MATHEMATICS DIAGNOSTIC

Simple Derivatives and integrals

Using the following two equations, eliminate v and solve for r in terms of n, h, ε_0, m & e:

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$

$$mvr = \frac{nh}{2\pi}$$

What are the solutions to the following differential equations? (k is a real constant)

$$y''(x) = -k^2 y(x)$$

$$y''(x) = +k^2 y(x)$$

Take the following derivatives:

$$\frac{d}{dx}\sin(ax)$$

$$\frac{d}{dx}e^{\alpha x}$$

$$\frac{d}{dx}e^{ikx}$$

$$\frac{d}{dx}e^{-\beta x^2}$$

Compute the following integrals:

$$\int_0^L \sin(\pi x/L)dx$$

$$\int_0^L \sin(\pi x/L)\cos(\pi x/L)dx$$

Can you rewrite the following expression in terms of sines and cosines?

$$e^{ikx}$$

What is the real part of the above expression? the imaginary part? (k and x are real)

COMPLEX NUMBERS

A complex number **z** has both a ***real*** and an ***imaginary*** part:

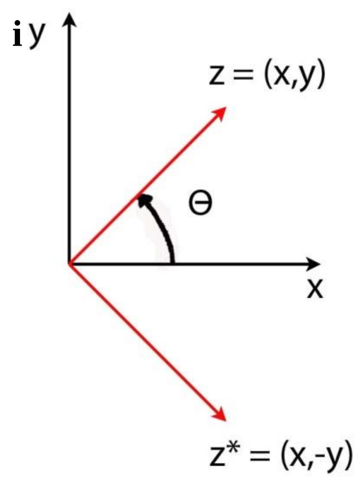
$z = x + iy$

where **x** and **y** are ***real numbers***. **x** is the ***real*** part of **z**, and **y** is the ***imaginary*** part of **z**.

We can think of **z** in the same way as a 2-D vector (ordered pair):

$\vec{r} = (x, y)$

$z = x + iy = (x, y)$



The ***complex conjugate*** (**z***) of **z** is defined as:

$z^* = x - iy = (x, -y)$

z and **z*** both have the same ***magnitude*** (length), but **z*** is the reflection of **z** across the **x-axis**.

z* is found from **z** by changing **i** to **–i** (in other words, **i*** = **-i**).

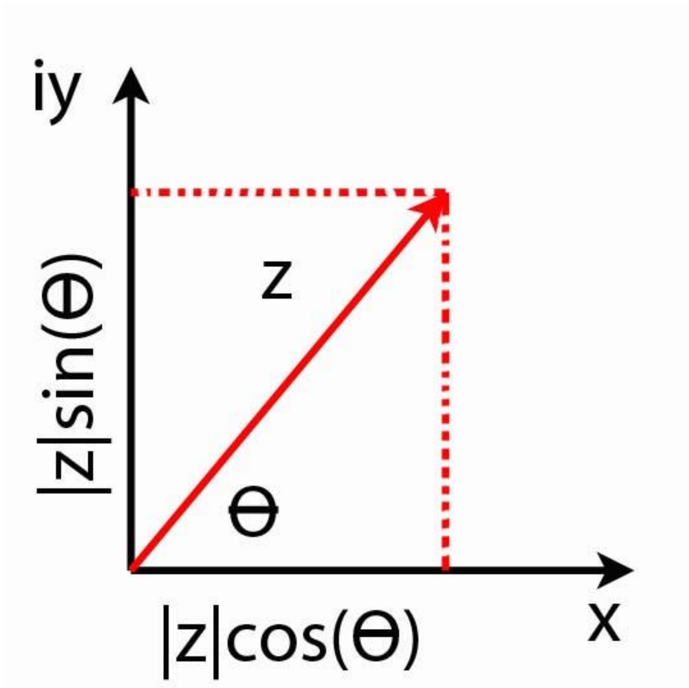
We define **i** as:

