{\varepsilon_j - \varepsilon_k}, \tag{48}$$

and $\gamma_k = \sqrt{s_k}$. Considering infinite-dimensional representations of the $su(2)$ algebras, where the lowest weight states are still well-defined [41], we treat γ_k as arbitrary real numbers. Matrix Hamiltonians $\tilde{H}_j$, termed type 1 or maximal, emerge independently from a rigorous notion of quantum integrability proposed in Refs. [42–44]. Their defining feature is that this is the maximal set of real symmetric matrices linear in a real parameter (B) that mutually commute, are linearly independent, and possess no B-independent symmetries. Other families of mutually commuting real symmetric $N \times N$ matrices contain fewer than N linearly independent members [43,44].

It is straightforward to specialize the off-shell Bethe Ansatz equations (18)–(21) of Section 2 to the present case. We have

$$\tilde{\Phi} \equiv |\tilde{\Phi}(\lambda,\boldsymbol{\varepsilon})\rangle = \sum_{j=1}^{N}\frac{\gamma_j|j\rangle}{\lambda - \varepsilon_j}, \tag{49}$$

$$\tilde{H}_j\tilde{\Phi} = h_j\tilde{\Phi} + \frac{f\gamma_j|j\rangle}{\lambda-\varepsilon_j}, \quad h_j = \frac{\gamma_j^2}{\varepsilon_j-\lambda}, \quad f = 2B - \sum_j \frac{\gamma_j^2}{\varepsilon_j-\lambda}. \tag{50}$$

The second step is to map $\tilde{H}_1$ in Eq. (48) to Demkov–Osherov, bow-tie, and generalized bow-tie models.

4.1. Demkov–Osherov model

To obtain the Demkov–Osherov model, we set

$$\gamma_1 = 1, \quad \varepsilon_1 = 0, \quad \gamma_k = -\frac{p_k}{a_k}, \quad \varepsilon_k = -\frac{1}{a_k}, \quad 2B = t - \sum_{k=2}^{N} \frac{p_k^2}{a_k}, \tag{51}$$

where $k = 2, \dots, N$. The matrix $\tilde{H}_1$ in Eq. (48) then turns into the Demkov–Osherov model, i.e.

$$\tilde{H}_1 = H_{\mathrm{DO}} = t|1\rangle\langle 1| + \sum_{k=2}^{N} \left(p_k|1\rangle\langle k| + p_k|k\rangle\langle 1| + a_k|k\rangle\langle k|\right), \tag{52}$$

while the remaining $\tilde{H}_j$ are its commuting partners

$$\tilde{H}_j = (t-a_j)|j\rangle\langle j| - p_j|1\rangle\langle j| - p_j|j\rangle\langle 1| + \sum_{k\neq j} \frac{p_jp_k|j\rangle\langle k| + p_jp_k|k\rangle\langle j| - p_k^2|j\rangle\langle j| - p_j^2|k\rangle\langle k|}{a_k-a_j}. \tag{53}$$

Importantly, in this parameterization mutually commuting matrix Hamiltonians H_{DO} and $\tilde{H}_j$ also satisfy the zero curvature condition (16), namely,

$$\frac{\partial \tilde{H}_j}{\partial a_k} = \frac{\partial \tilde{H}_k}{\partial a_j}, \quad \frac{\partial \tilde{H}_j}{\partial t} = \frac{\partial H_{\mathrm{DO}}}{\partial a_j}, \quad j,k = 2,\dots,N. \tag{54}$$

We also need to redefine the spectral parameter and rescale the off-shell Bethe state in Eq. (49) as follows

$$\eta = -\frac{1}{\lambda}, \quad \Phi_{\mathrm{DO}} = -\frac{\tilde{\Phi}}{\eta}. \tag{55}$$

