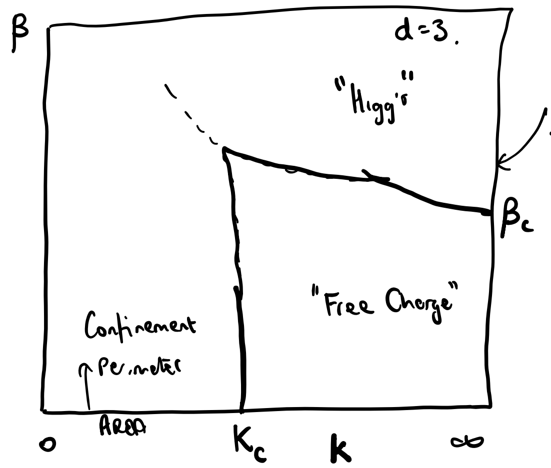
L20: THE Z2 Gauged Ising Model

Fradkin + Shenker PRD, 19, 3682 (1979)

Gauge Field

$$\boxed{\mathcal{H} = -\beta \sum \sigma_i u_{ij} \sigma_j - K \sum_{p=\square} \prod_{ij \in p} u_{ij}}$$

Scalar "Matter Field"



$\langle \sigma_i \sigma_j \rangle \xrightarrow{|i-j|\to\infty}$ const.

K finite $\langle \sigma_i \prod u_{ij} \sigma_j \rangle$ decays + dilute.

$\langle \sigma_i \sigma_j \rangle \sim e^{-|i-j|/\xi}$

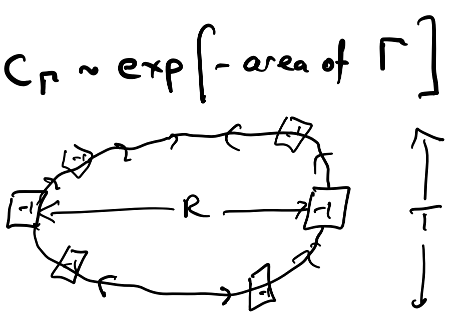
$$C_\Gamma = \left\langle \prod_{ij \in p} u_{ij} \right\rangle$$

Wilson loop "integral"

$\beta = 0, \quad K < K_c$

$C_\Gamma \sim \exp[-\text{area of } \Gamma]$

$\sigma_j \longleftarrow \bullet \longrightarrow \sigma_i$



$$W(R) \sim -\frac{1}{T} \ln C_\Gamma \propto L$$

Linearly confined.

$K > K_c$ $\qquad C_\Gamma \sim \exp\left(-\text{perimeter of } \Gamma\right)$

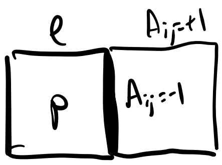
"Perimeter law"

$$A_{ij} = \sigma_i u_{ij} A_j$$

Duality

$$-\frac{H}{T} = \beta_e \sum_e A_{ij} + \beta_p \sum_p \left(A_{ij} A_{je} A_{em} A_{mn}\right)$$

2D

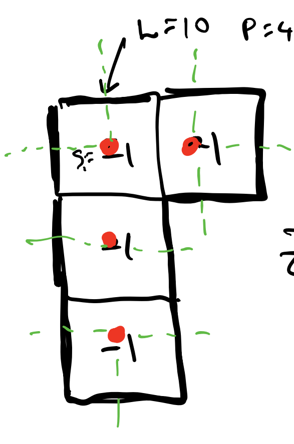


$$Z = \frac{1}{2^{2N}} \sum_{A_{ij}} e^{-H/T}$$

$$= \frac{1}{2^{2N}} e^{(\beta_e N_e + \beta_p N_p)} \prod_e \left(1\left(\frac{1+A_{ij}}{2}\right) + e^{-2\beta_e}\left(\frac{1-A_{ij}}{2}\right)\right) \prod_p \left(\left(\frac{1+\prod A}{2}\right) + e^{-2\beta_p}\left(\frac{1-\prod A}{2}\right)\right)$$

$$= \frac{1}{2^{2N}} \prod_e \left(\cosh\beta_e + \sinh\beta_e A_{ij}\right) \prod_p \left(\cosh\beta_p + \sinh\beta_p A_{ij} A_{jk} A_{ke} A_{ep}\right)$$