$i \equiv \sqrt{-1}$

so that

$i^2 = -1$, $i^3 = -i$, $i^4 = +1$, etc…

The length (magnitude) of a vector is found (via the Pythagorean Theorem) by summing the squares of the legs of the triangle with the vector as the hypotenuse:



$$|z|^2 = z^* z = (x - iy)(x + iy)$$

$$= x^2 + ixy - ixy - i^2 y^2$$

$$= x^2 + y^2$$

OR

$$z = x + iy = |z|\left[\cos(\theta) + i\sin(\theta)\right]$$

$$|z|^2 = z^* z = |z|\left(\cos(\theta) - i\sin(\theta)\right) \cdot |z|(\cos(\theta) + i\sin(\theta)$$

$$= |z|^2\left[\cos^2(\theta) + i\cos(\theta)\sin(\theta) - i\cos(\theta)\sin(\theta) - i^2\sin^2(\theta)\right]$$

$$= |z|^2\left[\cos^2(\theta) + \sin^2(\theta)\right] = |z|^2$$

Note that the magnitude (length) of a complex number is always a positive real number.

EXPONENTIALS

$e^0 = 1$

$e^x \cdot e^y = e^{x+y}$

$\frac{e^x}{e^y} = e^x \cdot e^{-y} = e^{x-y}$

If we use the identity:

$e^{i\theta} \equiv \cos(\theta) + i\sin(\theta)$

then any complex number can be written as:

$z = x + iy = |z|[\cos(\theta) + i\sin(\theta)] = |z|e^{i\theta} = |z|\exp(i\theta)$

$z = |z|\exp(i\theta)$

$z^* = |z|\exp(-i\theta)$

$|z|^2 = z^* z = |z|\exp(-i\theta) \cdot |z|\exp(i\theta)$

$= |z|^2 \exp(-i\theta + i\theta) = |z|^2 \exp(0) = |z|^2$

$\exp(ikx) = \cos(kx) + i\sin(kx)$ represents a wave that oscillates in both the real and imaginary directions:

TAYLOR EXAPNASION

f(x)= $f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots,$

$$where: \quad f' \equiv \frac{df}{dx}; \quad f'' \equiv \frac{d^2 f}{dx^2}; \quad f^n \equiv \frac{d^n f}{dx^n}$$

and *n*! denotes the <u>factorial</u> of *n*: $n! = 1 \times 2 \times 3 \times \ldots \times n$ $\quad 0! \equiv 1$

.

In the more compact <u>sigma notation</u>, this can be written as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

.

Example: f(x)=e^x ; if a =0 $\mapsto$ f(a)=e^0=1 $\quad f^{(n)}(a=0)=\frac{d^n}{dx^n}\left(e^x\right)\Big| = 1$

)

$$\Rightarrow \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots.$$

Taylor expansion- powerful tool for approximations

If x-a=ε <<1 then all higher powers:

$\varepsilon^2, \varepsilon^3, \varepsilon^4$,... are negligibly small and can be ignored! (note - x is dimensionless) Examples:

$$e^{0.1} \sim e^0 + 0.1 + \frac{(0.1)^2}{2} + \ldots \approx 0.1$$

$$\sin(0.1rad) \sim \sin(0) + \cos(0) \times 0.1 + \ldots \approx 0.1rad$$

Examples:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$Euler's equation \Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$$

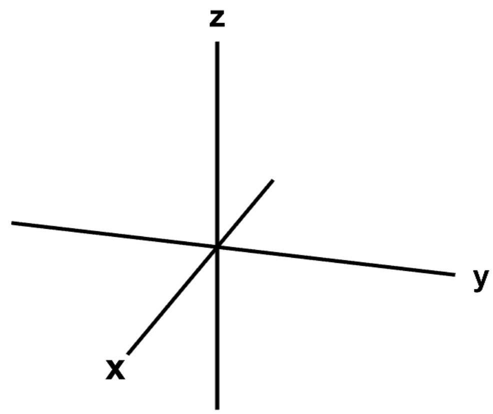
Vectors

$$\vec{F} \equiv \boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$$

$$\vec{i} \equiv \boldsymbol{i} \equiv \hat{x} \quad \textit{unit vecor along x}$$

$$\vec{j} \equiv \boldsymbol{j} = \hat{y} \quad \textit{unit vecor along y}$$

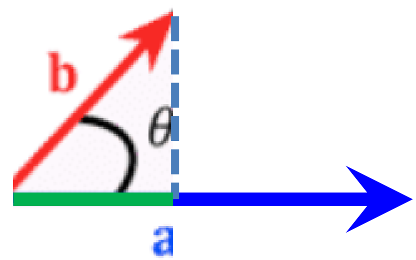
$$\vec{k} \equiv \boldsymbol{k} \equiv \hat{z} \quad \textit{unit vecor along z}$$



