Chapter 1

Phil Anderson's Magnetic Ideas in Science

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In Philip W. Anderson's research, magnetism has always played a special role, providing a prism through which other more complex forms of collective behavior and broken symmetry could be examined. I discuss his work on magnetism from the 1950s, where his early work on antiferromagnetism led to the pseudospin treatment of superconductivity - to the 70s and 80s, highlighting his contribution to the physics of local magnetic moments. Phil's interest in the mechanism of moment formation, and screening evolved into the modern theory of the Kondo effect and heavy fermions.

1. Introduction

This article is based on a talk I gave about Phil Anderson's contributions to our understanding of magnetism and its links with superconductivity, at the 110th Rutgers Statistical mechanics meeting. This event, organized by Joel Lebowitz, was a continuation of the New Jersey celebrations began at "PWA90: A lifetime in emergence", on the weekend of Phil Anderson's 90th birthday in December 2014. My title has a double-entendre, for Phil's ideas in science have a magnetic quality, and have long provided inspiration, attracting students such as myself, to work with him. I first learned about Phil Anderson as an undergraduate at Cambridge in 1979, some three years after he had left for Princeton. Phil had left behind many legends at Cambridge, one of which was that he had ideas of depth and great beauty, but also that he was very hard to understand. For me, as with many fellow students of Phil, the thought of working with an advisor with some of the best ideas on the block was very attractive, and it was *this* magnetism that brought me over to New York Harbor nine months later, to start a Ph. D. with Phil at Princeton.

One of the recurrent themes of Anderson's work, is the importance of using models as a gateway to discovering general mechanisms and principles, and throughout his career, models of magnetism played a key role. In his book "Basic concepts

of condensed matter physics"[1] , Anderson gives various examples of such basic principles, such as adiabatic continuation, the idea of renormalization as a way to eliminate all but the essential degrees of freedom, and most famously, the link between broken symmetry and the idea of *generalized rigidity*, writing

> *"We are so accustomed to the rigidity of solid bodies that is hard to realize that such action at a distance is not built into the laws of nature It is strictly a consequence of the fact that the energy is minimized when symmetry is broken in the same way throughout the sample*
>
> *The generalization of this concept to all instances of broken symmetry is what I call generalized rigidity. It is responsible for most of the unique properties of the ordered (broken-symmetry) states: ferromagnetism, superconductivity, superfluidity, etc."*

Yet in the 50s, when Phil began working on magnetism, these ideas had not yet been formed: the term *broken symmetry* was not yet in common usage, renormalization was little more than a method of eliminating divergences in particle physics and beyond the Ising and Heisenberg models, there were almost no other simple models for interacting electrons. Phil's studies of models of magnetism spanning the next three decades played a central role in the development of his thoughts on general principles and mechanisms in condensed matter physics, especially those underlying broken symmetry.

I'll discuss three main periods in Phil's work as shown in the time-line of Fig. 1, and arbitrarily color coded as the "blue", "orange" and "green" period. My short presentation is unfortunately highly selective but I hope it will give a useful flavor to the reader of the evolution of ideas that have accompanied Phil's work in magnetism.
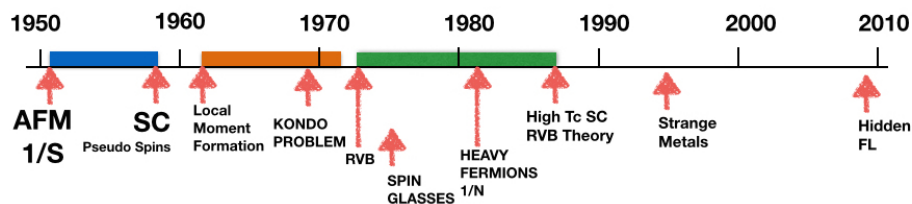

Fig. 1.   Three periods of Anderson's research into magnetism selectively discussed in this article. Blue period: from antiferromagnetism to superconductivity. Orange period: theory of local moment formation and the Kondo problem. Green period: from resonating valence bonds (RVB) to high temperature superconductivity.

2. Blue Period: Antiferromagnetism and Superconductivity

Today it is hard to imagine the uncertainties connected with antiferromagnetism and broken symmetry around 1950. While Néel and Landau[2,3] had independently predicted "antiferromagnetism", with a staggered magnetization (↑↓↑↓↑↓), as the classical ground-state of the Heisenberg model with positive exchange interaction,

$$H = J \sum_{(i,j)} \vec{S}_i \cdot \vec{S}_j, \qquad (J > 0), \qquad (1)$$

the effects of quantum fluctuations were poorly understood. Most notably, the one-dimensional $S = 1/2$ model had been solved exactly by Bethe in 1931,[4] and in his *Bethe Ansatz* solution, it was clear there was no long range order, indicating that at least in one dimension, quantum fluctuations overcome the long-range order. This issue worried Landau so much, that by the 1940's he had abandoned the idea of antiferromagnetism in quantum spin systems.[5] Phil Anderson reflects on this uncertainty in his 1952 article "An Approximate Quantum Theory of the Antiferromagnetic Ground State",[6] writing

> *"For this reason the very basis for the recent theoretical work which has treated antiferromagnetism similarly to ferromagnetism remains in question. In particular, since the Bethe-Hulthén ground-state is not ordered, it has not been certain whether an ordered state was possible on the basis of simple $\vec{S}_i \cdot \vec{S}_j$ interactions"*

The situation began to change in 1949, with Shull and Smart's[7] detection of antiferromagnetic order in MnO by neutron diffraction, which encouraged Anderson to turn to the unsolved problem of zero-point motion in antiferromagnets. Early work on spin-wave theory by Heller, Kramers and Hulthén had treated spin waves as classical excitations, but later work by Klein and Smith[8] had noted that quantum zero point motions in a spin S ferromagnet correct the ground-state energy by an amount of order $1/S$, a quantity that becomes increasingly small as the size of the spin increases. It is this effect that increases the ground-state energy of a ferromagnet from its classical value $E \propto -J\langle \vec{S}^2 \rangle = -JS(S+1)$ to its exact quantum value $E \propto -JS^2$.