$$= \left(\frac{\cosh^2\beta_e \cosh\beta_p}{4}\right)^N \prod \left(1 + \tanh\beta_e A_{ij}\right) \prod \left(1 + \tanh\beta_p (A_{ij} \cdots A_{ei})\right)$$



$$= \left(\frac{\cosh^2\beta_e \cosh\beta_p}{4}\right)^N \sum_{\{L,P\}} (\tanh\beta_e)^L (\tanh\beta_p)^P$$

$$t\, c^2 = sc = \frac{\sinh 2\beta}{2}$$

$$Z = \left(\frac{\cosh^2\beta_e \cosh^2\beta_p}{4}\right)^N \sum_{\{s_i = \pm 1\}} \exp\left[\sum_e \tfrac{1}{2}\ln\tanh\beta_e (1 - s_i s_j) + \sum_p \frac{\ln\tanh\beta_p}{2}(1 - s_i)\right]$$

$$F(\beta_e, \beta_p) = \frac{1}{2}\ln\left[\left(\frac{\sinh 2\beta_e}{2}\right)^2 \left(\frac{\sinh 2\beta_p}{2}\right)\right] + F_e[\beta_*, H]$$

$$L = \sum \frac{1 - s_i s_j}{2}$$

$$P = \sum \left(\frac{1 - s_i}{2}\right)$$

$$Z \propto \sum_{\{s_i = \pm 1\}} \exp\left[\sum \beta_* \, s_i s_j + \sum_j h s_j\right]$$

$$\beta_* = -\frac{1}{2} \ln \tanh \beta_e \qquad H = -\frac{1}{2} \ln \tanh \beta_p$$

No finite T limit unless $H = 0 \Rightarrow K = \infty$

Then $\beta_c = -\frac{1}{2} \ln \tanh \beta_c$

$$e^{-2\beta_c} = \frac{1 - e^{-2\beta_c}}{1 + e^{-2\beta_c}} \qquad x = e^{-2\beta_e}$$

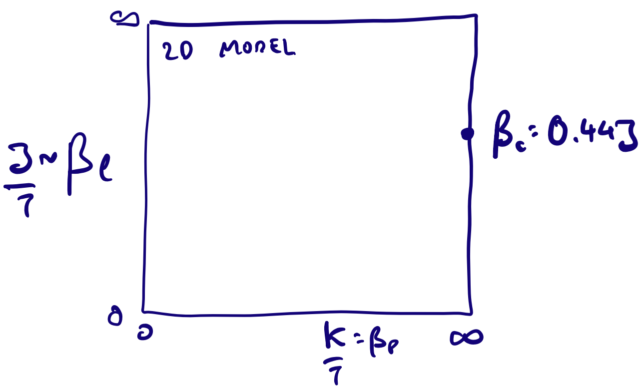
$$x + x^2 = 1 - x$$

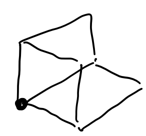
$$\Rightarrow x^2 + 2x - 1 = 0$$

$$\Rightarrow \left(\frac{1}{x} - x\right) = 2$$

$\Rightarrow \sinh 2\beta_c = 1$ Ising P.T

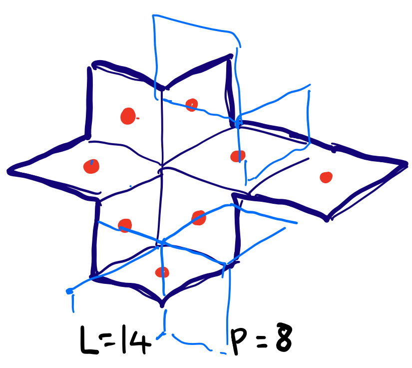
$\Rightarrow T_c = 2.263$





$\underline{\underline{3D}}$

$$Z_{3D} = \left(\frac{\cosh^3 \beta_e \cosh^3 \beta_p}{8}\right)^N \sum_{\{L,P\}} \underbrace{(\tanh \beta_e)^L}_{(s_i s_j s_k s_l)} \underbrace{(\tanh \beta_p)^P}_{A_{ij}}$$



