Letter

Singularities in the Loschmidt echo of quenched topological superconductors

Sankalp Gaur [1], Victor Gurarie [1], and Emil A. Yuzbashyan [2]

[1] *Department of Physics and Center for Theory of Quantum Matter, University of Colorado, Boulder, Colorado 80309, USA*

[2] *Department of Physics and Astronomy, Center for Materials Theory, Rutgers University, Piscataway, New Jersey 08854, USA*

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We study the Loschmidt echo in the quenched two-dimensional p-wave topological superconductor. We find that if this superconductor is quenched out of the critical point separating its topological and nontopological phases into either of the two gapful phases, its Loschmidt echo features singularities occurring periodically in time where the second derivative of the Loschmidt echo over time diverges logarithmically. Conversely, we give arguments towards s-wave superconductors not having singularities in their Loschmidt echo regardless of the quench. We also demonstrate that the conventional mean-field theory calculates classical echo instead of its quantum counterpart, and show how it should be modified to capture the full quantum Loschmidt echo.

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About a decade ago, the Loschmidt echo was proposed as yet another way of characterizing the dynamics of quenched quantum systems [1–3]. Consider a quantum system residing in a state $|\Psi_i\rangle$, e.g., in the ground state. Suppose the system Hamiltonian suddenly changes triggering time evolution. After some time t the time-evolved state is projected back onto $|\Psi_i\rangle$. The Loschmidt echo is defined as [4]

$$\mathcal{Z}(t) = \langle \Psi_i | e^{-i\hat{H}t} | \Psi_i \rangle. \tag{1}$$

Of particular interest is the question of whether $\mathcal{Z}(t)$ may be nonanalytic as a function of t, in a way reminiscent of the thermal partition function of quantum systems being nonanalytic when they undergo phase transitions.

This question was explored in the literature for a wide variety of quantum systems. Here, we would like to demonstrate that a two-dimensional (2D) p-wave superconductor when quenched out of a critical point separating its two phases [5] (BEC, or strongly paired, or nontopological phase and BCS, or weakly paired, or topological phase) into either of its two phases features the Loschmidt echo with periodically occurring singularities. Specifically, $\mathcal{Z}(t)$ varies as $(t - t_n)^2 \ln |t - t_n|$ near $t_n = (n + 1/2)\pi/|\xi_0|$, where $|\xi_0|$ is the energy of the zero momentum excitation in the superconductor, and $\partial^2 \mathcal{Z}/\partial t^2$ diverges logarithmically at $t = t_n$ as a result.

We also argue that an s-wave superconductor, in any number of dimensions, is unlikely to have any singularities in its Loschmidt echo. To arrive at these results, we show that the conventional mean-field theory used to describe time evolution of out of equilibrium superconductors is not applicable for the purpose of calculating their Loschmidt echo. We give a prescription how this problem can be corrected and show that the time evolution in this problem is equivalent to the Hamiltonian evolution of complex classical spins satisfying certain boundary conditions. This construction generalizes classical Anderson pseudospins (which are real vectors) used in previous work to describe the time evolution of superconductors.

In the previous work of some of us [6] we claimed that s-wave superconductors did have singularities in their echo. We demonstrate here that the quantity evaluated in that work was in fact the *classical echo*, as opposed to the true quantum echo we calculate below. We also note that Loschmidt echo for topological insulators/superconductors was explored in the literature for quite some time [7]. However, that analysis involved imposing a gap function on a superconductor externally as opposed to determining it self-consistently, and therefore cannot be realized by interacting fermions.

We begin our analysis with writing down the Hamiltonian for the 2D chiral p-wave superconductor [8–10]

$$\hat{H} = \sum_{\mathbf{p}} \xi_p \, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \frac{g}{V} \sum_{\mathbf{p},\mathbf{q}} p\, q\, e^{i(\phi_{\mathbf{p}} - \phi_{\mathbf{q}})} \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} \hat{a}_{-\mathbf{q}} \hat{a}_{\mathbf{q}}, \tag{2}$$

where p and ϕ_p are the polar coordinates describing the vector $\mathbf{p}$, $\hat{a}_{\mathbf{p}}$ and $\hat{a}^\dagger_{\mathbf{p}}$ are fermionic creation and annihilation operators, $\xi_p = p^2/(2m) - \mu$ with μ being the chemical potential and m the mass of the fermions, g is the coupling constant, and V is the volume of the system. It is customary to absorb the angles $\phi_{\mathbf{p}}$, $\phi_{\mathbf{q}}$ into the fermionic creation and annihilation operators. Equation (2) becomes

$$\hat{H} = \sum_{\mathbf{p}} \xi_p \, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \frac{g}{V} \sum_{\mathbf{p},\mathbf{q}} p\, q\, \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} \hat{a}_{-\mathbf{q}} \hat{a}_{\mathbf{q}}. \tag{3}$$