Now we make replacements (51) and (55) in Eqs. (49) and (50) to obtain

$$\Phi_{\mathrm{DO}} \equiv |\Phi_{\mathrm{DO}}(\eta, \boldsymbol{a})\rangle = |1\rangle - \sum_{j=2}^{N} \frac{p_j|j\rangle}{a_j-\eta}, \quad \boldsymbol{a} = (a_2,\dots,a_N), \tag{56}$$

$$\tilde{H}_j\Phi_{\mathrm{DO}} = h_j\Phi_{\mathrm{DO}} + \frac{fp_j|j\rangle}{\eta-a_j}, \quad H_{\mathrm{DO}}\Phi_{\mathrm{DO}} = h_{\mathrm{DO}}\Phi_{\mathrm{DO}} + f|1\rangle, \tag{57}$$

$$h_{\mathrm{DO}} = \eta, \quad h_j = \frac{p_j^2}{a_j} - \frac{p_j^2}{a_j-\eta}, \quad f = t - \eta + \sum_{j=2}^{N} \frac{p_j^2}{a_j-\eta}. \tag{58}$$

As in previous sections we introduce a function $\mathcal{S}_{\mathrm{DO}}$, such that

$$\frac{\partial \mathcal{S}_{\mathrm{DO}}}{\partial a_j} = h_j, \quad \frac{\partial \mathcal{S}_{\mathrm{DO}}}{\partial t} = h_{\mathrm{DO}}, \quad \frac{\partial \mathcal{S}_{\mathrm{DO}}}{\partial \eta} = f, \tag{59}$$

where $j = 2, \dots, N$. We have

$$\mathcal{S}_{\mathrm{DO}}(\eta, \boldsymbol{a}, t) = \eta t - \frac{\eta^2}{2} + \sum_{j=2}^{N} p_j^2 \ln\left(\frac{a_j}{a_j-\eta}\right). \tag{60}$$

By analogy with the solution (24) of KZ equations, the solution of the non-stationary Schrödinger equation for the Demkov–Osherov model (52) as well as the rest of Eqs. (13) for $x_j = a_j$ and $\hat{H}_j = \tilde{H}_j$ reads

$$\Psi_{\rm DO}(t, \boldsymbol{a}) = \oint_\gamma d\eta e^{-i\mathcal{S}_{\rm DO}(\eta,\boldsymbol{a},t)} |\Phi_{\rm DO}(\eta, \boldsymbol{a})\rangle. \tag{61}$$

Note that replacing λ with η and rescaling the wave-function in Eq. (55) is important for this scheme to work. These steps ensure the consistency of Eqs. (59), $\partial h_j/\partial\eta = \partial f/\partial a_j$ and $\partial h_{\rm DO}/\partial\eta = \partial f/\partial t$, and that $\Psi_{\rm DO}(t, \boldsymbol{a})$ is indeed the solution.

As discussed in the Introduction, by construction $\Psi_{\rm DO}(t(\tau), \boldsymbol{a}(\tau))$ is also the solution of the non-stationary Schrödinger equation

$$i\frac{d\Psi}{d\tau} = H\Psi, \tag{62}$$

for the Hamiltonian

$$H = \frac{dt}{d\tau}H_{\rm DO} + \sum_{j=2}^N \frac{da_j}{d\tau}\tilde{H}_j. \tag{63}$$

For example, choosing linear $t(\tau)$ and $a_j(\tau) \propto \tau$, we obtain various models of the form $A + B\tau + C/\tau$, where A, B, and C are τ-independent $N \times N$ matrices.