Dot (Scalar) product

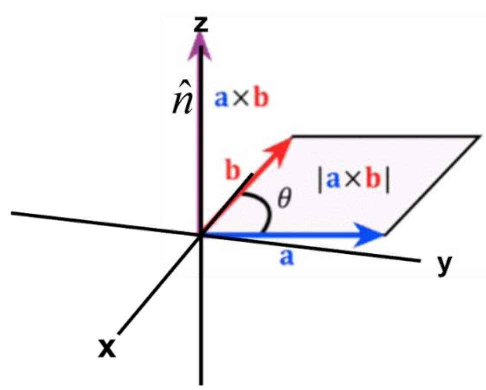
$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta$$

Also: $$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$



Cross (vector) product

$$\vec{a} \times \vec{b} = \hat{n}|\vec{a}| \cdot |\vec{b}| \sin\theta$$

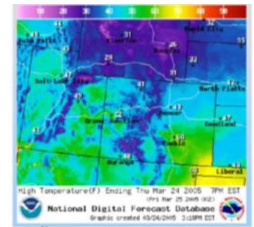


$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\vec{i} - (A_x B_z - A_z B_x)\vec{j} + (A_x B_y - A_y B_x)\vec{k}$$

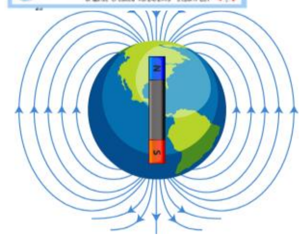
Scalar and Vector fields, Gradient, Divergence

Scalar field : Associates a number with every point in space



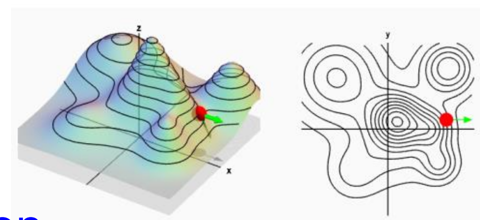
$$F(x,y,z)$$

Vector field : Associates a vector with every point in space



$$\vec{F}(x,y,z) \equiv \boldsymbol{F}(x,y,z) = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$$

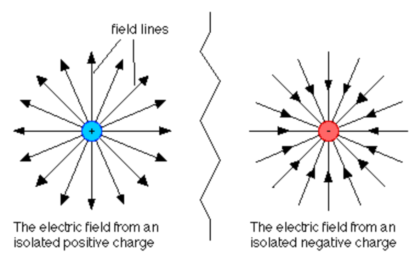
Gradient: Turns a scalar field into a vector field



$$\nabla F = \frac{dF}{dx}\boldsymbol{i} + \frac{dF}{dy}\boldsymbol{j} + \frac{dF}{dz}\boldsymbol{k}$$ Vector field

Gives the "direction and rate of increase" of a function

Divergence: Turns a vector field into a scalar field

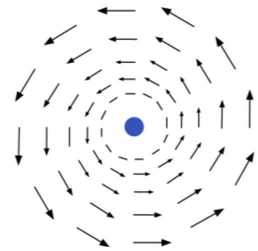


$$\nabla \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$ Scalar field

Gives the extent to which the vector field behaves like a source at a given point- the extent to which there are more of the field vectors exiting from a point than entering it.

Gauss law: $$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_o}$$

Curl: Measures tendency of a vector field to rotate, gives axis and rate of rotation



$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{\boldsymbol{\imath}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\boldsymbol{k}}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Suppose ***F is*** the velocity field of a fluid flowing past a small ball making it rotate. The rotation axis points in the direction of the curl and the angular speed of the rotation is half the magnitude of the curl.

Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Vector Identities

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla \cdot \mathbf{A}) = 0$$

https://en.wikipedia.org/wiki/Vector_calculus_identities

<u>**Divergence (Gauss) theorem**</u>

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S \mathbf{F} \cdot d\mathbf{S}$$

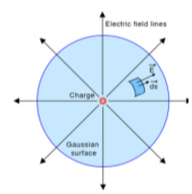
Volume integral

Closed Surface integral = flux through the surface

The surface integral of a vector field over a closed surface- called the "flux" through the surface- is equal to the volume integral of the divergence over the region enclosed by the surface.