It is also customary to use mean-field theory to study this problem, which results in the simplified

$$\hat{H} = \sum_{\mathbf{p}} \xi_p \, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \Delta \sum_{\mathbf{p}} p\, \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} - \bar{\Delta} \sum_{\mathbf{p}} p\, \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}}. \tag{4}$$

Here, Δ and $\bar{\Delta}$ are generally time-dependent gap functions. These can be related back to the fermions via the so-called gap equation. We will find that the gap equation that we need here is surprisingly subtle. Because of that, we will postpone its discussion until later and for now treat Δ and $\bar{\Delta}$ as some given time-dependent functions.

It is well known in the theory of superconductivity that the eigenstates of Eq. (4) can be written in the following form,

$$|\Psi\rangle = \prod_{\mathbf{p}} (u_{\mathbf{p}} + v_{\mathbf{p}}\, \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}^{\dagger}_{-\mathbf{p}})|0\rangle, \tag{5}$$

where $|0\rangle$ is a vacuum. Normalization of this wave function requires that

$$|u_{\mathbf{p}}|^2 + |v_{\mathbf{p}}|^2 = 1. \tag{6}$$

In this Letter we consider the standard quench problem. We initiate the system in the ground state $|\Psi_i\rangle$ of the form (5) parametrized by $u_{i\mathbf{p}}$ and $v_{i\mathbf{p}}$. The coupling constant g of the Hamiltonian is suddenly changed to a new value so that $|\Psi_i\rangle$ is no longer its eigenstate. The state now evolves forward with the new Hamiltonian. Knowing the time-dependent state $|\Psi(\tau)\rangle$, we can calculate the Loschmidt echo according to Eq. (1), or with $\mathcal{Z}(t) = \langle \Psi_i|\Psi(t)\rangle$. We would like to understand if there are values of "Loschmidt time" t at which $\mathcal{Z}(t)$ is singular.

We note that evolving the state $|\Psi\rangle$ forward in time is equivalent to having time dependent $u_{\mathbf{p}}(\tau)$ and $v_{\mathbf{p}}(\tau)$, satisfying the initial conditions $u_{\mathbf{p}}(0) = u_{i\mathbf{p}}$, $v_{\mathbf{p}}(0) = v_{i\mathbf{p}}$. By applying the Hamiltonian (4) to the state (5) we derive their equations of motion,

$$i\dot{u}_{\mathbf{p}} = -\xi_p u_{\mathbf{p}} - \bar{\Delta}\, p\, v_{\mathbf{p}}, \quad i\dot{v}_{\mathbf{p}} = \xi_p v_{\mathbf{p}} - \Delta\, p\, u_{\mathbf{p}}. \tag{7}$$

To calculate the Loschmidt echo, we evolve $|\Psi\rangle$ to the Loschmidt time t and project it back onto itself, with the result

$$\mathcal{Z} = \prod_{\mathbf{p}} 2S_{\mathbf{p}}, \quad \ln \mathcal{Z} = V \int d^2p \ln(2S_{\mathbf{p}}), \tag{8}$$

where

$$2S_{\mathbf{p}} = u^*_{i\mathbf{p}} u_{\mathbf{p}}(t) + v^*_{i\mathbf{p}} v_{\mathbf{p}}(t). \tag{9}$$

A typical mechanism for $\mathcal{Z}$ to become singular is for $S_{\mathbf{p}}$ to vanish at some critical value $t = t_c$, at some value p_c. Let us show that the $S_0 \equiv \lim_{\mathbf{p}\to 0} S_{\mathbf{p}}$ can vanish in a particularly simple way. Indeed, the equations of motion of u_0, v_0 (also understood as limits when $\mathbf{p}$ is taken to zero) can be solved directly, as they decouple from the functions $\Delta(\tau)$, $\bar{\Delta}(\tau)$, with the result

$$u_0 = u_{i0}\, e^{i\xi_0\tau}, \quad v_0 = v_{i0}\, e^{-i\xi_0\tau}. \tag{10}$$