4.2. Bow-tie model

The bow-tie model

$$H_{\rm bt} = \sum_{k=2}^N \left(p_k|1\rangle\langle k| + p_k|k\rangle\langle 1| + r_k t|k\rangle\langle k|\right), \tag{64}$$

obtains from the Demkov–Osherov model (52) via a substitution

$$a_j = r_j t + t. \tag{65}$$

Indeed, we find

$$H_{\rm bt} = H_{\rm DO} - t\mathbb{1}. \tag{66}$$

This defines the mapping from one of the Gaudin magnets $\hat{H}_1^G$ to the bow-tie model via maximal Hamiltonians (48) and the Demkov–Osherov model. Correspondingly, remaining Gaudin magnets map to commuting partners of $H_{\rm bt}$. Specifically, setting $a_j = r_j t + t$ in Eq. (53), we get

$$\tilde{H}_j = -r_j t|j\rangle\langle j| - p_j|1\rangle\langle j| - p_j|j\rangle\langle 1| + \sum_{k\neq j} \frac{p_j p_k|j\rangle\langle k| + p_j p_k|k\rangle\langle j| - p_k^2|j\rangle\langle j| - p_j^2|k\rangle\langle k|}{(r_k - r_j)t}. \tag{67}$$

It is not immediately clear how to identify the set of variables for which the zero curvature condition (16) holds. We found the following parameterization:

$$r_j = \alpha_j^2, \quad p_j = \alpha_j\beta_j, \quad j = 2, \dots, N. \tag{68}$$

The bow-tie model in this parameterization reads

$$H_{\rm bt} = \sum_{k=2}^N \left(\alpha_k\beta_k|1\rangle\langle k| + \alpha_k\beta_k|k\rangle\langle 1| + \alpha_k^2 t|k\rangle\langle k|\right). \tag{69}$$

Moreover, we have to rescale the commuting partners (67),

$$I_j = -\frac{t\tilde{H}_j}{\alpha_j} = \alpha_j t^2 |j\rangle\langle j| + \beta_j t\,(|1\rangle\langle j| + |j\rangle\langle 1|) + \sum_{k\neq j} \frac{\alpha_j\alpha_k\beta_j\beta_k(|j\rangle\langle k| + |k\rangle\langle j|) - \alpha_k^2\beta_k^2|j\rangle\langle j| - \alpha_j^2\beta_j^2|k\rangle\langle k|}{\alpha_j(\alpha_j^2-\alpha_k^2)}. \tag{70}$$

Now the bow-tie model and this set of its commuting partners fulfill the necessary zero curvature condition, i.e.

$$\frac{\partial I_j}{\partial \alpha_k} = \frac{\partial I_k}{\partial \alpha_j}, \quad \frac{\partial I_j}{\partial t} = \frac{\partial H_{\rm bt}}{\partial \alpha_j}, \quad j,k = 2,\ldots,N. \tag{71}$$

It is even less obvious how to make appropriate modifications in the construction of the solution of the multi-time Schrödinger equations of the previous subsection. Simply making the replacements $a_j = \alpha_j^2 t + t, p_j = \alpha_j\beta_j$ and rescaling $\tilde{H}_j$ in Eqs. (56)–(58) does not work. In addition, we have to redefine the parameter η as $\eta = \kappa^2 t + t$, rescale the off-shell Bethe state $\Phi_{\rm DO}$, and redefine the function f. We found that the following construction works:

$$\Phi_{\rm bt} \equiv |\Phi_{\rm bt}(\kappa,\boldsymbol{\alpha})\rangle = t|1\rangle - \sum_{j=2}^{N} \frac{\alpha_j\beta_j|j\rangle}{\alpha_j^2-\kappa^2}, \quad \boldsymbol{\alpha} = (\alpha_2,\ldots,\alpha_N), \tag{72}$$

$$I_j\Phi_{\rm bt} = m_j\Phi_{\rm bt} + \frac{f\kappa\beta_j|j\rangle}{\kappa^2-\alpha_j^2}, \quad H_{\rm bt}\Phi_{\rm bt} = h_{\rm bt}\Phi_{\rm bt} - f\kappa|1\rangle, \tag{73}$$

$$h_{\rm bt} = \kappa^2 t, \quad m_j = \frac{\alpha_j\beta_j^2}{\alpha_j^2-\kappa^2} - \frac{\alpha_j\beta_j^2}{\alpha_j^2+1}, \quad f = \kappa t^2 - \sum_{j=2}^{N} \frac{\alpha_j^2\beta_j^2}{(\alpha_j^2-\kappa^2)\kappa}. \tag{74}$$