A key result of Anderson's work is an explanation for the survival of antiferromagnetic order in two and higher dimensions, despite its absence in the Bethe chain. His expression for the reduced sublattice magnetization (Fig. 2a) of a bipartite antiferromagnet, is

$$\langle S_z \rangle = \sqrt{S(S+1) - \delta S_\perp^2}$$
$$= S \left[1 - \frac{1}{2S} \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{\sqrt{1 - \gamma_{\mathbf{q}}^2}} - 1 \right) + O(\frac{1}{S^2}) \right] \qquad (2)$$

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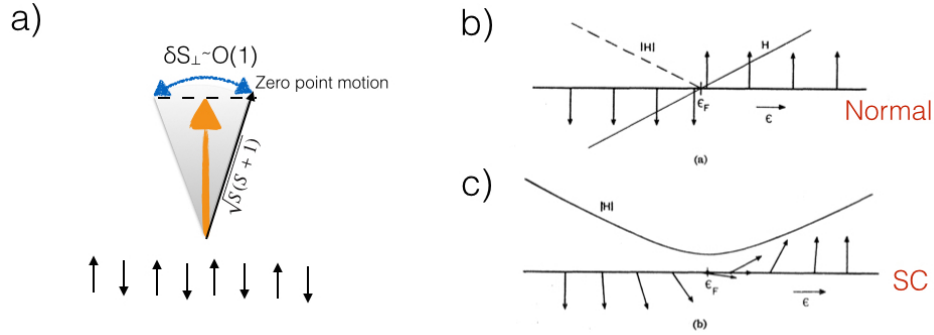

Fig. 2.   (a) Reduction of staggered moment from length $\sqrt{S(S+1)}$ to semi-classical value $\sqrt{S(S+1)} - O(1)$. (b) In Anderson's domain wall interpretation of superconductivity the normal Fermi liquid is a sharp domain wall where the Weiss field H vanishes at the Fermi surface; (c) in the superconductor the pseudospins rotate smoothly and Weiss field never vanishes, giving rise to a finite gap.

where $\gamma_{\mathbf{q}} = \frac{1}{d} \sum_{l=1,d} \cos q_l$. A similar result was independently discovered by Ryogo Kubo.[9]   In an antiferromagnet, the staggered magnetization does not commute with the Hamiltonian and thus undergoes continuous zero point fluctuations that reduce its magnitude (Fig. 2a). Since $\sqrt{1 - \gamma_{\mathbf{k}}^2} \sim |\mathbf{q}|$ at small wavevector $\mathbf{q}$, these fluctuations become particularly intense at long wavelengths, with a reduction in magnetization

$$\Delta M \sim \int \frac{d^d q}{q} \sim \begin{cases} \infty & (d=1) \\ \text{finite} & (d \geq 2) \end{cases} \tag{3}$$

In this way, Anderson's model calculation could account for the absence of long range order in the Bethe chain as a result of long-wavelength quantum fluctuations and the stability of antiferromagnetism in higher dimensions.

At several points in his paper, Phil muses on the paradox that the ground-state of an antiferromagnet is a singlet, with no preferred direction, a thought he would return to in his later work on resonating valence bonds with Patrick Fazekas.[10] For the moment however, Phil resolves the paradox by estimating that the time for an antiferromagnet to invert its spins by tunnelling is macroscopically long, so that the sublattice magnetization becomes an observable classical quantity. Phil's semiclassical treatment of the antiferromagnet would later set the stage for Duncan Haldane[11] to carry out a semi-classical treatment of the one dimensional Heisenberg model, revealing an unexpected topological term. But in the near future, Anderson's study of antiferromagnetism had influence in a wholly unexpected direction: superconductivity.

One of the issues that was poorly understood following the Bardeen Cooper Schrieffer (BCS) theory of superconductivity, was the question of charge fluctuations. In an insulator the charge gap leads to a dielectric with no screening. How-

ever, were the superconducting gap to have the same effect, it would eliminate the weak electron phonon attraction. It was thus essential to show that both screening and the longitudinal plasma mode survive the formation of the BCS gap. In his 1958 paper "Random-Phase Approximation in the Theory of Superconductivity",[12] Phil writes

> *"Both for this reason, and because it seems optimistic to assume that the collective [charge fluctuation] and screening effects (which are vital even in determining the phonon spectrum) will be necessarily unaffected by the radical changes in the Fermi sea ..., it is desirable to have a theory of the ground-state of a superconductor which can simultaneously handle these collective effects in the best available approximation ..."*

Phil's experience with antiferromagnetism enabled him to make a new link between magnetism and superconductivity. He observed that if one considered a pair to be a kind of "down-spin" and the absence of a pair to be a kind of "up spin" in particle hole space,

$$
\begin{aligned}
\text{no pair:} && | \Uparrow \rangle &\equiv |0\rangle, \\
\text{pair:} && | \Downarrow \rangle &\equiv c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |0\rangle
\end{aligned}
\tag{4}
$$

then the BCS ground-state is revealed as a kind of Bloch domain wall (Fig. 2b,c) formed around the Fermi surface.[12] This new interpretation forges a link between between superconductivity and antiferromagnetism, enabling the pairing field to be identified behaves as a transverse Weiss field in particle-hole space. Moreover the analogy works at a deeper level, because like in quantum antiferromagnetism, the superconducting order parameter is non-conserved, allowing it to fluctuate and importantly, to deform in response to an electric field, preserving the screening.

Let us look at this in a little more detail. BCS theory involves three key operators, the number operator $(n_{k\uparrow} + n_{-k\downarrow})$, the pair creation and pair annihilation operators, $b^\dagger_k = c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}$, $b_k = c_{-k\downarrow} c_{k\uparrow}$. The key observation was to identify these operators as the components of a pseudo-spin. In the subspace where $n_{k\uparrow} + n_{-k\downarrow}$ is either 0or 2, Anderson's defined the pseudospin as

$$
2s_z = 1 - n_k - n_{-k} = \begin{array}{c} \\ \text{empty} \\ \text{full} \end{array} \overset{\text{empty full}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}},
\tag{5}
$$

so that a fully occupied k state is a "down" pseudo-spin, and an empty k-state is an "up" spin. Similarly, the the raising and lowering operators are respectively, the pair destruction and creation operators

$$
b_k \equiv s_{xk} + i s_{yk} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad b^\dagger_k \equiv s_{xk} - i s_{yk} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\tag{6}
$$

In this language, the BCS reduced Hamiltonian

$$\mathcal{H}_{\text{RED}} = -2\sum_k \epsilon_k s_{zk} - \sum_{k,k'} V_{k,k'}\vec{s}_{\perp k} \cdot \vec{s}_{\perp k'} \tag{7}$$

is a kind of magnet that resides in momentum space. Anderson showed that in this language, the metal is a sharp domain wall along the Fermi surface (Fig. 2b) while the superconductor has a soft "Bloch domain wall" (Fig 2c) in which the pseudo-spins rotate continuously from down (full) to sideways (linear combination of full and empty) to up (empty). By calculating the spin-wave fluctuations Anderson was able to show that with the Coulomb interaction included, the longitudinal electromagnetism of the metal, its screening and plasma modes, are unaffected by the superconducting gap.

Anderson's pseudo-spin reformulation of BCS had a wide influence. Two years later, Nambu extended the pseudo-spin approach to reformulate Gor'kov's Green function approach using his now famous "Nambu matrices".[13] Perhaps most important of all, by making the analogy between superconductivity and magnetism, the community took a cautious step closer to regarding the superconducting phase as a palpable, detectable variable (with the caveats of gauge fixing). This new perspective, especially the link between phase, supercurrents and gauge invariance, would soon culminate in Anderson's ideas on how gauge particles acquire mass - the *Anderson Higgs* mechanism (see Witten's article in this volume).