Substituting into Eq. (9) and taking into account Eq. (6) we find

$$2S_0 = \cos(\xi_0 t) + i(u^*_{i0}u_{i0} - v^*_{i0}v_{i0})\sin(\xi_0 t). \tag{11}$$

S_0 vanishes as a function of t only if

$$|u_{i0}|^2 = |v_{i0}|^2. \tag{12}$$

Therefore let us restrict our attention to this case. It is well known that the p-wave superconductor we study here can be in the weakly coupled phase or strongly coupled phase depending on the sign of $\xi_0 = -\mu$. The condition (12) holds true in the ground state at the critical point between the phases only. Therefore, from now on we consider $|\Psi_i\rangle$ to be the ground state of a critical p-wave superconductor, while the Hamiltonian after the quench will describe the strongly coupled $\xi_0 < 0$ or weakly coupled $\xi_0 > 0$ phase. In other words, the chemical potential effectively changes from $\mu_i = 0$ to a nonzero μ as a result of the quench. It is necessary to keep μ in the mean-field Hamiltonian (4) to make sure we have the correct average fermion number (see below and also Ref. [6]).

With S_0 turning to zero at times $t_n = \pi(n + 1/2)/|\xi_0|$ with integer n, $\mathcal{Z}$ can now be singular at $t = t_n$. However, S_0 becoming zero at these times is not by itself a sufficient condition for $\mathcal{Z}$ to be singular. To understand if it becomes singular at these times, we need to examine not just the point $p = 0$ in the momentum space but also its vicinity. Fortunately in this region we can solve the equations of motion (7) perturbatively, using p as a small parameter. The solution reads

$$\begin{aligned} u_{\mathbf{p}}(\tau) &= \left(u_{i\mathbf{p}} + i v_{i\mathbf{p}}\, p \int_0^{\tau} d\tau'\, \bar{\Delta}(\tau') e^{-2i\xi_p\tau'}\right) e^{i\xi_p\tau}, \\ v_{\mathbf{p}}(\tau) &= \left(v_{i\mathbf{p}} + i u_{i\mathbf{p}}\, p \int_0^{\tau} d\tau'\, \Delta(\tau') e^{2i\xi_p\tau'}\right) e^{-i\xi_p\tau}. \end{aligned} \tag{13}$$

This allows us to calculate, from Eq. (9),

$$\begin{aligned} 2S_{\mathbf{p}} \approx\ & u^*_{i\mathbf{p}} u_{i\mathbf{p}}\, e^{i\xi_p t} + v^*_{i\mathbf{p}} v_{i\mathbf{p}}\, e^{-i\xi_p t} \\ & + i v^*_{i\mathbf{p}} u_{i\mathbf{p}}\, p\, f_{\mathbf{p}} e^{-i\xi_p t} + i u^*_{i\mathbf{p}} v_{i\mathbf{p}}\, p\, \bar{f}_{\mathbf{p}} e^{i\xi_p t}, \end{aligned} \tag{14}$$

where

$$f_{\mathbf{p}} = \int_0^t d\tau\, \Delta(\tau) e^{2i\xi_p\tau}, \quad \bar{f}_{\mathbf{p}} = \int_0^t d\tau\, \bar{\Delta}(\tau) e^{-2i\xi_p\tau}. \tag{15}$$

Equation (14) is an expansion in powers of p, therefore $u_{i\mathbf{p}}$, $v_{i\mathbf{p}}$, and ξ_p themselves need to be expanded in powers of p. Generally, taking into account Eq. (12), this expansion has the form

$$2|u_{i\mathbf{p}}|^2 = 1 + \alpha p + \cdots, \quad 2|v_{i\mathbf{p}}|^2 = 1 - \alpha p + \cdots, \tag{16}$$

where α is some (real) constant. This leads to

$$2S_p \approx \cos(\xi_0 t) + \tfrac{1}{2}(i\bar{f}_0 + \alpha) p e^{i\xi_0 t} + \tfrac{1}{2}(i f_0 - \alpha) p e^{-i\xi_0 t}. \tag{17}$$

Let us examine the vicinity of the point in time where S_0 vanishes. Expanding in powers of $t - t_n$ and p we find

$$\begin{aligned} 2S_p &\approx (-1)^n[|\xi_0|(t_n - t) + \operatorname{sgn}(\xi_0)\beta p], \\ \beta &= i\alpha + (f_0 - \bar{f}_0)/2. \end{aligned} \tag{18}$$