As before, we define the Yang–Yang action $\mathcal{S}_{\rm bt}$ through equations

$$\frac{\partial \mathcal{S}_{\rm bt}}{\partial \alpha_j} = m_j, \quad \frac{\partial \mathcal{S}_{\rm bt}}{\partial t} = h_{\rm bt}, \quad \frac{\partial \mathcal{S}_{\rm bt}}{\partial \kappa} = f, \tag{75}$$

where $j = 2,\ldots,N$. These equations are consistent because $\partial m_j/\partial t = \partial h_{\rm bt}/\partial\alpha_j = 0$, $\partial m_j/\partial\kappa = \partial f/\partial\alpha_j$, and $\partial h_{\rm bt}/\partial\kappa = \partial f/\partial t$.

Integration results in

$$\mathcal{S}_{\rm bt}(\kappa,\boldsymbol{\alpha},t) = \frac{\kappa^2 t^2}{2} - \ln\kappa \sum_{j=2}^{N}\beta_j^2 + \frac{1}{2}\sum_{j=2}^{N}\beta_j^2 \ln\left(\frac{\alpha_j^2-\kappa^2}{\alpha_j^2+1}\right), \tag{76}$$

and as in previous examples we write the wave-function in the form

$$\Psi_{\rm bt}(t,\boldsymbol{\alpha}) = \oint_\gamma d\kappa\, e^{-i\mathcal{S}_{\rm bt}(\kappa,\boldsymbol{\alpha},t)}|\Phi_{\rm bt}(\kappa,\boldsymbol{\alpha})\rangle. \tag{77}$$

This wave-function solves the following system of multi-time Schrödinger equations

$$i\frac{\partial \Psi_{\rm bt}}{\partial t} = H_{\rm bt}\Psi_{\rm bt}, \quad i\frac{\partial \Psi_{\rm bt}}{\partial \alpha_j} = I_j\Psi_{\rm bt}, \tag{78}$$

which includes the non-stationary Schrödinger equation for the bow-tie model. As before, we verify this directly. For example,

$$
\begin{aligned}
i\frac{\partial \Psi_{\rm bt}}{\partial \alpha_j} - I_j \Psi_{\rm bt} &= \oint_\gamma d\kappa\, e^{-i\mathcal{S}_{\rm bt}} \left[i\frac{\partial |\Phi_{\rm bt}\rangle}{\partial \alpha_j} - \frac{f\kappa\beta_j |j\rangle}{\kappa^2-\alpha_j^2}\right] = \\
&- \oint_\gamma d\kappa\, e^{-i\mathcal{S}_{\rm bt}} \left[i\frac{(\alpha_j^2+\kappa^2)\beta_j|j\rangle}{\kappa^2-\alpha_j^2} + \frac{f\kappa\beta_j |j\rangle}{\kappa^2-\alpha_j^2}\right] = \\
&-i\oint_\gamma d\kappa \frac{\partial}{\partial\kappa}\left(e^{-i\mathcal{S}_{\rm bt}} \frac{\kappa\beta_j|j\rangle}{\kappa^2-\alpha_j^2}\right) = 0.
\end{aligned}
\tag{79}
$$

4.3. Generalized bow-tie model

The generalized bow-tie model is the following $N \times N$ matrix Hamiltonian:

$$
\begin{aligned}
H_{\rm gbt} \quad = \quad & \frac{\varepsilon}{2}|a\rangle\langle a| - \frac{\varepsilon}{2}|b\rangle\langle b| + \\
& \sum_{k=3}^{N} \left(q_k|a\rangle\langle k| + q_k|k\rangle\langle a| + q_k|b\rangle\langle k| + q_k|k\rangle\langle b| + r_k t|k\rangle\langle k| \right),
\end{aligned}
\tag{80}
$$

where $|a\rangle$, $|b\rangle$, and $|k\rangle$ is a set of N orthonormal states. 'Generalized' in the name of this model seems somewhat misleading, because it is in fact a *particular case* the usual bow-tie model after a time-independent basis change.