Membrane + Surface tension

$L = 14 = P'$

$P = 8 = L'$

??

$$\beta_e^* = -\frac{1}{2} \ln(\tanh \beta_p)$$

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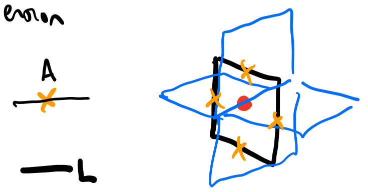
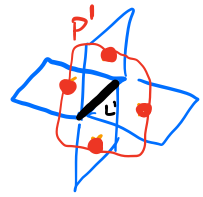
+ Line Tension



$L=4 \quad P=1$

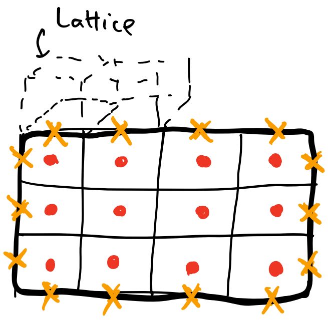


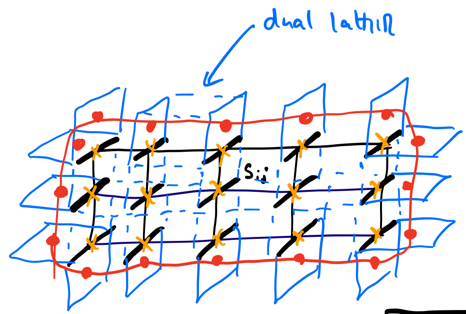
$P'=4 \quad L'=1$

s_{ij}

$$P = \sum \tfrac{1}{2}\left[1 - s_{ij}\right]$$

$$L = \sum \tfrac{1}{2}\left[1 - s_{ij}s_{jk}s_{kl}s_{li}\right]$$





$$A_{ij}A_{jk}A_{kl}A_{li} \rightarrow \tilde{A}$$

$$A_{ij} \rightarrow \tilde{A}\tilde{A}\tilde{A}\tilde{A}$$

$P = 12$	× → •	$L' = 12$
$L = 14$	• → ×	$P' = 14$
Original variables	Duality.	Dual Variables.

The configurations of the dual variables select the "membranes" of visons, bounded by links. The links of the dual variables select the plaquets of the original system, while the dual plaquets select the links

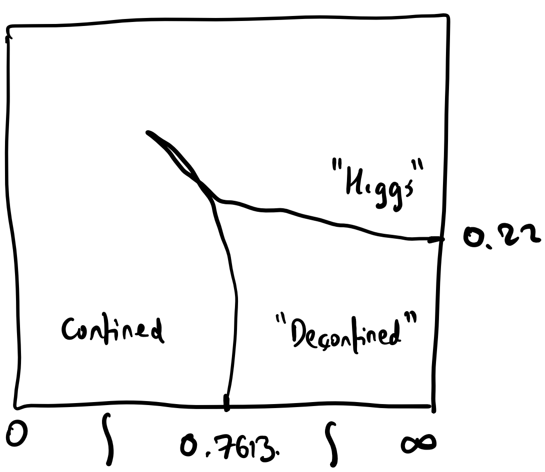
every plaquet becomes a bond
every bond becomes a plaquet

$$F[\beta_e, \beta_p] = \frac{3}{2} \ln\left(\sinh 2\beta_e \sinh 2\beta_p\right) + F\left[-\tfrac{1}{2}\ln\tanh\beta_p, -\tfrac{1}{2}\ln\tanh\beta_e\right]$$

$$\beta_p = \frac{k}{T} \quad \beta_e = 0 \quad \longrightarrow \quad \beta_p = \infty, \quad \beta_e = -\tfrac{1}{2}\ln\tanh\left(\frac{K}{T}\right)$$

$$\beta_{ec}^{3D} = \frac{1}{2 \times (2.25516)} = 0.22$$

$$\beta_p^c = 0.7613$$



Area Perimeter.