3. Orange Period: Moment Formation and the Kondo Problem

3.1. *Superexchange*

Towards the end of the 1950's, Anderson began to turn his attention towards the microscopic origins of antiferromagnetism. In his 1959 paper "New Approach to the Theory of Superexchange Interactions"[14] Phil argues that origin of antiferromagnetism is *the Mott mechanism*, i.e the Coulomb cost of doubly occupied orbitals. Phil writes

> "In such a simple model all the degenerate states in the ground-state manifold have exactly one electron per ion, while all the excited states with one transferred electron have energy U. Between an pair of ions at a distance $\mathbf{R} - \mathbf{R}'$ there is only one (hopping term) $b_{\mathbf{R}-\mathbf{R}'}$; this must act to return the state to one of the ground manifold... so that

$$\Delta E = constant + \sum_{\mathbf{R},\mathbf{R}'} \frac{2|b_{\mathbf{R}-\mathbf{R}'}|^2}{U}\mathbf{S}_{\mathbf{R}} \cdot \mathbf{S}'_{\mathbf{R}'}, \tag{8}$$

> This is the antiferromagnetic exchange effect."

This paper contains the origins of our modern understanding of Mott insulators, including an early formulation of the Hubbard model with Anderson's hallmark use of U to denote the onsite Coulomb repulsion.

3.2. *Anderson's model for local moment formation*

While the notion of local moments is rooted in early quantum mechanics, the mechanism of moment formation was still unknown in the 1950s. At this time, experiments at Orsay at Bell Labs started to provide valuable new insights. In Orsay, Jacques Friedel and André Blandin proposed that virtual bound-states develop around localized d-states in a metal, arguing that ferro-magnetic exchange forces then split these resonances to form local moments. Recalling the first time he encountered this idea, Phil writes:[15,16]

> In the Fall of '59, a delightful little discussion meeting on magnetism in metals was held in Brasenose College, Oxford. ... Blandin presented the idea of virtual states and I the conceptual basis for antiferromagnetic s-d exchange, without any understanding, at least on my part, that the two ideas belonged together. The only immediate positive scientific result of the meeting was that I won a wager on the sign of the Fe hyperfine field on the basis of these ideas."

Around this time, Bernd Matthias's group at at Bell Labs discovered that the development of a localized moment on iron atoms depends on the metallic environment - for example, iron impurities dissolved in niobium do not develop a local moment, yet they do so in the niobium-molybdenum alloy, $Nb_{1-x}Mo_x$ once the concentration of molybdenum exceeds 40% ($x > 0.4$). Anderson was intrigued by this result and realized that while it was probably connected to the virtual bound-state ideas of Friedel and Blandin, ferromagnetic exchange was two weak to drive moment formation. Once again, he turned to the Mott mechanism as a driver and the key element of his theory "Localized Magnetic States in Metals", is the repulsion between *anti-parallel* electrons in the same orbital, given by the Coulomb repulsion integral,

$$U = \int |\phi_{loc}(1)|^2 e^2 r_{12}^{-1} |\phi_{loc}(2)|^2 d\tau. \tag{9}$$

Phil emphasizes this point, writing

> "the formal theory is much more straightforward if one includes U in the manner in which we do it, as a repulsion of opposite-spin electrons in ϕ_{loc}, not as an attraction of parallel ones"

Another new element of Anderson's theory of moment formation, not contained in earlier theories, was the explicit formulation of his model as a quantum field theory.

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The heart of the Anderson model is a hybridization term

$$H_{sd} = \sum_{\mathbf{k},\sigma} V_{d\mathbf{k}}(c^{\dagger}_{\mathbf{k}\sigma}c_{d\sigma} + c^{\dagger}_{d\sigma}c_{\mathbf{k}\sigma}), \tag{10}$$

which mixes s and d electrons, generting a virtual bound-state of width $\Delta = \pi\langle V^2\rangle\rho$, where $\rho(\epsilon)$ is the conduction electron density of states and $\langle V^2\rangle$ the Fermi surface average of the hybridization, and the onsite Coulomb interaction,

$$H_{corr} = Un_{d\uparrow}n_{d\downarrow}, \tag{11}$$

where U is as given in (9). With these two terms, Phil unified the Freidel-Blandin virtual bound-state resonance with the "Mott mechanism" he had already introduced for insulating antiferromagnets. Using a mean-field Hartree-Fock treatment of his model, Anderson shows that if

$$U > \pi\Delta \tag{12}$$

the virtual bound state resonance splits into two. One of the aspects of the paper that may have been confusing at the time, was that taken literally, this suggested a real phase transition into a local moment state. Anderson clearly did not see it this way,

> "It is the great conceptual simplification of the impurity problem that is is possible to separate the question of the existence of the "magnetic state" entirely from the actually irrelevant question of whether the final state is ferromagnetic, antiferromagnetic or paramagnetic."

Today we understand that Anderson's mean-field description of magnetic moments captures the physics at intermediate time scales, describing a cross-over in the renormalization trajectory as it makes a close fly-by past the repulsive local moment captured in Phil's mean field theory.

3.3. *The Kondo Model*

A central prediction of Phil's 1961 paper was that the residual interaction between the local moment, and the surrounding electrons, the *s-d* interaction is *antiferromagnetic*. By freezing the local moment, Phil was able to calculate the small shifts in the conduction electron energies, demonstrating they were indeed antiferromagnetic. He writes

> " Thus any g shifts caused by free electron polarization will tend to have antiferromagnetic sign."

Phil had of course guessed this in his 1959 bet at Brasenose College Oxford!

The conventional wisdom of the time expected a ferromagnetic s-d exchange. Indeed, the *s-d model* describing the interaction of local moments with conduction

electrons had been formulated by Clarence Zener[17] and written down in second-quantized notation in the 1950s, by Tadao Kasuya,[18] but with a *ferromagnetic* interaction, derived from exchange. [a]

On the other side of the world in Tokyo, one person, Jun Kondo realized that Anderson's prediction of an antiferromagnetic s-d coupling would have experimental consequences, and his efforts to reveal them led to the solution of a 30 year old mystery. Following Anderson's prediction, Kondo now wrote down a simple model for the *antiferromagnetic* interaction,

$$H_K = \sum \epsilon_k c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + J \vec{S}_d \cdot \vec{\sigma}(0), \tag{14}$$

where $\vec{\sigma}(0)$ is the electron spin density at the site of the magnetic moment, and he set out to examine the consequences of the antiferromagnetic exchange. This led Kondo to calculate the magnetic scattering rate $\frac{1}{\tau}$ of electrons to cubic order in the s-d interaction. To his surprise, the cubic term contained a logarithmic temperature dependence:[20]

$$\frac{1}{\tau} \propto \left[J\rho + 2(J\rho)^2 \ln \frac{D}{T} \right]^2 \tag{15}$$

where ρ is the density of states of electrons in the conduction sea and D is the half-width of the electron band. Kondo noted that if the s-d interaction were positive and antiferromagnetic, then as the temperature is lowered, the coupling constant, the scattering rate and resistivity start to rise. This meant that once the magnetic scattering overcame the phonon scattering, the resistance would develop a resistance minimum. Such resistance minima had been seen in metals for more than 30 years.[21,22] Through Kondo and Anderson's work, this thirty year old mystery could be directly interpreted as a a direct consequence of the predicted antiferromagnetic s-d interaction with local magnetic moments.