We now know, quite generally, the behavior of $S_{\mathbf{p}}$ in the vicinity of the point $p = 0$ and time t_n where a singularity of $\mathcal{Z}$ can occur. We can estimate the contribution of small $\mathbf{p}$ to the Loschmidt echo (8) by writing

$$\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial t} = 2\pi V \int_0^{p_0} \frac{p\, dp}{t - t_n - \beta p/\xi_0}, \tag{19}$$

where p_0 is some momentum beyond which the expansion (18) no longer holds. The integral is easy to evaluate and produces

$$\frac{1}{V\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial t} = \frac{2\pi \xi_0^2 (t - t_n)}{\beta^2} \ln\left[\frac{\xi_0(t - t_n)}{\beta p_0}\right] \tag{20}$$

as the singular contribution to the Loschmidt echo (with the second derivative of $\ln \mathcal{Z}$ over time t and therefore also with $\partial^2\mathcal{Z}/\partial t^2$ having a logarithmic singularity).

Equation (20) constitutes the main result of this Letter. A 2D p-wave superconductor when quenched out of its critical point into either its weak (topological) or strong (nontopological) phase has periodic singularities in its Loschmidt echo where its second derivative over time diverges logarithmically. The singularity is driven by the behavior of small momentum fermions. The parameter α which appears in the calculations above is controlled by the initial amplitudes $u_{i\mathbf{p}}$, $v_{i\mathbf{p}}$. Those can be found as the ground state of the Hamiltonian before quench is known. Being at the critical point where $\xi_0 = -\mu_i = 0$, it must have the form

$$\hat{H}_i = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \Delta_i \sum_{\mathbf{p}} p\, \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} - \bar{\Delta}_i \sum_{\mathbf{p}} p\, \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}}, \quad (21)$$

where Δ_i and $\bar{\Delta}_i$ are the equilibrium gap functions. Calculating its ground state is a standard exercise. Evaluating $u_{i\mathbf{p}}$ and $v_{i\mathbf{p}}$ leads to $\alpha = 1/(4m\sqrt{\bar{\Delta}_i \Delta_i})$.

We would like to point out that the ratio of the amplitudes $v_{i\mathbf{p}}/u_{i\mathbf{p}}$ goes either to infinity or to zero in the two phases of the 2D chiral p-wave superconductors as $p \to 0$, determining the topological properties [5] of these phases. The only exception is the critical point between the phases where this ratio is 1. Therefore, the criticality in the Loschmidt echo directly reflects the fact that before the quench the system is at the critical point between the two phases with different topology.

The question still remains whether $S_{\mathbf{p}}$ can also vanish at other values of momenta, perhaps leading to other singularities in $\mathcal{Z}$ which we also need to explore. We would like to argue that this does not happen. To do this, we need to understand further how Δ and $\bar{\Delta}$ are related to the fermions. Usually in the problem of quantum quench where the goal is to calculate $|\Psi(\tau)\rangle$ after the quench, the following equation is used for Δ,

$$\Delta(\tau) = \frac{g}{V} \sum_{\mathbf{p}} p \langle \Psi(\tau) | \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} | \Psi(\tau) \rangle = \frac{g}{V} \sum_{\mathbf{p}} p\, u^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau), \quad (22)$$

and its complex conjugate for $\bar{\Delta}(\tau)$. Substituting this into Eq. (7) produces the equations of motion for $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$. They are nonlinear but known to be integrable. Their solution can be found for a variety of initial conditions and allows to calculate $|\Psi(t)\rangle$ in many interesting cases [10].

However, we would like to argue that these equations are not suitable for calculating the Loschmidt echo (1). The resulting wave function $|\Psi(t)\rangle$, while well suited for calculating the expectations of local observables in the problem and finding their time dependence, produces wrong results if used to evaluate overlaps of $|\Psi_i\rangle$ and $|\Psi(t)\rangle$ occurring in the calculation of $\mathcal{Z}$.

While leaving the detailed construction for the Supplemental Material [11], we note that the Loschmidt echo is simply a matrix element of the evolution operator. These can be calculated using a conventional Feynman functional integral, avoiding the intricacies involved in the Schwinger-Keldysh functional integral construction. The saddle point approximation with respect to Δ and $\bar{\Delta}$ calculated in the framework of the conventional Feynman functional integral produces (see also Ref. [12]),

$$\Delta(\tau) = \frac{g}{V\mathcal{Z}} \sum_{\mathbf{p}} p \langle \Psi_i | e^{-i\hat{H}(t-\tau)} \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} e^{-i\hat{H}\tau} | \Psi_i \rangle, \quad (23)$$

and similarly for $\bar{\Delta}(\tau)$. This equation replaces Eq. (22) for the purpose of determining Δ and $\bar{\Delta}$.