Indeed, let

$$
|1\rangle = \frac{|a\rangle + |b\rangle}{\sqrt{2}}, \quad |2\rangle = \frac{|a\rangle - |b\rangle}{\sqrt{2}}. \tag{81}
$$

The generalized bow-tie model (80) becomes

$$
H_{\rm gbt} = \frac{\varepsilon}{2}|1\rangle\langle 2| + \frac{\varepsilon}{2}|2\rangle\langle 1| + \sum_{k=3}^{N}\left(\sqrt{2}q_k|1\rangle\langle k| + \sqrt{2}q_k|k\rangle\langle 1| + r_k t|k\rangle\langle k|\right). \tag{82}
$$

This is just the bow-tie Hamiltonian (64) with the following choice of parameters:

$$
p_2 = \frac{\varepsilon}{2}, \quad r_2 = 0, \quad p_k = \sqrt{2}q_k, \quad k = 3, \ldots, N. \tag{83}
$$

However, slight adjustments are necessary in the construction of the previous subsection, since $r_2 = 0$ implies $\alpha_2 = 0$. As a result, I_2 in Eq. (70) and β_2 are not well-defined. Instead of I_2 we introduce

$$
\begin{aligned}
\tilde{I}_2 = -\frac{t\tilde{H}_2}{\varepsilon} = -\frac{t}{2}(|1\rangle\langle 2| + |2\rangle\langle 1|) + \frac{1}{\varepsilon}\left(\sum_{m=3}^{N}\beta_m^2\right)|2\rangle\langle 2| + \\
\frac{\varepsilon}{4}\sum_{j=3}^{N}\frac{1}{\alpha_j^2}|j\rangle\langle j| + \sum_{j=3}^{N}\frac{\beta_j}{2\alpha_j}(|1\rangle\langle j| + |j\rangle\langle 1|).
\end{aligned}
\tag{84}
$$

This is the linear in t commuting partner for the generalized bow-tie model constructed in Ref. [15]. In the remaining I_j in Eq. (70) we simply set

$$
\alpha_2 = 0, \quad \alpha_2\beta_2 = \frac{\varepsilon}{2}, \tag{85}
$$

so that

$$I_j = \alpha_j t^2 |j\rangle\langle j| + \beta_j t\left(|1\rangle\langle j| + |j\rangle\langle 1|\right) - \frac{\varepsilon\beta_j}{2\alpha_j^2}(|2\rangle\langle j| + |j\rangle\langle 2|) - \frac{\beta_j^2}{\alpha_j}|2\rangle\langle 2| - \frac{\varepsilon^2}{4\alpha_j^3}|j\rangle\langle j| + \sum_{k\neq j,2}\frac{\alpha_j\alpha_k\beta_j\beta_k(|j\rangle\langle k| + |k\rangle\langle j|) - \alpha_k^2\beta_k^2|j\rangle\langle j| - \alpha_j^2\beta_j^2|k\rangle\langle k|}{\alpha_j(\alpha_j^2-\alpha_k^2)}, \tag{86}$$

for $j = 3,\ldots,N$. Zero curvature conditions (71) hold as before, except ones involving α_2 and I_2 are replaced with

$$\frac{\partial I_j}{\partial\varepsilon} = \frac{\partial\tilde{I}_2}{\partial\alpha_j}, \quad \frac{\partial\tilde{I}_2}{\partial t} = \frac{\partial H_{\text{gbt}}}{\partial\varepsilon}, \quad j = 3,\ldots N. \tag{87}$$