3.4. *Kondo's result poses a problem*

After Kondo and Anderson's work, the community quickly realized that the "Kondo effect" raised a major difficulty. You can see from (15) that at the "Kondo temperature" $T \sim T_K$ where $2J\rho \ln(D/T_K) \sim 1$, or

$$T_K \sim D e^{-1/(2J\rho)} \tag{16}$$

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[a]Curiously, in his 1961 paper, Phil does not mention super-exchange as the origin of the antiferromagnetic s-d interaction, despite his development of this idea in his 1959 paper on insulating antiferromagnets. Perhaps it was felt that metals are different. It was not until the work of Schrieffer and Wolff[19] that the Kondo interaction was definitively identified, using a careful canonical transformation, as a form of super-exchange interaction, of magnitude

$$J \sim \frac{4\langle V^2_{d\mathbf{k}} \rangle}{U}. \tag{13}$$

the Kondo log becomes comparable with the bare interaction, so that at lower temperatures perturbation theory fails. What happens at lower temperatures once perturbation theory fails? This is the "Kondo problem".

By the late 1960's, from the work of early pioneers on the Kondo problem, including Alexei Abrikosov, Yosuke Nagaoka, Harry Suhl, Bob Schrieffer and Kei Yosida, much had been learned about the Kondo problem. It had become reasonably clear that at low temperatures the Kondo coupling constant grew to strong coupling, to form a spin singlet, but the community was divided over whether the residual scattering would be singular, or whether it would be analytic, forming an "Abrikosov Suhl" resonance. The problem also lacked lacked a conceptual framework and there were no controlled approximations.

3.5. *How a Catastrophe led to new insight*

The solution to Kondo problem required a new concept - the renormalization group. Today we know the Kondo effect as an example of asymptotic freedom - a running coupling constant that flows from weak coupling at high energies, to strong coupling at low energies, ultimately binding the local moment into a singlet with electrons in the conduction sea. In the late 60's, renormalization had entered condensed matter physics as a new tool for statistical physics. Phil and his collaborators now brought the renormalization group to quantum condensed matter by mapping the Kondo problem onto a one-dimensional Ising model with long range interactions.

Phil entered the field from an unexpected direction after discovering an effect known as the *orthogonality* or *X-ray* catastrophe. Phil's 1967 paper "Infrared Catastrophe in Fermi Gases with Local Scattering Potentials",[23] was stimulated by a conversation with John Hopfield, who speculated that the introduction of an impurity potential into a Fermi gas produces a new ground-state $|\phi^*\rangle$ that is orthogonal to the original ground-state $\phi_0\rangle$. Phil examined this idea in detail, and showed that when a local scattering potential suddenly changes, in the thermodynamic limit, the overlap between the original and the new Fermi gas ground-states identically vanishes $\langle\phi_0|\phi^*\rangle = 0$. For example, when an X-ray ionizes an atom in a metal, the ionic potential suddenly changes and this causes the conduction sea to evolve from its original ground-state $|\phi_0\rangle$ into a final-state $|\phi_f(t)\rangle = e^{-iHt}|\phi_0\rangle$.[24] In fact, Phil showed that the resulting relaxation is *critical*, with power-law decay in the overlap amplitude

$$G(t) = \langle\phi_0|\phi_f(t)\rangle \sim \frac{e^{-i\Delta E_g t}}{t^\epsilon}, \tag{17}$$

where ΔE_g is difference between final and initial ground-state energies. The absence of a characteristic time-scale indicates that the relaxation into the final-state ground-state is infinitely slow. By Fourier transform, this implies a singular density of states[24]

$$\rho(E) \sim \int dt\, G(t) e^{iEt} \sim (E - E_g)^{-1+\epsilon}. \tag{18}$$

This topic was also studied by Mahan[24] who linked the subject with X-ray line-shapes. Nozières and de Dominicis[25] later found an exact solution to the integral equations of the orthogonality catastrophy. The X-ray catastrophe is also responsible for the singular Green's functions of electrons in a one-dimensional Luttinger Liquid.[26–28]

One of the key conclusions of this work was that the orthogonality catastrophe *occurs* in the Kondo problem. Phil recognized that each time a local moment flips, the Weiss field it exerts on conduction electrons reverses, driving an orthogonality catastrophe in the "up" and "down" electron fluids. In its anisotropic form, the Kondo interaction takes the form

$$H_K = J_z S_{dz}\sigma_z(0) + J_\perp \left[S_+\sigma_-(0) + S_-\sigma_+(0) \right] \tag{19}$$

where $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ and $S_\pm = S_{dx} \pm iS_{dy}$ are the local lowering and raising operators for the mobile conduction and localized d-electrons respectively. From the work of Nozières and de Dominicis, the amplitude for two spin flips at times t_1 and t_2 is

$$(J_\perp)^2 \left(\frac{\tau_0}{t_2 - t_1} \right)^{2-2\epsilon} = (J_\perp)^2 \exp\left[-(2-\epsilon)\ln\left(\frac{t_2 - t_1}{\tau_0} \right) \right]. \tag{20}$$

where τ_0 is the short-time (ultra-violet) cut-off and $\epsilon \sim 2J_z\rho$ is determined by the change in the scattering phase shift of the up and down Fermi gases, each time the local spin reversed. This suggested that the quantum spin flips in a Kondo problem could be mapped onto the statistical mechanics of a 1D Coulomb gas of "kinks" with a logarithmic interaction.

3.6. *The Anderson-Yuval solution to the Kondo problem*

Working with graduate student Gideon Yuval[29,30] and a little later, Bell Labs colleague Don Hamann, Phil's team took up the task of organizing and summing the X-ray divergences of multiple spin-flip processes as a continuous time path-integral. With some considerable creativity, it became possible to map the quantum partition function of the Kondo model onto the *classical* partition function of a Coulomb gas of kinks. By regarding the kinks as domain walls in a one dimensional Ising model, they could further map the problem onto a one-dimensional Ising Ferromagnet spin chain with a $1/r^2$ interaction,

$$\frac{H}{T} = -(2-\epsilon)\sum_{i>j} \frac{S_i^z S_j^z}{|i-j|^2} - \mu \sum_i \sum_i S_i^z S_{i+1}^z, \tag{21}$$

where the Ising spins can have values $S_j = \pm 1/2$ at each site; the position j along the chain is really the imaginary time $\tau = j\tau_0$ measured in units of the short-time cut-off, with periodic boundary conditions and a total length determined by the inverse temperature, $L = \frac{\hbar\tau_0}{k_B T}$. The tuning parameter $\epsilon = J_z\rho$ is determined by the Ising part of the exchange interaction, while the transverse interaction $J_\perp$ sets

$\mu = -2 \ln J_\perp \rho$, the chemical potential of domain-wall kinks in the ferromagnetic spin chain. The larger $J_\perp \rho$, the more kinks are favored.