To elucidate the meaning of Eq. (23) we introduce a new wave function defined by

$$|\tilde{\Psi}(\tau)\rangle = e^{i\hat{H}(t-\tau)} |\Psi_i\rangle. \quad (24)$$

Just as $|\Psi(\tau)\rangle$, it is described by the amplitudes $\tilde{u}_{\mathbf{p}}(\tau)$, $\tilde{v}_{\mathbf{p}}(\tau)$. They satisfy the same equations of motion (7) as $u_{\mathbf{p}}(\tau)$, $v_{\mathbf{p}}(\tau)$, but with different boundary conditions. Whereas $u_{\mathbf{p}}(0) = u_{i\mathbf{p}}$, $v_{\mathbf{p}}(0) = v_{i\mathbf{p}}$, the boundary conditions on these new amplitudes are imposed at $t = \tau$, $\tilde{u}_{\mathbf{p}}(t) = u_{i\mathbf{p}}$, $\tilde{v}_{\mathbf{p}}(t) = v_{i\mathbf{p}}$. In terms of these we find

$$\frac{1}{\mathcal{Z}} \langle \Psi_i | e^{-i\hat{H}(t-\tau)} \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} e^{-i\hat{H}\tau} | \Psi_i \rangle = \frac{\tilde{u}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}}, \quad (25)$$

where $S_{\mathbf{p}}$ can also be expressed in terms of these amplitudes by

$$2S_{\mathbf{p}}(t) = \tilde{u}^*_{\mathbf{p}}(\tau) u_{\mathbf{p}}(\tau) + \tilde{v}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau). \quad (26)$$

Note that $S_{\mathbf{p}}$ do not depend on τ, thus they can be calculated at arbitrary τ. In particular, substituting $\tau = t$ we see that the definitions (26) and (9) coincide. With the help of these relations we find

$$\Delta(\tau) = \frac{g}{V} \sum_{\mathbf{p}} \frac{p\, \tilde{u}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}}, \quad \bar{\Delta}(\tau) = \frac{g}{V} \sum_{\mathbf{p}} \frac{p\, \tilde{v}^*_{\mathbf{p}}(\tau) u_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}}. \quad (27)$$

These new gap equations replace the old Eq, (22). Thus we now need to solve equations of motion (7) supplemented by Eq. (27). These equations constitute the second main result of this Letter. Note that unlike in Eq. (22), here $\bar{\Delta}$ is not equal to the complex conjugate of Δ.

Solving these new equations of motion is an interesting problem by itself, which will be left as a subject for future work. Here, we would just like to see whether they are compatible with the singularities in $\mathcal{Z}$ that we found earlier. In the quench scenario considered earlier, we had $S_{\mathbf{p}}$ vanish for small $\mathbf{p}$ according to Eq. (18). Substituting this into Eqs. (27) we find

$$\Delta \sim \int \frac{\tilde{u}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau)\, p^2 dp}{t - t_n + \beta p}. \quad (28)$$

This shows that as a function of t, Δ will have a divergent second derivative. This by itself does not affect the earlier established fact of the divergent second derivative of $\mathcal{Z}$.

However, up until now we only analyzed vanishing of S_0. What if $S_{\mathbf{p}}$ vanishes for some nonzero p_c? If this were to happen, then the gap equation should be expected to read

$$\Delta \sim \int \frac{\tilde{u}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau)\, p^2 dp}{t - t_c + \beta(p - p_c)}, \quad (29)$$

where the expression to be integrated is an approximation valid in the vicinity of $p \sim p_c$. Taking into account that β is generally complex and recalling the standard formula $\mathrm{Im}\, 1/(x \pm i\epsilon) = \mp i\pi\delta(x)$, we see that Δ will then generally be a discontinuous function of t. If Δ is discontinuous, so will be $\mathcal{Z}$ [13]. On the other hand, $\mathcal{Z}$ is closely related to the partition functions of quantum systems. Partition functions of thermal systems cannot be discontinuous functions of temperature. Likewise we expect that the Loschmidt echo cannot be a discontinuous function of time t, thus we conclude that the scenario where $S_{\mathbf{p}}$ vanishes for some nonzero p cannot be realized.