Intuitively, it states that "the sum of all sources of the field in a region (with sinks regarded as negative sources) gives the net flux out of the region".

Example – Gauss law in E&M



Electric flux through a surface is proportional to the total charge enclosed by the surface.

Integral form: $$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{in}}{\varepsilon_0}$$

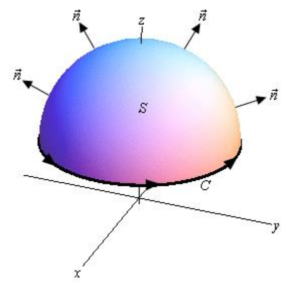
$$\oiint_S \vec{E} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{E} dV$$

$$\frac{Q_{in}}{\varepsilon_0} = \iiint_V \rho(\vec{r}) dV$$

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ Differential form

ρ is the Charge density

Stokes theorem



$$\oint_C \vec{A}\,.\,\vec{dl} = \iint_S (\vec{\nabla} \times \vec{A})\ .\,d\vec{S}$$

Line integral Surface integral

The line integral of a vector field **A** around any closed curve is equal to the surface integral of the curl of **A** taken over any surface **S** of which the curve is a bounding edge.

Example – Faraday’s law in E&M

The electromotive force around a closed path is equal to the negative of the time rate of change of the magnetic flux enclosed by the path

Integral form:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint_C \vec{E} \cdot d\vec{l} = \iint_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{S}$$

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S}$$

$\Longrightarrow$

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Differential form

DIFFERENTIAL EQUATIONS

- e^x is the only function whose derivative is equal to itself:

$$\frac{d}{dx}y(x) = y(x) \quad \Leftrightarrow \quad y(x) = Ae^x$$

$$\frac{d}{dx}(A\ \exp(+kx)) \ = \ A\frac{d}{dx}\exp(+kx) = A\ \exp(+kx)\cdot\ \frac{d}{dx}(+kx) = \ +k\cdot A\ \exp(+kx)$$

$$\frac{d^2}{dx^2}(A\ \exp(ikx)) \ = \ \frac{d}{dx}(ik\cdot A\ \exp(ikx)) = i^2k^2\cdot A\ \exp(ikx) = \ -k^2A\ \exp(ikx)$$

- $\frac{\mathrm{d}^2}{\mathrm{dx}^2}y(x) = -k^2y(x)$ has two independent solutions:

  $$y(x) = A\cos(kx) \qquad \& \qquad y(x) = B\cos(kx)$$

  The most general solution is a superposition (linear combination) of the two independent solutions:

  $$y(x) = A\cos(kx)\ +\ B\sin(kx)$$

  The same equation has for solutions $y(x) = C\ \exp(+ikx)$ & $y(x) = D\ \exp(-ikx)$ or, more generally, a linear combination of the two:

  $$y(x) = C\ \exp(+ikx)\ +\ D\ \exp(-ikx)$$

  These two "different" solutions are equivalent, via the relations:

  $$\exp(+ikx) = \cos(kx) + i\sin(kx) \qquad \& \qquad \exp(-ikx) = \cos(kx) - i\sin(kx)$$

  $$\cos(kx)\ =\ \frac{e^{ikx} + e^{-ikx}}{2} \qquad \& \qquad \sin(kx)\ =\ \frac{e^{ikx} - e^{-ikx}}{2i}$$

  The minus sign in front of k^2 indicates the independent solutions will be oscillatory. This should make sense since the second derivative tells us about the curvature of the function. When sine (or cosine) is positive, it has negative curvature; when negative, the function has positive curvature, so it is always curving back on itself, hence oscillatory solutions like sine or cosine.

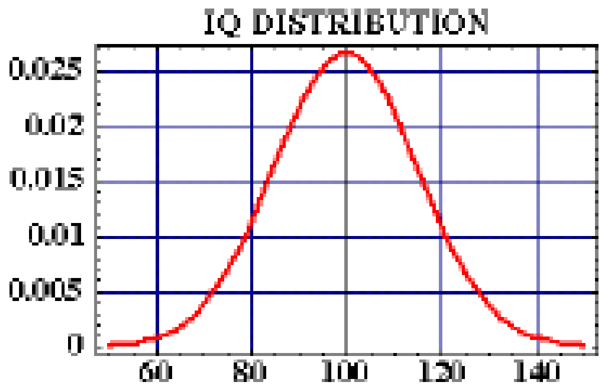
- $\frac{d^2}{dx^2} y(x) = +k^2 y(x)$ has for solutions $y(x) = A \exp(+kx)$ & $y(x) = B \exp(-kx)$

  The most general solution is a superposition: $y(x) = A \exp(+kx) + B \exp(-kx)$

  The plus sign in front of k^2 indicates the independent solutions are going to be either exponentially growing or exponentially decaying – the curvature is always positive, the functions are always either monotonically increasing or monotonically decreasing.