Suddenly a complex quantum problem became a tractable statistical mechanics model. It meant one could adapt the renormalization group from statistical physics to examine how the effective parameters of the Kondo parameter changed at lower and lower temperatures. By integrating out the effects of two closely separated pairs of spin flips, Anderson, Yuval and Hamann[31] derived the scaling equations

$$\frac{\partial J_z}{\partial \ln \tau_0} = J_\perp^2,$$
$$\frac{\partial J_\perp}{\partial \ln \tau_0} = J_\perp J_z. \tag{22}$$

Under these scaling laws, $J_\perp^2 - J_z^2$ is conserved, giving rise to the famous scaling trajectories shown in Fig. 3. There are two fixed points:

(a) $\epsilon \sim J_z \rho < 0$ Ferromagnetic ground state $\equiv$ unscreened local moment.
(b) $\epsilon \sim J_z \rho > 0$ "Kink liquid" $\equiv$ screened local moment, where the Kondo temperature sets the typical kink separation $l_0 \sim T_K$.
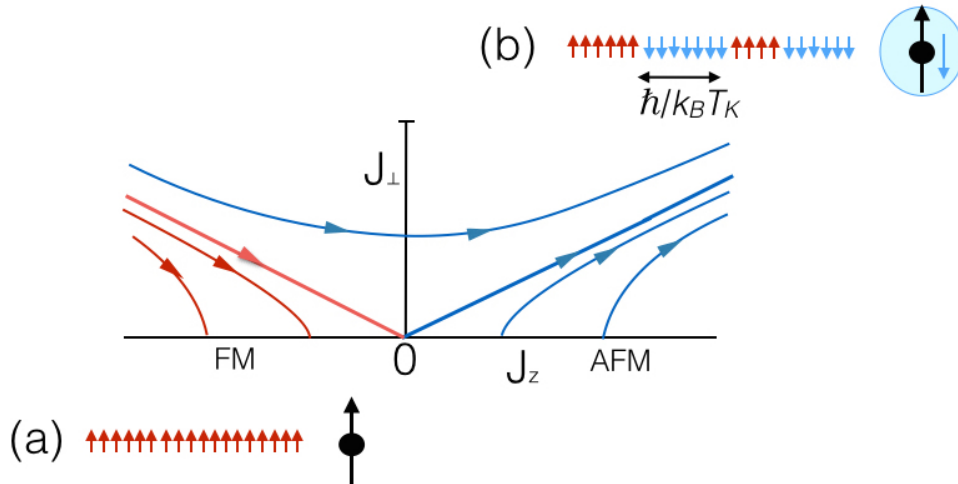

Fig. 3.   The Anderson-Yuval-Hamann scaling curves[31] for the Kondo model. (a) Red trajectories correspond to an unscreened moment, or a kink-free Ferromagnetic ground-state in the magnetic analogy. (b) Blue trajectories correspond to a fully screened local moment, or a "kink condensate" where the average kink separation is determined by the inverse Kondo temperature.

In this way, the Coulomb gas of kinks had a phase transition at $J_z = 0$. For negative $J_z < 0$, the kinks are absent, but for positive J_z, the chemical potential of the kinks grows so that they proliferate, forming a kink-liquid. Although Anderson,

Yuval and Hamann were unable to completely solve the strong coupling problem, the problem was solvable for so called *Toulouse limit*, where $\epsilon = 2J_z\rho = 1$, and in this limit, it could be shown that the strong coupling limit was free of any singular scattering. In their paper,[31] the authors conclude that

> " *The most interesting question on the Kondo effect has been from the start whether it did or did not fit into the structure of usual Fermi gas theory: In particular, does a true infrared singularity occur as in the x-ray problem, or does the Kondo impurity obey phase-space arguments as $T \to 0$ and give no energy dependences more singular than E^2 (or T^2), and is [the susceptibility] $\chi(T=0)$ finite? The result we find is that the usual antiferromagnetic case in fact <u>does</u> fit after the time scale has been revised to τ_κ, i.e. that it behaves like a true bound-singlet as was conjectured originally by Nagaoka.* "

i.e the authors conclude that ground-state of the Kondo problem is a Fermi liquid.

During this period, Phil wrote series of informal papers in Comments in Solid State Physics[15,32–34] that provided a very personalized update on the progress. The last of these papers, "Kondo effect IV: out of the wilderness",[34] summarizes what become the *status quo* in this problem. Phil writes

> "*In conclusion then, the status is this: we understand very clearly the physical nature of the Kondo problem, which is beautufully expressed in Fowler's picture of scaling: electrons of high enough energy interact with the weak, bare interaction and the bare Kondo spin, but as we lower the energy the effects of the other electrons gradually strengthen the effective interaction until finally, at energies near T_K, the effective interaction starts to get so large that we must allow the local spin to bind a compensating spin to itself, and the Kondo spin effectively disappears, being replaced by a large resonant nonmagnetic scattering effect. My own opinion is that the low temperature behavior is totally non-singular, the Kondo impurity looking simply like a localized spin fluctuation site, but others believe that there may remain a trace of singular behavior.*"

The influence of these ideas ran far and wide:

(1) It introduced scaling theory to quantum systems. The project started by Anderson and Yuval later culminated in Wilson's numerical renormalization work.[35]

(2) The key conclusion about the non-singular character of the ground-state, confirmed confirmed by Wilson, later became the basis for Nozières' strong coupling treatment of the key low temperature properties of the Kondo
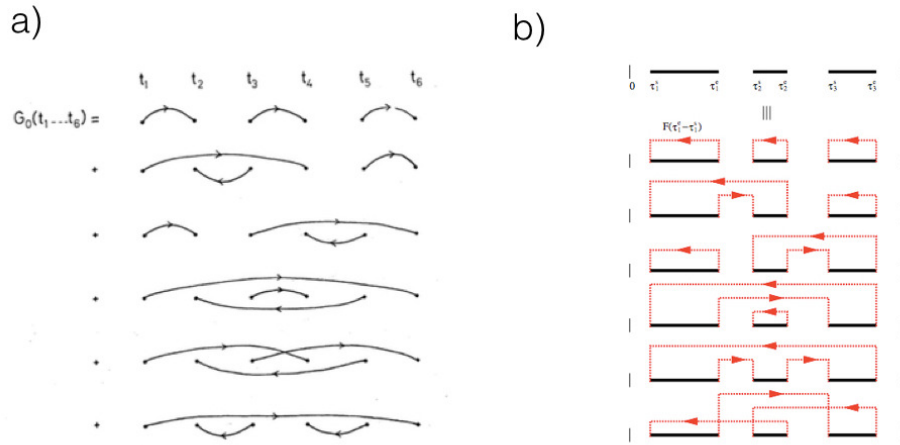
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a)   b)

Fig. 4.   Figure contrasting the use of the Yuval-Anderson approach to impurity models[29] with the modern continuous time Monte Carlo methods:[36] (a) illustrative sixth order diagram from the original Anderson-Yuval paper,[29] (b) the same set of diagrams used in.[36] The black arrowed lines in (a) and the red-arrowed lines in (b) describe conduction electrons propagating between spin flips.

problem,[37] which by mapping the physics onto a local Fermi liquid, accounted for the two-fold enhancement of the Sommerfeld-Wilson ratio observed in Wilson's numerical work.[35]

(3) The statistical mechanics of the problem, with scale-dependent interactions between topological defects, provided inspiration and ground-work for Kosterlitz and Thouless's scaling solution[38] to the Berezinski-Kosterlitz-Thouless transition in the 2D xy antiferromagnet, whose scaling flows replicate the Anderson-Yuval-Hamann diagram.