$$f = F(\beta_e, \beta_p) - \frac{3}{2}\ln\left[(1+e^{2\beta_e})(1+e^{2\beta_p})\right]$$

$$f(\xi_e, \xi_p) = f(\ln 2 - \xi_p, \ln 2 - \xi_e)$$

$$\xi_e = \ln(1+e^{-2\beta_e})$$
$$\xi_p = \ln(1+e^{-2\beta_p})$$

$$e^{\xi_e} = 1+e^{-2\beta_e} \Rightarrow 1-e^{\xi_e} = -e^{-2\beta_e}$$

$$-\frac{1}{2}\ln\tanh\beta_e = -\frac{1}{2}\ln\left[\left(\frac{1-e^{-2\beta_e}}{1+e^{-2\beta_e}}\right)\right]$$

$$\beta_p^* = -\frac{1}{2}\ln\left[\left(\frac{2-e^{\xi_e}}{e^{\xi_e}}\right)\right]$$

$$e^{-2\beta_p^*} = \left(\frac{2-e^{\xi_e}}{e^{\xi_e}}\right) = -1+2e^{-\xi_e}$$

$$1+e^{-2\beta_p^*} = +2e^{-\xi_e}$$

$$\xi_p^* = \ln(1+e^{-2\beta_p^*}) = +\ln 2 - \xi_e.$$

$$f(\xi_e,\xi_p) + \frac{3}{2}\ln\left[(1+e^{2\beta_e})(1+e^{2\beta_p})\right] = \frac{3}{2}\ln\left[\sinh 2\beta_e \sinh 2\beta_p\right]$$
$$+ f\left[\ln 2 - \xi_p, \ln 2 - \xi_e\right]$$
$$+ \frac{3}{2}\ln\left(1+e^{2\beta_e^*}, 1+e^{2\beta_p^*}\right)$$

$$f(\xi_e,\xi_e) = f(\ln 2-\xi_p, \ln 2-\xi_e) + \frac{3}{2}\ln\left[\underbrace{\left(\frac{\sinh 2\beta_e}{1+e^{2\beta_e}}(1+e^{2\beta_p^*})\right)}_{0}\left(\frac{\sinh 2\beta_p}{1+e^{2\beta_p}}(1+e^{2\beta_e^*})\right)\right]$$

$$\boxed{f(\xi_e, \xi_p) = f(\ln 2-\xi_p, \ln 2-\xi_e)}$$

But

$$\frac{1}{2}\left(\frac{e^{2\beta_e}-e^{-2\beta_e}}{1+e^{2\beta_e}}\right)\left(1+e^{2\beta_p^*}\right) = \frac{e^{2\beta_e}-e^{-2\beta_e}}{1+e^{2\beta_e}} \frac{e^{2\beta_e}}{e^{2\beta_e}-1}$$

$$e^{-2\beta_e^*} = \tanh\beta_e = \left(\frac{e^{2\beta_e}-1}{e^{2\beta_e}+1}\right) = \frac{(e^{2\beta_e}+1)(e^{2\beta_e}-1)}{(1+e^{2\beta_e})(e^{2\beta_e}-1)} = 1!$$

$$\Longrightarrow 1+e^{2\beta_p^*} = 1+\frac{e^{2\beta}+1}{e^{2\beta}-1} = \frac{2e^{2\beta_e}}{(e^{2\beta_e}-1)}$$

If $\xi_e+\xi_p = \ln 2$ then $\xi_e' = \xi_e$ $\xi_{p'} = \xi_p$.

Self dual. $\xi_e+\xi_p = \ln(1+e^{-2\beta_e})(1+e^{-2\beta_p}) = \ln 2$

$$\Rightarrow \boxed{(1+e^{-2\beta_e})(1+e^{-2\beta_p}) = 2}$$

$$\sqrt{2}-1 = e^{-2\beta} \quad -2\beta = \ln(\sqrt{2}-1)$$

$$\boxed{\beta_e = \beta_p = -\ln\left(\frac{1}{\sqrt{2}-1}\right)}$$

$$\beta_e = -\frac{1}{2}\ln\left[\frac{2}{e^{-2\beta_p}+1}-1\right] = -\frac{1}{2}\ln\left[\frac{-e^{-2\beta}+1}{1+e^{-2\beta}}\right]$$

$$\beta_e = -\frac{1}{2}\ln\left[\tanh\beta_p\right]$$