Gaussian Integrals

An apocryphal story is told of a math major showing a psychology major the formula for the infamous bell-shaped curve or gaussian, which purports to represent the distribution of intelligence and such:



The formula for a normalized gaussian looks like this:

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

The psychology student, unable to fathom the fact that this formula contained π, the ratio between the circumference and diameter of a circle, asked "Whatever does π have to do with intelligence?" The math student is supposed to have replied, "If your IQ were high enough, you would understand!" The following derivation shows where the π comes from.

Laplace (1778) proved that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \tag{1}$$

This remarkable result can be obtained as follows. Denoting the integral by I, we can write

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy \qquad (2)$$

where the dummy variable y has been substituted for x in the last integral. The product of two integrals can be expressed as a double integral:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

The differential $dx\, dy$ represents an elementof area in cartesian coordinates, with the domain of integration extending over the entire xy-plane. An alternative representation of the last integral can be expressed in plane polar coordinates r, θ. The two coordinate systems are related by

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad (3)$$

so that

$$r^2 = x^2 + y^2 \qquad (4)$$

The element of area in polar coordinates is given by $r\, dr\, d\theta$, so that the double integral becomes

$$I^2 = \int_0^{\infty} \int_0^{2\pi} e^{-r^2}\, r\, dr\, d\theta \qquad (5)$$

Integration over θ gives a factor 2π. The integral over r can be done after the substitution $u = r^2$, $du = 2r\, dr$:

$$\int_0^{\infty} e^{-r^2}\, r\, dr = \frac{1}{2} \int_0^{\infty} e^{-u}\, du = \frac{1}{2} \qquad (6)$$

Therefore $I^2 = 2\pi \times \frac{1}{2}$ and Laplace's result (1) is proven.

A slightly more general result is

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \left(\frac{\pi}{\alpha}\right)^{1/2} \tag{7}$$

obtained by scaling the variable x to $\sqrt{\alpha}x$.

We require definite integrals of the type

$$\int_{-\infty}^{\infty} x^n\, e^{-\alpha x^2}\, dx, \qquad n = 1, 2, 3 \ldots \tag{8}$$

for computations involving harmonic oscillator wavefunctions. For odd n, the integrals (8) are all zero since the contributions from $\{-\infty, 0\}$ exactly cancel those from $\{0, \infty\}$. The following stratagem produces successive integrals for even n. Differentiate each side of (7) wrt the parameter α and cancel minus signs to obtain

$$\int_{-\infty}^{\infty} x^2\, e^{-\alpha x^2}\, dx = \frac{\pi^{1/2}}{2\,\alpha^{3/2}} \tag{9}$$

Differentiation under an integral sign is valid provided that the integrand is a continuous function. Differentiating again, we obtain

$$\int_{-\infty}^{\infty} x^4\, e^{-\alpha x^2}\, dx = \frac{3\,\pi^{1/2}}{4\,\alpha^{5/2}} \tag{10}$$

The general result is

$$\int_{-\infty}^{\infty} x^n\, e^{-\alpha x^2}\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (n+1)\,\pi^{1/2}}{2^{n/2}\alpha^{(n+1)/2}}, \quad n = 0, 2, 4 \ldots \tag{11}$$

Delta functions

The Kronecker delta (named after Leopold Kronecker) is a function of two variables, usually just non-negative integers. The function is 1 if the variables are equal, and 0 otherwise:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Another useful representation is the following form:

$$\delta_{nm} = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i \frac{k}{N}(n-m)}$$

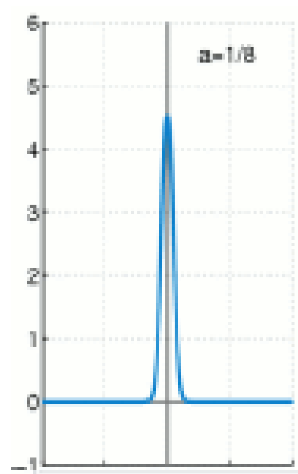
This can be derived using the formula for the finite geometric series.