(4) Phil's belief that Kondo model would be exactly solvable was dramatically confirmed by the independent Bethe Ansatz solutions of Natan Andrei and Paul Weigman[39,40]

(5) Modern continuous-time Montecarlo solvers for computational Dynamical Mean Field Theory approaches to materials research are a direct descendant of the Anderson-Yuval mapping of quantum impurity models to statistical mechanics in time (see Fig. 4).[36,41]

4. Green period: Mixed valence and the large N expansion.

The solution to the Kondo problem resurrected the question about when fluctuations (quantum, random) defeat anti-ferromagnetic order, and when they do, what replaces it? Leaving the Kondo wilderness behind, Anderson's magnetic life devel-

oped in directions that explore this question. One way to avoid magnetism is by enhancing zero-point fluctuations with frustration, as in a triangular lattice antiferromagnet, and it was this direction Phil explored with Patrik Fazekas,[10] leading them to apply Pauling's resonating valence bond idea to spin liquids (see Fig. 5 a). A second way is through quenched disorder, as in dilute magnetic alloys, and this led Phil, with Sam Edwards to invent the concept of the spin glass[42,43] (see Fig. 5 b). A third direction, is through quantum fluctuations induced by the Kondo effect and valence fluctuations. This returned Anderson to the unfinished business of the Anderson and Kondo lattices (see Fig. 5 c). The first two directions are discussed in the excellent articles by Ted Kirpatrick, Patrick Lee and Mohit Randeria in this volume. Here, I will focus the discussion on Phil's contributions to our understanding of the Kondo lattice and valence fluctuations.

In the 1970's experimentalists started to investigate the fate of dilute magnetic alloys as the magnetic atoms become more concentrated. In transition metal alloys, the RKKY interaction between the magnetic ions overcomes the Kondo effect, giving rise to spin glasses.[42,44] But in rare earth and actinide intermetallic compounds, the Kondo effect and associated valence fluctuations are strong enough to overcome the magnetism, even in fully concentrated lattices of local moments, leading to a wide variety of *heavy fermion* materials. Phil's insights played a vital role in the development of the field.
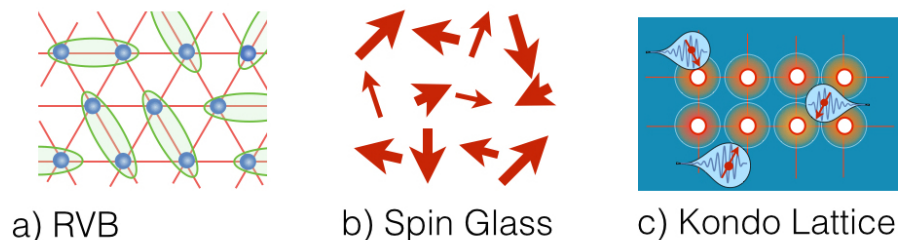

a) RVB   b) Spin Glass   c) Kondo Lattice

Fig. 5.   Three new directions for Anderson's research in the 1970s: a) the resonating valence bond ground state for the frustrated triangular lattice,[10] b) the spin-glass ground-state for a frustrated disordered array of spins[43] and c) the problem of mixed valence, where mobile heavy electrons move through a lattice of Kondo screened local moments.

Early experimental progress in the new field of *mixed valence* was rapid and chaotic. A plethora of new intermetallic compounds were discovered which display local moment physics at high temperatures, but which instead of magnetically ordering at low temperatures, form an alternative ground-state. Already in 1969, the group of Ted Geballe at Bell Labs had discovered SmB_6,[45] in which the magnetic Sm ions avoid ordering by developing a narrow-gap insulator, now called a "Kondo insulator". In 1975, an ETH Zurich-Bell Labs collaboration discovered the first heavy fermion metal $CeAl_3$.[46] The amazing thing about these two materials, is

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that both display the same sort of Kondo resistance scattering at high temperatures, but at low temperatures two materials respond differently - with the resistivity sky-rocketing in SmB_6, but collapsing into a coherent low temperature metal in $CeAl_3$. Three years later, Frank Steglich discovered the first heavy fermion superconductor $CeCu_2Si_2$,[47] though it took a number of years for the community to change their mind-set and accept this pioneering discovery.
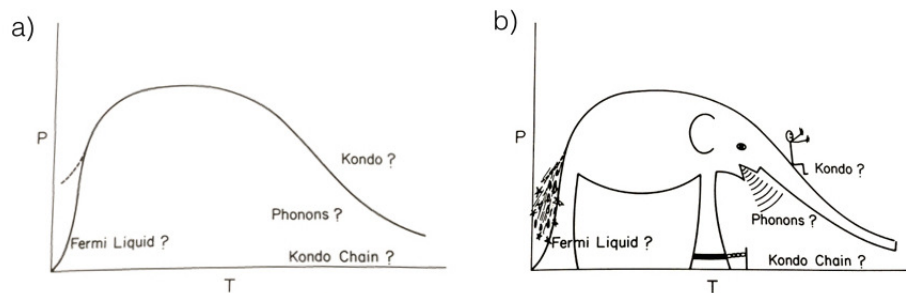

Fig. 6.   Sketches from Phil Anderson's "Epilogue" from the 1976 Rochester Conference on Mixed Valence.[48] (a) Resistivity stereotypical of systems such as $CeAl_3$. (b) Elephantine version of Fig a). (Art work by PWA). Reproduced from Valence Instabilities and Narrow Band Phenomenon, P. W. Anderson, editor Ron Parks, p 389-396 (1977) with permission from the author.

Despite this rapid progress on the experimental front, theoretical progress was flummoxed by the difficulty of making the transition to the dense "Kondo lattice" problem, lacking both the conceptual and mathematical framework. Phil's input, particularly his summary talks at the 1976 Rochester and 1980 Santa Barbara meetings on mixed valence had a profound impact.

Ron Parks and Chandra Varma organized the first conference on mixed valence at the University of Rochester in November 1976 and invited Phil to give the summary. As part of this summary, Phil roasted the theory community by sketching the resistivity of heavy fermion metals (see Fig 6a) in the guise of an elephant. Recalling an Indian parable about an elephant and seven blind men, one who pulls its tail and says its a rope, the other who says its leg is tree and so on, Phil introduced his elephantine sketch of Kondo lattice resistivity, with Jun Kondo sliding down the elephant's trunk and a Fermi liquid coming out of its behind! (See Fig. 6b). [b] The main point of the figure however, was to urge the theory community to unify its understanding of these diverse phenomena.

In the transcript, describing the Kondo elephant, Phil remarks

> *"Now we come to the heart and core of the elephant, the part*
> *which nobody has really done, which was first mentioned at least*

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[b]Did this sketch reveal Phil's subconscious discomfort with Landau Fermi liquid theory ?