It is straightforward to make appropriate replacements in the remaining formulas of the previous section. Let us only give the final answer

$$\Psi_{\text{gbt}}(t,\boldsymbol{\alpha},\varepsilon) = \oint_\gamma d\kappa\, e^{-i\mathcal{S}_{\text{gbt}}(\kappa,\boldsymbol{\alpha},t)}|\Phi_{\text{gbt}}(\kappa,\boldsymbol{\alpha},\varepsilon)\rangle, \tag{88}$$

where $\boldsymbol{\alpha} = (\alpha_3,\ldots,\alpha_N)$,

$$\mathcal{S}_{\text{gbt}}(\kappa,\boldsymbol{\alpha},t) = \frac{\kappa^2 t^2}{2} - \frac{\varepsilon^2}{8\kappa^2} - \frac{\varepsilon^2}{8} - \ln\kappa\sum_{j=3}^N\beta_j^2 + \frac{1}{2}\sum_{j=3}^N\beta_j^2\ln\left(\frac{\alpha_j^2-\kappa^2}{\alpha_j^2+1}\right), \tag{89}$$

and

$$\Phi_{\text{gbt}} \equiv |\Phi_{\text{bt}}(\kappa,\boldsymbol{\alpha},\varepsilon)\rangle = t|1\rangle + \frac{\varepsilon}{\kappa^2}|2\rangle - \sum_{j=3}^N\frac{\alpha_j\beta_j|j\rangle}{\alpha_j^2-\kappa^2}. \tag{90}$$

5. Many-body extension of the Demkov–Osherov model

In this section we analyze a many-body extension of the Demkov–Osherov model to a system of spinless fermions interacting with a time-dependent impurity level [32]. The Hamiltonian is

$$\hat{H}_f = t\hat{n}_1 + \sum_{k=2}^N p_k\hat{c}_1^\dagger\hat{c}_k + p_k\hat{c}_k^\dagger\hat{c}_1 + a_k(1-u\hat{n}_1)\hat{n}_k, \quad \hat{n}_j \equiv \hat{c}_j^\dagger\hat{c}_j. \tag{91}$$

When the interaction $u = 0$ and there is only one fermion in the system, this is the Demkov–Osherov Hamiltonian (52). We will derive this model together with its commuting partners from Gaudin magnets. We, however, will not pursue a solution of its non-stationary Schrödinger equation here as we did for the other models.

We start by casting Gaudin magnets (1) into a more convenient form,

$$\hat{H}_j^G = 2B\hat{s}_j^z - \sum_{k\neq j}\frac{\frac{1}{2}(\hat{s}_j^+\hat{s}_k^- + \hat{s}_j^-\hat{s}_k^+) + \hat{s}_j^z\hat{s}_k^z}{\varepsilon_j-\varepsilon_k}, \quad [\hat{H}_i^G,\hat{H}_j^G] = 0. \tag{92}$$

Let us represent spins in terms of bosons via a variant of Holstein–Primakoff transformation,

$$\hat{s}_j^- = \sqrt{2s_j}\left(1-\frac{\hat{b}_j^\dagger\hat{b}_j}{2s_j}\right)^{1/2}\hat{b}_j, \quad \hat{s}_j^+ = \sqrt{2s_j}\hat{b}_j^\dagger\left(1-\frac{\hat{b}_j^\dagger\hat{b}_j}{2s_j}\right)^{1/2}, \quad \hat{s}_j^z = \hat{b}_j^\dagger\hat{b}_j - s_j. \tag{93}$$

Expansion in inverse spin magnitudes yields,

$$\hat{s}_j^+\hat{s}_k^- = 2\sqrt{s_js_k}\left(\hat{b}_j^\dagger\hat{b}_k - \frac{(\hat{b}_j^\dagger)^2\hat{b}_j\hat{b}_k}{4s_j} - \frac{\hat{b}_j^\dagger\hat{b}_k^\dagger(\hat{b}_k)^2}{4s_j} + \ldots\right). \tag{94}$$