Dirac delta function,

https://en.wikipedia.org/wiki/Dirac_delta_function

Dirac delta distribution (δ distribution), is a generalized function or distribution over the real numbers, whose value is zero everywhere except at zero, and whose integral over the entire real line is equal to one.

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$



The Dirac delta as the limit as $a \to 0$ (in the sense of distributions) of the sequence of zero-centered normal distributions $\delta_a(x) = \frac{1}{|a|\sqrt{\pi}} e^{-(x/a)^2}$

Given any continuous function f(x):

$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\mathrm{d}x = f(x_0)$$

The delta function satisfies the following scaling property for a non-zero scalar α:

$$\int_{-\infty}^{\infty} \delta(\alpha x)\,dx = \int_{-\infty}^{\infty} \delta(u)\,\frac{du}{|\alpha|} = \frac{1}{|\alpha|}$$

and so

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}.$$

Convolution Property

Convolution of a function f with a delta function at x_0 is equivalent to shifting f by x_0.

$$f(x) * \delta(x - x_0) = f(x - x_0)$$

Identity 1

A nascent delta function is the sinc function as the width of the sinc goes to zero:

$$\lim_{\varepsilon \to 0} \frac{\sin(x/\varepsilon)}{\pi x} = \lim_{a \to \infty} \frac{\sin ax}{\pi x} = \delta(x)$$

Identity 2

$$\int_{-\infty}^{\infty} \cos(2\pi v x)\mathrm{d}x = \delta(v)$$

Identity 3

The Fourier transform of one is the delta function:

Identity 4 – the Dirac Comb

The following identity is useful in the derivation of the diffraction pattern for a periodic pattern with pitch p.

$$p \sum_{n=-\infty}^{\infty} e^{-i2\pi n v p} = \sum_{m=-\infty}^{\infty} \delta\left(v - \frac{m}{p}\right)$$

The function on the right-hand side o is called a *Dirac comb* of period p.

Dirac Delta Function

The Dirac delta function (also called the *unit impulse function*) is a mathematical abstraction which is often used to describe (i.e. approximate) some physical phenomenon. The main reason it is used has to do with some very convenient mathematical properties which will be described below. In optics, an idealized point source of light can be described using the delta function. Of course, real points of light will have finite width, but if the point is narrow enough, approximating it with a delta function can be very useful.

C.1 Definition

The Dirac delta function is in fact not a function at all, but a distribution (a generalized function, such as a probability distribution) that is also a measure (i.e. it assigns a value to a function) – terms that come from probability and set theory. However, for our purposes it will suffice to consider it a special function with infinite height, zero width and an area of 1. It can be considered the derivative of the Heaviside step function.

To help think about the Dirac delta function, consider a rectangle with one side along the x-axis centered about $x = x_o$ such that the area of the rectangle is 1 (this is equivalent to a uniform probability distribution). Obviously there are many such rectangles, as shown in Figure C.1. We can construct a Dirac delta function by starting with a square of height and width of 1. If we halve the width and double the height, the area will remain constant. We can repeat this process as many times as we wish. As the width goes to zero, the height will become infinite but the area will remain 1. Any unit area rectangle, centered at x_o, can be expressed as

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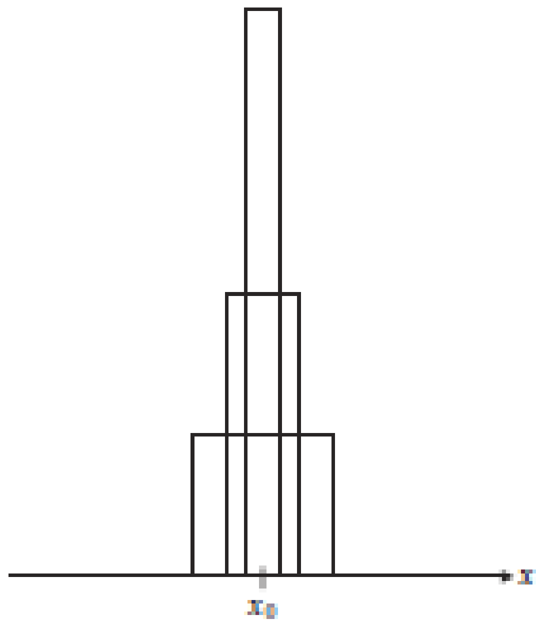