> *as a serious problem here in this conference, namely the Kondo*
> *lattice, which Seb Doniach has made a start on. What you really*
> *have here is a lattice full of these objects that fluctuate back and*
> *forth from one valence to another. There are the phonons, there is*
> *the fact that the electrons fluctuate by tossing electrons into the d*
> *level on the next site which can then go down into the f levels on yet*
> *another site. So the things which toss the valence back and forth*
> *are definitely coupled between one site and another. The net result*
> *of doing this is something that most of the experiments have to tell*
> *us about : that this probably renormalizes to a very heavy Fermi*
> *liquid theory with some kind of strong antiferromagnetic prejudice*
> *in that the f-like objects in the Fermi liquid somehow lost all of their*
> *desire to be magnetic and don't very easily order anymore. This is*
> *an extremely hard problem, it's a problem in the same category of*
> *problems which are failing to be done in field theory these days."*

In this brief paragraph, Phil has laid out his view of the physical framework needed to understand heavy fermion materials. Despite his cartoon, he did emphasize that the low temperature ground-state would be a renormalized heavy Fermi liquid. He also notes the parallel between strongly correlated materials and the challenges of field theory, a parallel that would inspire many younger physicists in the decades to come. Later in the talk, Phil discusses the possibility of further instabilities in the Fermi liquid, and expresses the view that these will be more than just antiferromagnets:

> *"Once you get down to this Fermi liquid, it seems that there is*
> *a serious question of what then happens? What does the result-*
> *ing heavy Fermi liquid do with itself, what further transformation*
> *might it undergo? There are several possibilities. ... There is no*
> *reason at all why it shouldn't localize and maybe there are cases*
> *where it localizes. Kasuya gave an argument for one of then. A*
> *second possibility a whole series of experiments seem to indicate is*
> *that some phase transition takes place in many cases. the question*
> *is: what is the nature of these phase transitions? I for one am not*
> *ready to accept the idea that they are all simple magnetic phase*
> *transitions .... Maybe there is some kind of d to f excitonic phase*
> *transition that either does or does not leave some Fermi surface*
> *behind. Maybe there's a density wave. What else? "*

Curiously though, reflecting the continued mind-set of the community Phil does not mention the possibility of superconductivity, reflecting the fact that Steglich's 1979 work was not yet widely accepted [c]

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[c]Indeed, although Phil didn't know it, superconductivity had been seen at Bell Labs three years earlier in the heavy fermion material UBe_{13},[49] but mis-interpreted as an artifact of uranium

In 1980, Walter Kohn, Brian Maple and Werner Hanke at the Institute for Theoretical Physics, Santa Barbara (now the Kavli Institute for Theoretical Physics) organized a six month workshop on valence fluctuations, culminating in a conference in January 1981. In the summary of the conference, Phil continued on the theme of the link between field theory and strongly correlated electrons, introducing for the first time, the seminal idea that a *large N expansion*, akin to that used in particle physics, might be useful.

Phil magnetic life already had two links with the idea of a large N expansion. Of course, his early work on spin-wave theory was based on a $1/S$ expansion, but more recently, his work with Sam Edwards on the infinite range spin glass had involved the replica trick, replacing the disorder-averaged Free energy $\overline{-T \ln Z}$ with the $N \to 0$ limit of the disorder-averaged partition function average of N replicas,

$$\overline{\ln Z} = \lim_{N \to 0} \overline{(Z^N - 1)}. \tag{23}$$

To take the $N \to 0$ limit requires that one first solve the problem at large N to extrapolate back to zero.

But the context of heavy fermions Phil noticed that there was already a large finite N to expand in. Rare earth atoms are strongly spin-orbit coupled, and so, ignoring crystal field effects, they have a large spin degeneracy $N = 2j + 1$, where $j = 5/2$ or $7/2$ for individual f-electrons. Phil realized that the parameter $1/N$ could act as an effective small parameter for resuming many body effects:

> " The most important one, . . . is the importance of what you might call the large N limit; it was only at this conference that I, at least, realized that we have been going through a case of parallel evolution with non-Abelian gauge theory. This really has great resemblances to what one does in the intermediate valence problem, and it is interesting that the gauge theorists have found that their best controlled approximations are in a limit which they call large N - which is large order of the group, large degeneracy of the particles, and in our case that has to do with large values of the degeneracies of the states. This is the number that Ramakrishnan called n_λ. I'm going to talk later about how many different kinds of roles that plays."

Later in the same article, Phil expands on this idea and how it can be used for scaling. He makes two key observations:

- That valence fluctuations are N-fold enhanced by the large orbital degeneracy of f-electrons.
- That intersite interactions are reduced by a factor of $1/N$ relative to onsite interactions.

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filaments.

Summarizing a full page of discussion, Phil writes

> *"So we find again and again that we are gaining from this degeneracy factor and it may make the problem a lot simpler than such apparently easier problems like the Kondo problem."*

Phil's new proposal had an electrifying effect on the fledgling strongly correlated electron theory community, for it undid the log-jam, providing for the first time, a controllable expansion parameter for dealing with the mixed valent and Kondo lattices. In the immediate future, A wide range of large N treatments of the Anderson model followed, including work by Ramakrishnan and Sur,[50,51] Gunnarson and Schonhammer[52] and by Zhang and Lee.[53] Phil's observations also inspired a search for a more field theoretic way to formulate the Kondo and mixed valence problems, leading to the pioneering work by Nicolas Read and Dennis Newns[54,55] on the large N Kondo model and my own *slave boson* approach[56] to mixed valence developed under Phil's generous tutelage, in which the Gutzwiller projected f-electron operator is factorized in terms of an Abrikosov pseudo-fermion and a slave boson operator $X_{\sigma 0} = f_{\sigma}^{\dagger} b$. With this device, one could see for the first time, that the no-double occupancy constraint gives rise to locally conserved charges (here $n_b + n_f = Q$) and corresponding gauge fields.

5. Superconductivity and Magnetism come together

The theoretical perspective of condensed matter physics has dramatically transformed over the period of Phil's research. Fifty years ago, magnetism and superconductivity were regarded as mutually exclusive forms of order. Yet, gradually, starting in the 1970s, the discovery of new kinds of pair condensate, of superfluid He-3,[58] of heavy fermion,[47] organic,[59] high temperature cuprate[60] and iron-based superconductors,[61] has indicated a more intimate connection between magnetism and superconductivity.

Phil's ideas have evolved during this same period, and as they have done so, they have often transformed our scientific consensus. Phil started in the 1950s accounting for the stability of antferromagnetic order against quantum fluctuations, at the time itself contraversial. Through a journey via the Kondo effect, spin liquids and spin glasses he was led to consider states of matter in which conventional magnetism is absent and the magnetic degrees of freedom drive new kinds of electronic ground-states, especially superconductivity.

I'd like to selectively mention three exciting areas of evolving and currently unresolved controversy connected with Phil's ideas.

