682A. Exercises 2. Due Mar 23rd

1. Carry out a Hubbard Stratonovich transformation for the following interaction Hamiltonians, factorizing the terms in brackets, and write one or two sentences interpreting the effective Hamiltonian that it gives rise to

   (a) An attractive interaction between particles in a superconductor

   $$H_I = -g \int d^3x (\psi^\dagger{}_\uparrow(x)\psi^\dagger{}_\downarrow(x))(\psi_\downarrow(x)\psi^\dagger{}_\uparrow(x))$$

   (b) In a "spin liquid" with antiferromagnetic interaction written

   $$H_I = -\frac{J_H}{2} \sum_{(i,j),\sigma,\sigma'} \left(f^\dagger{}_{i\sigma} f_{j\sigma}\right)\left(f^\dagger{}_{j\sigma'} f_{i\sigma'}\right)$$

   (c) A "Kondo interaction" between a conduction electron field ψ_α and a boson field b_α

   $$H_I = -\frac{J}{2} \sum_{\alpha,\beta} \left(\psi^\dagger{}_\alpha b_\alpha\right)\left(b^\dagger{}_\beta \psi_\beta\right).$$

   (Beware $b^\dagger{}_\beta \psi_\beta$ is a fermionic combination.)

2. The interaction between an electron gas and the potential field is given by

   $$S_{EM} = \int_0^\beta d\tau \int d^3x \left[-e\rho(x)\phi(x) - \frac{\epsilon_0 (\nabla\phi)^2}{2}\right] \tag{1}$$

   (a) Write down the path integral for a Coulomb gas of quantum electrons, involving an integral over the electron and potential fields.

   (b) What equation does the potential ϕ satisfy at the saddle point of this path integral?

   (c) Why is the coefficient of the $(\nabla\phi)^2$ term negative? (Give a physical interpretation). If you eliminate the particle density in terms of the potential, what is the energy density associated with the resulting electric field $\vec{E} = -\vec{\nabla}\phi$?

   (d) Write down the effective action of the system when the fermions have been integrated out and interpret your result in terms of Feynman diagrams.

   (e) Suppose the potential field acquires a "mass" term:

   $$S_{EM} = \int_0^\beta d\tau \int d^3x \left[-e\rho(x)\phi(x) - \epsilon_0 \frac{\phi(-\nabla^2 + \kappa^2)\phi}{2}\right] \tag{2}$$

   What form does S_{EM} take (in real space) when one integrates out the potential field?

3. Consider the Euclidean action for a single Harmonic oscillator

$$S_E = \int_0^\beta d\tau \, [\bar{a}\, (\partial_\tau + \omega)\, a]$$

(a) By re-writing the fields a and $\bar{a}$ in terms of their momentum and position co-ordinates (here we take $\hbar = m = 1$)

$$\begin{aligned} a &= \sqrt{\frac{\omega}{2}}\left(x + i\frac{p}{\omega}\right), \\ \bar{a} &= \sqrt{\frac{\omega}{2}}\left(x - i\frac{p}{\omega}\right), \end{aligned} \tag{3}$$

derive the action in terms of p and x. (Hint: terms like $\int d\tau x \partial_\tau x = \frac{1}{2}\int d\tau \partial_\tau (x^2) = 0$ vanish because they are perfect differentials of functions periodic in β).

(b) Carry out the Gaussian integral over the momentum field p and derive the corresponding action in terms of $\dot{x}$ and x.

4. **Mean field theory for antiferromagnetic Spin Density Wave** Develop the mean-field theory for a three dimensional tight-binding cubic lattice with commensurate antiferromagnetic order parameter

$$\mathbf{M}_j = \mathbf{M} e^{i\mathbf{Q}\cdot\mathbf{R}_j} \tag{4}$$

where $\mathbf{Q} = (\pi, \pi, \pi)$.

(a) Show that the mean-field free energy can be written in the form

$$H_{MF} = \sum_{\mathbf{k}\in\frac{1}{2}BZ} \psi^\dagger{}_{\mathbf{k}} \begin{pmatrix} \epsilon_{\mathbf{k}} - \mu & \mathbf{M}\cdot\vec{\sigma} \\ \mathbf{M}\cdot\vec{\sigma} & \epsilon_{\mathbf{k+Q}} - \mu \end{pmatrix} \psi_{\mathbf{k}} + \mathcal{N}_s \frac{M^2}{2I} \tag{5}$$

where $M = |\mathbf{M}|$ is the magnitude of the staggered mangetization, $\psi_{\mathbf{k}}$ denotes the four-component spinor

$$\psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{\mathbf{k}\downarrow} \\ c_{\mathbf{k+Q}\uparrow} \\ c_{\mathbf{k+Q}\downarrow} \end{pmatrix}, \tag{6}$$

$\epsilon_{\mathbf{k}} = -2t(c_x + c_y + c_z)$, ($c_l \equiv \cos k_l$, $l = x, y, z$) is the kinetic part of the energy and the summation is restricted to the magnetic Brillouin zone, (One half the original Brillouin zone.)

(b) On a tight binding lattice the kinetic energy has the "nesting" property that $\epsilon_{\mathbf{k}+\mathbf{Q}} = -\epsilon_{\mathbf{k}}$. Show that the energy eigenvalues of the mean-field Hamiltonian have a BCS form

$$E_{\mathbf{k}\pm} = \pm\sqrt{\epsilon_{\mathbf{k}}^2 + M^2} - \mu. \tag{7}$$

corresponding to an excitation spectrum with gap M. Notice that the gap is offset by an amount μ.

(c) Show that the mean-field free energy takes the form

$$F = \sum_{\mathbf{k},p=\pm 1} -T \ln\left[2\cosh\left(\frac{\beta E_{\mathbf{k}p}}{2}\right)\right] + \mathcal{N}_s\left(\frac{M^2}{2I} - 2\mu\right) \tag{8}$$

(d) By minimizing the free energy with respect to M, show that the gap equation for M is given by

$$\frac{1}{2}\sum_{\mathbf{k},p=\pm 1} \tanh\left(\frac{\sqrt{\epsilon_{\mathbf{k}}^2 + M^2} - \mu p}{2T}\right)\frac{1}{\sqrt{\epsilon_{\mathbf{k}}^2 + M^2}} = \frac{1}{I}. \tag{9}$$

(e) Show that at half filling, the nesting guarantees that a transition to a spin-density wave will occur for arbitrarily small interaction strength I.

(f) Calculate the phase diagram assuming that the order remains commensurate at finite doping.