Figure C.1 *Geometrical construction of the Dirac delta function*

$$\delta_\varepsilon(x-x_o)=\begin{cases}0, & x<x_o-\dfrac{\varepsilon}{2}\\ \dfrac{1}{\varepsilon}, & x_o-\dfrac{\varepsilon}{2}<x<x_o+\dfrac{\varepsilon}{2}\\ 0, & x>x_o+\dfrac{\varepsilon}{2}\end{cases}=\frac{1}{\varepsilon}rect\left[\frac{x-x_o}{\varepsilon}\right] \tag{C.1}$$

where *rect* is the common rectangle function. The Dirac delta function, located at $x = x_o$, can be defined as the limiting case as ε goes to zero.

$$\delta(x-x_o)=\lim_{\varepsilon\to 0}\delta_\varepsilon(x-x_o) \tag{C.2}$$

Although a rectangle is used here, in general the Dirac delta function is any pulse in the limit of zero width and unit area. Thus, the Dirac delta function can be defined by two properties:

$$\delta(x)=0 \quad \text{when} \quad x\neq 0 \tag{C.3}$$

$$\int_{-\infty}^{\infty}\delta(x)\mathrm{d}x=1 \tag{C.4}$$

Any function which has these two properties is the Dirac delta function. A consequence of Equations (C.3) and (C.4) is that $\delta(0)=\infty$.

The function $\delta_\varepsilon(x)$ is called a 'nascent' delta function, becoming a true delta function in the limit as ε goes to zero. There are many nascent delta functions, for example, the

Gaussian pulse (a normal probability distribution, letting the standard deviation go to zero).

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} e^{-\pi x^2/\varepsilon^2} \tag{C.5}$$

Extending this form to two dimensions,

$$\delta(x,y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} e^{-\pi(x^2+y^2)/\varepsilon^2} = \delta(x)\delta(y) \tag{C.6}$$

Generalizations to more dimensions are straightforward. Other nascent delta functions include the Airy disk function, the sinc function (see section C.2.4), and the Bessel function of order $1/\varepsilon$. In general, any probability density function with a scale parameter ε is a nascent delta function as ε goes to zero.

C.2 Properties and Theorems

The following sections will state some important identities and properties of the Dirac delta function, providing proofs for some of them.

C.2.1 Sifting Property

For any function $f(x)$ continuous at x_o,

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_o)\mathrm{d}x = f(x_o) \tag{C.7}$$

It is the sifting property of the Dirac delta function that gives it the sense of a measure – it measures the value of $f(x)$ at the point x_o.

Proof

Since the delta function is zero everywhere except at $x = x_o$, the range of the integration can be changed to some infinitesimally small range ε around x_o.

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_o)\mathrm{d}x = \int_{x_o-\varepsilon}^{x_o+\varepsilon} f(x)\delta(x - x_o)\mathrm{d}x \tag{C.8}$$

Over this very small range of x, the function $f(x)$ can be thought to be constant and can be taken out of the integral.

$$\int_{x_o-\varepsilon}^{x_o+\varepsilon} f(x)\delta(x - x_o)\mathrm{d}x = f(x_o)\int_{x_o-\varepsilon}^{x_o+\varepsilon} \delta(x - x_o)\mathrm{d}x \tag{C.9}$$

From the definition of the Dirac delta function, the integral on the right-hand side will equal 1, thus proving the theorem. In fact, Equation (C.7) can be used as an alternate definition of the Dirac delta function. Any function $\delta(x-x_o)$ which satisfies the sifting property is the Dirac delta function.