(1) <u>Mott versus Landau.</u> From the very outset, Phil he has emphasized the importance of the *Mott mechanism*, namely the exclusion of double occupancy of atomic orbitals. One of the questions he has emphasized, is whether the Landau quasiparticle description of electrons can survive the imposition
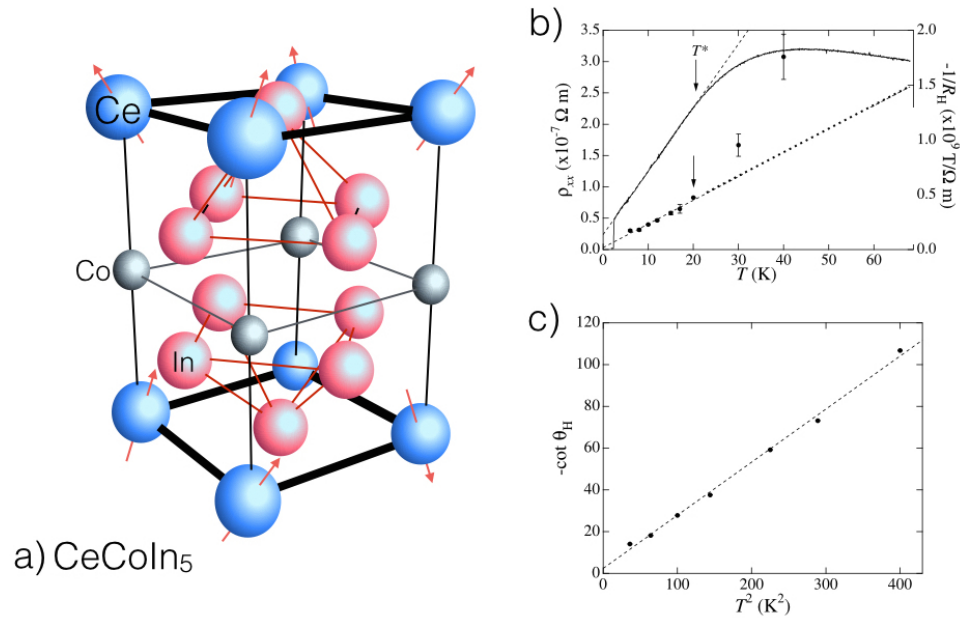
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Fig. 7.   The heavy fermion superconductor CeCoIn$_5$ showing a) the structure of this layered compound (b) the resistivity and inverse Hall constant, which are both linearly proportional to the temperature. The linear temperature dependence of the resistivity indicates a linear temperature dependence of the electron scattering rate. The temperature dependence of the Hall constant indicates that the Hall transport relaxation rate and the linear transport relaxataion rate are not equal after.[57] c) the cotangent of the Hall angle, showing the T^2 dependence of the Hall scattering rate after.[57]

of these severe constraints, suggesting instead that new kinds of metallic ground-states must inevitably develop in which the excitations have zero overlap with non-interacting electrons, and thus can not be regarded as Landau quasiparticles. Central to Phil's arguments, is the idea that teh of electrons to highly constrained electron fluids leads to a many-body X-ray catastrophe that leads to the innevitable demise of the Landau quasiparticle to form *strange metals*,[62,63] and sometimes, *hidden Fermi liquids*,[64,65] which resemble the Landau Fermi liquid thermodynamically, but without overlap with the original electron fields.

(2) Quantum Criticality versus Strange Metals There are now many example of metals which exhibit highly unusual transport and thermodynamic properties which defy a Landau Fermi liquid description, such as the optimally doped normal state of the cuprate superconductors,[66] MnSi under pressure and various heavy fermion materials,[67] such as CeCoIn$_5$[57,68] (see Fig. 7) and YbAlB$_4$[69] which each exhibit unusual linear or power-law tempera-

ture dependencies in the resitivity. One of the key discussions about these materials is whether such non-Fermi liquid behavior is generated by the vicinity to a *quantum critical point*, or whether, as Phil believes, the unusual metallic behavior is related to a new kind *strange metal phase*.[65] The recent discovery of a pressure-independent anomalous metal phase in YbAlB$_4$ may be an example of such a strange metal phase.[69]

(3) <u>Fabric versus Glue.</u> The conventional view of unconventional superconductors argues that they should be regarded as magnetic analogs of phonon-mediated superconductors, in which the soft magnetic fluctuations provide the pairing *glue*. Phil has argued[70] for a different picture, in which preformed, resonating valence bonds, on doping, provide the underlying *fabric* for a pair condensate. These opposing ideas continue to be lively debated in the context of high-temperature cuprate superconductors. Another place they may be important, is in heavy fermion superconductors, where the Kondo effect can play the same role as doping, forcing valence bonds out into the conduction sea to form pairs.

In his 2006 paper, *"The strange metal is a projected Fermi liquid with edge singularities"*,[63] Phil summarizes his point of view, writing

> *"This strange metal phase continues to be of much theoretical interest. Here we show it is a consequence of projecting the doubly occupied amplitudes out of a conventional Fermi- sea wavefunction (Gutzwiller projection), requiring no exotica such as a mysterious quantum critical point. Exploiting a formal similarity with the classic problem of Fermi-edge singularities in the X-ray spectra of metals, we find a Fermi-liquid-like excitation spectrum, but the excitations are asymmetric between electrons and holes, show anomalous forward scattering and the renormalization constant Z = 0."*

One of the most fascinating, and still unsolved aspects of these discussions above concerns the apparent development of two transport lifetimes in the electronic conductivity:[66,71] a transport scattering lifetime, inversely proportional to temperature $\tau_{tr}^{-1} \propto k_B T$ and a Hall scattering time, inversely proportional the square of the temperature $\tau_H^{-1} \propto T^2$. In a modified Drude formalism, the linear and Hall conductivities are given by

$$\sigma_{xx} = \frac{ne^2}{m}\tau_{tr}$$
$$\sigma_{xy} = \frac{ne^2}{m}\tau_{tr}(\omega_c \tau_H), \tag{24}$$

giving rise to a resistivity $\rho_{xx} \propto \tau_{tr}^{-1} \sim T$ and a Hall angle which satifies $\cot\theta_H = \sigma_{xx}/\sigma_{xy} \propto \tau_H^{-1} \propto T^2$. There are now three separate classes of mate-

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rial where this behavior has been seen: the cuprate metals,[66] the 115 heavy fermion superconductors CeCoIn$_5$[57] and electrons fluids at two dimensional oxide interfaces (SrTiO$_3$/RTiO$_3$ (R=Gd,Sm)).[72] The remarkable aspect of these metals, is that the two relaxation times enter *multiplicatively* into their Hall conductivity, $\sigma_{xy} \propto \tau_{tr}\tau_H$. Since since σ_{xy} is a zero momentum probe of the current fluctuations at the Fermi surface, this suggests that electrons are subject to two separate relaxation times at the very same point on the Fermi surface, linked by the current operator. Phil's ideas on this subject[71] have inspired a range of new theories,[73–75] but we still await a final understanding.

Like many in our community, I've often marvelled at Phil Anderson's ability to radically transform his viewpoints in response to new data and new insights. I've asked him what it would be like if he ever met his younger self for a physics discussion, and he agrees that he'd probably have quite a forceful disagreement on topics he originally pioneered and on which he now has a new perspective. Perhaps Tom Stoppard will write a play on this someday.

Phil, here's to the continuing success and inspiration of your magnetic ideas!

Acknowledgments

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