Note that the s-wave superconductors are described by the Hamiltonians very similar to the ones studied here, but without the extra factors of momenta in the interactions. Let us test if they could have a vanishing $S_{\mathbf{p}}$ at some critical p_c. Putting this superconductor in d-dimensional space for generality, the gap equation and the expression for the Loschmidt echo now read

$$\Delta \sim \int \frac{\tilde{u}^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau)p^{d-1}dp}{t-t_c+\beta(p-p_c)}, \quad \frac{\partial \mathcal{Z}}{\partial t} \sim \int \frac{p^{d-1}dp}{t-t_c+\beta(p-p_c)}.$$

Again, unless $p_c = 0$, the equations above lead to the discontinuity of Δ as a function of t and therefore the discontinuity in $\mathcal{Z}$, which we expect cannot happen. The only exception would be $p_c = 0$. However, here $p = 0$ is not special in the same way as in p-wave superconductors as the spin at $p = 0$ is coupled to the rest of the spins and we were unable to identify any initial conditions for which S_0 can vanish. We are also not aware of any alternative calculation showing vanishing of S_0 in s-wave superconductors. Overall, this leads to our conclusion that the Loschmidt echo in s-wave superconductors lacks any singularities.

On the contrary, we expect that the criticality in Loschmidt echo due to the $p = 0$ mode also manifests itself in higher-order superconductors, such as a 2D $d_{x^2-y^2} + id_{xy}$ superconductor. Exploring this could be the subject of further research.

Given the wave function $|\Psi(t)\rangle$ calculated using the standard approach of Eqs. (7) together with (22), one can ask whether its overlap with the initial wave function $|\Psi_i\rangle$ is still meaningful. Let us argue that

$$\mathcal{L} = |\langle \Psi_i|\Psi(t)\rangle|^2 \tag{30}$$

coincides with the classical echo defined as [14]

$$\mathcal{L} = \int d\mathbf{x}\, \rho(\mathbf{x}, 0)\rho(\mathbf{x}, t). \tag{31}$$

Here, $\mathbf{x}$ are the coordinates parametrizing the phase space of a classical system and $\rho(\mathbf{x}, t)$ is the classical distribution function, which is in general time dependent.

Indeed, equations of evolution of $|\Psi(t)\rangle$ (7) together with (22) are quasiclassical and should be equivalent to evolving the classical distribution function ρ. Therefore, it should not be surprising that Eqs. (30) and (31) coincide. Formal proof of that consists of identifying ρ for the interacting fermions system that we study here with the Wigner function computed from the quantum state of our system and showing formally that Eq. (30) reduces to (31); see the Supplemental Material [11] and Ref. [15] to see how this calculation can be carried out. The quantity $\mathcal{L}$ was calculated for s- and p-wave superconductors in Ref. [6] and found to have many singularities as a function of t. Thus we arrive at a striking conclusion: The classical echo (31) can be singular, even when the full quantum echo calculated here is not.

If the echo is measured experimentally, we need to account for the possibility that the initial state might be at a finite temperature T. This implies that a certain number of Bogoliubov excitations might be present in the initial state. The Bogoliubov excitations do not contribute to the echo; however, they reduce the number of Cooper pairs resulting in the same singularity as given in Eq. (20) but now suppressed by a weight factor $0 < w < 1$ multiplying Eq. (20). This is consistent with other studies of Loschmidt echo in systems kept initially at a finite temperature [16].

An interesting remaining question is the role of the chemical potential μ we introduced into the postquench Hamiltonian. Normally in dynamical problems with conserved total particle number μ is arbitrary as changing μ simply changes the phase of the time-dependent wave function. However, our initial wave function is not an eigenstate of the total particle number operator $\hat{N} = \sum_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}$. Note that $\langle\Psi(\tau)|\hat{N}|\Psi(\tau)\rangle$ can be expressed entirely in terms of $u_{i\mathbf{p}}$, $v_{i\mathbf{p}}$ characterizing the initial wave function which contains μ_i but is independent of μ. A more relevant expectation value which arises in the process of evaluating the Loschmidt echo and depends on μ is $\langle\Psi_i|e^{-i\hat{H}(t-\tau)}\hat{N}e^{-i\hat{H}\tau}|\Psi_i\rangle/\mathcal{Z}$. Therefore, μ must be chosen in such a way as to make this expectation equal to the desired fermion number, i.e., we need to introduce μ to ensure that we describe the time evolution with the correct number of fermions. More details of this formalism are given in the Supplemental Material [11].

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