C.2.2 Scaling Property

$$\delta(ax) = \frac{\delta(x)}{|a|} \tag{C.10}$$

C.2.3 Convolution Property

Convolution of a function f with a delta function at x_o is equivalent to shifting f by x_o.

$$f(x) * \delta(x - x_o) = f(x - x_o) \tag{C.11}$$

C.2.4 Identity 1

Another nascent delta function is the sinc function as the width of the sinc goes to zero:

$$\lim_{\varepsilon \to 0} \frac{\sin(x/\varepsilon)}{\pi x} = \lim_{a \to \infty} \frac{\sin ax}{\pi x} = \delta(x) \tag{C.12}$$

Proof

To prove identity 1, it is sufficient to show that this expression for the Dirac delta function satisfies sifting property:

$$\lim_{a \to \infty} \int_{-\infty}^{\infty} f(x) \frac{\sin ax}{\pi x} \mathrm{d}x = f(0) \tag{C.13}$$

Breaking the integral into three sections, the outer two of which avoid the problem of dividing by zero at $x = 0$,

$$\int_{-\infty}^{\infty} f(x) \frac{\sin ax}{\pi x} \mathrm{d}x = \int_{-\infty}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\infty} \tag{C.14}$$

The first and last integral on the right-hand side are zero by the Riemann–Lebesgue lemma (an important theorem of the Fourier integral that will not be discussed here). The center integral can be evaluated by taking ε to be very small (but not zero). Over this very small range, $f(x)$ will be about constant:

$$\int_{-\varepsilon}^{\varepsilon} f(x) \frac{\sin ax}{\pi x} \mathrm{d}x = f(0) \int_{-\varepsilon}^{\varepsilon} \frac{\sin ax}{\pi x} \mathrm{d}x \tag{C.15}$$

Taking the limit as a goes to infinity,

$$\lim_{a \to \infty} \int_{-\varepsilon}^{\varepsilon} \frac{\sin ax}{\pi x} \mathrm{d}x = \lim_{a \to \infty} \int_{-a\varepsilon}^{a\varepsilon} \frac{\sin x'}{\pi x'} \mathrm{d}x' = \int_{-\infty}^{\infty} \frac{\sin x'}{\pi x'} \mathrm{d}x' = 1 \tag{C.16}$$

Thus,

$$\lim_{a\to\infty}\int_{-\infty}^{\infty} f(x)\frac{\sin ax}{\pi x}\mathrm{d}x = f(0) \tag{C.17}$$

C.2.5 Identity 2

$$\int_{-\infty}^{\infty}\cos(2\pi\nu x)\mathrm{d}x = \delta(\nu) \tag{C.18}$$

Proof

The proof simply performs the integration and then applies identity 1.

$$\int_{-\infty}^{\infty}\cos(2\pi\nu x)\mathrm{d}x = \lim_{a\to\infty}\int_{-a}^{a}\cos(2\pi\nu x)\mathrm{d}x = \lim_{a\to\infty}\frac{\sin(2\pi\nu a)}{\pi\nu} = \delta(\boldsymbol{\nu}) \tag{C.19}$$

C.2.6 Identity 3 – $\mathcal{F}(1)$

The Fourier transform of one is the delta function:

$$\int_{-\infty}^{\infty} e^{-i2\pi\nu x}\mathrm{d}x = \delta(\nu) \tag{C.20}$$

Proof

Changing the exponential into a sine and cosine,

$$\int_{-\infty}^{\infty} e^{-i2\pi\nu x}\mathrm{d}x = \int_{-\infty}^{\infty}\cos(2\pi\nu x)\mathrm{d}x - i\int_{-\infty}^{\infty}\sin(2\pi\nu x)\mathrm{d}x \tag{C.21}$$

Since the sine is an odd function, the sine integral will vanish. Applying identity 2 to the cosine integral completes the proof.

C.2.7 Identity 4 – the Dirac Comb

The following identity is useful in the derivation of the diffraction pattern for a periodic line/space mask pattern with pitch p.

$$p\sum_{n=-\infty}^{\infty} e^{-i2\pi n p\nu} = \sum_{m=-\infty}^{\infty}\delta\left(\nu - \frac{m}{p}\right) \tag{C.22}$$

The function on the right-hand side of Equation (C.22) is called a *Dirac comb* of period p. This identity can be proved by recognizing that the Dirac comb is a periodic function that can be easily represented by a Fourier series. Direct calculation of the Fourier coefficients of the complex Fourier series produces Equation (C.22).

C.2.8 Relationship to the Heaviside Step Function

The Heaviside step function is defined as

$$u(x)=\begin{cases}0, & x<0\\ 1 & x\geq 1\end{cases} \tag{C.23}$$

The step function is related to the Dirac delta function by

$$\delta(x)=\frac{\mathrm{d}}{\mathrm{d}x}u(x) \quad \text{and} \quad u(x)=\int_{-\infty}^{x}\delta(t)\mathrm{d}t \tag{C.24}$$