Since all terms in the expansion of $\hat{H}_j^G$ contain even number of bosonic creation and annihilation operators, replacing bosons with fermions does not affect the commutation relation $[\hat{H}_i^G, \hat{H}_j^G] = 0$. Similarly, we are free to replace $\sqrt{s_j}$ with arbitrary real numbers γ_j, i.e.

$$\hat{b}_j^\dagger \to \hat{c}_j^\dagger, \quad \hat{b}_j \to \hat{c}_j, \quad \sqrt{s_j} = \gamma_j. \tag{95}$$

Then, all terms in brackets in Eq. (94) except the first one vanish and we obtain

$$\hat{H}_j^G = \hat{H}_j^{(0)} + \hat{H}_j^{(1)} + \hat{H}_j^{(2)}, \tag{96}$$

where $\hat{H}_j^{(0)}$ are constants given by the second term on the right hand side of Eq. (47) and

$$\hat{H}_j^{(1)} = 2B\hat{n}_j - \sum_{k\neq j} \frac{\gamma_j\gamma_k(\hat{c}_j^\dagger\hat{c}_k + c_k^\dagger\hat{c}_j) - \gamma_k^2\hat{n}_j - \gamma_j^2\hat{n}_k}{\varepsilon_j - \varepsilon_k}, \tag{97}$$

$$\hat{H}_j^{(2)} = -\sum_{k\neq j} \frac{\hat{n}_j\hat{n}_k}{\varepsilon_j - \varepsilon_k}. \tag{98}$$

We have

$$[\hat{H}_i^{(1)}, \hat{H}_j^{(1)}] = 0, \quad [\hat{H}_i^{(1)} + \hat{H}_i^{(2)}, \hat{H}_j^{(1)} + \hat{H}_j^{(2)}] = 0. \tag{99}$$

The first relation holds because it corresponds to the leading order in the expansion in inverse spin magnitudes. Independently, it follows from the fact that $\hat{H}_j^{(1)}$ are the type 1 Hamiltonians (48) dressed with fermions, i.e. $\hat{H}_j^{(1)} = \sum_{l,m}(\tilde{H}_j)_{lm}\hat{c}_l^\dagger\hat{c}_m$. Further, Eqs. (99) imply that operators $\hat{K}_j = H_j^{(1)} + u\hat{H}_j^{(2)}$ mutually commute. Note that we equivalently acquire a parameter u in front of $\hat{H}_j^{(2)}$ via a simple rescaling $\varepsilon_i \to u^{-1}\varepsilon_i$ and $\gamma_i \to u^{-1/2}\gamma_i$.

Thus, we have derived the following set of mutually commuting Hamiltonians from Gaudin magnets:

$$\hat{K}_j = 2B\hat{n}_j - \sum_{k\neq j} \frac{\gamma_j\gamma_k(\hat{c}_j^\dagger\hat{c}_k + c_k^\dagger\hat{c}_j) - \gamma_k^2\hat{n}_j - \gamma_j^2\hat{n}_k + u\hat{n}_j\hat{n}_k}{\varepsilon_j - \varepsilon_k}. \tag{100}$$

Finally, under the transformation (51), $\hat{K}_1$ turns into the fermion model (91), while the remaining $\hat{K}_j$ are its commuting partners. We read them off Eq. (53), replacing $|i\rangle \to \hat{c}_i$ and adding the term (98) proportional to u, where $(\varepsilon_j - \varepsilon_k)^{-1} = a_j a_k(a_k - a_j)^{-1}$. We obtain

$$\begin{aligned}\tilde{H}_j &= (t - a_j)\hat{n}_j - p_j(\hat{c}_1^\dagger\hat{c}_j - \hat{c}_j^\dagger\hat{c}_1) + \\ &\sum_{k\neq j} \frac{p_j p_k(\hat{c}_j^\dagger\hat{c}_k + c_k^\dagger\hat{c}_j) - p_k^2\hat{n}_j - p_j^2\hat{n}_k - u a_j a_k\hat{n}_j\hat{n}_k}{a_k - a_j}.\end{aligned} \tag{101}$$

6. Discussion

We have constructed complete sets of solutions of the non-stationary Schrödinger equation for a number of time-dependent models. Among them are two interacting many-body Hamiltonians — the driven inhomogeneous Dicke (9) and the time-dependent BCS (5) models. The former describes molecular production in an atomic Fermi gas swept through a narrow Feshbach resonance. The latter include, in particular, the BCS Hamiltonian (6) with a coupling constant inversely proportional to time as well as periodically driven (Floquet) BCS models, e.g., for $B = B_0 \cos \nu t$ in Eq. (5).

It is instructive to assess our results in the context of the theory of exactly solvable multi-level Landau–Zener problems. These are Hamiltonians of the form $A + Bt$, where A and B are time-independent real-symmetric matrices of arbitrary size. The problem is considered solvable if one is able to determine transition probabilities between states at $t = \pm\infty$ explicitly in terms of matrix elements of A and B. Over the years, only a few nontrivial,[4] solvable Hamiltonians have been identified. The main ones are the Demkov–Osherov (10) bow-tie (11), and the generalized bow-tie (80) models. It turns out that the key special property of these models is the presence of nontrivial commuting partners [15]. Moreover, we have demonstrated in this paper that all these models map to a particular sector of Gaudin magnets (1) and presented a new, improved construction of their commuting partners.

Building on the presence of commuting partners in nontrivial solvable Landau–Zener models, Ref. [16] proposed a method of determining the transition probabilities based on zero curvature conditions (15) and (16). Here we have seen that the Demkov–Osherov, bow-tie, and the generalized bow-tie models as well as the inhomogeneous Dicke Hamiltonian indeed satisfy these conditions. Moreover, we solved the non-stationary Schrödinger equation for these models by paralleling the off-shell Bethe Ansatz solution of the Knizhnik–Zamolodchikov equations. The Demkov–Osherov and bow-tie models give rise via the procedure outlined at the end of Section 4.1 (see also Ref. [16]) to derivative solvable Hamiltonians of the form $A + Bt$ and $A + Bt + C/t$, which look very similar to those introduced in Refs. [45–48]. However, most interesting would be to identify nontrivial solvable Hamiltonians of this form[5] that do not reduce to Gaudin magnets and whose non-stationary Schrödinger equations are not of Knizhnik–Zamolodchikov type.

More work is needed to extract various physical information from exact solutions of the non-stationary Schrödinger equations presented in this paper. For example, an interesting quantity to evaluate in the Dicke model is the quantum mechanical average number of bosons $\langle \hat{n}_b(t) \rangle$ as a function of time. Several predictions are available for this quantity in the limit $t \to +\infty$ for both homogeneous [49–53] and inhomogeneous [25] Dicke models, such as, e.g., the breakdown of the adiabaticity [50]. A potentially useful tool for such a calculation are the matrix elements of $\hat{n}_b$ and spin operators between the off-shell Bethe Ansatz states for the inhomogeneous Dicke and Gaudin Hamiltonians [54,55]. It might be then possible to obtain simple expressions for large time asymptotes of $\langle \hat{n}_b(t) \rangle$ and other quantities of interest in the thermodynamic limit. Let us also note in this connection the semiclassical asymptotic expansion for the solution of Knizhnik–Zamolodchikov equations [31]. Another application is to determine the scattering matrix and related observables with the help of the machinery developed for evaluating transition functions between asymptotic solutions of Knizhnik–Zamolodchikov equations as discussed in the Introduction.

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[4] See the footnote 1.

[5] Here we mean ‘scalable’ models uniformly defined for arbitrary matrix size or particle (spin) number, rather than models integrable only for a given fixed matrix size or particle (spin) number.

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