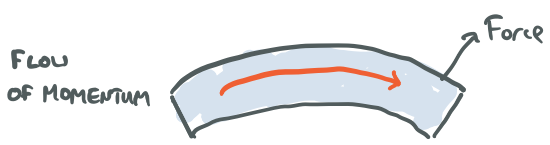
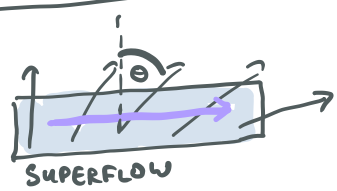
When we are dealing with macroscopic ensembles of particles, we seek a compact way of representing the thermal and quantum fluctuations so that we can calculate their statistical mechanics & their dynamics; we also seek to explore broken symmetry phases of matter, such as a superconductor or a superfluid, where the field operator acquires an expectation value

$\vec{k} = \vec{\nabla}\varphi$

$$\boxed{\begin{aligned} \Psi(x) &= \langle \hat{\psi}(x) \rangle = \sqrt{n_s(x)}\, e^{i\varphi(x)} \qquad \text{Superfluid e.g } {}^4\text{He} \\ \vec{V}_s &= \frac{\hbar \vec{k}_s}{m} = \frac{\hbar}{m} \nabla \varphi(x) \end{aligned}}$$



$E \propto u^2$



Strained wavefunction.

$E \propto (\nabla\phi)^2$

"Rigidity"

$$\boxed{\begin{aligned} \Psi(x) &= \langle \hat{\psi}_\downarrow(x) \hat{\psi}_\uparrow(x) \rangle = \sqrt{n_s}\, e^{i\varphi(x)} \qquad \text{Superconductor} \\ \vec{J}_s &= n_s \vec{V}_s = \frac{n_s}{m} \Big[\underbrace{\hbar \vec{\nabla}\varphi - q\vec{A}}_{\text{GAUGE INVARIANT}} \Big] \qquad \text{Supercurrent} \end{aligned}}$$

$\psi \to \psi\, e^{i\chi}$

$\varphi \to \varphi + 2\chi$

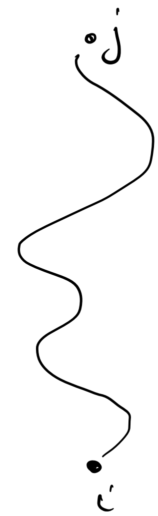
$\nabla\varphi \to \nabla\varphi + 2\nabla\chi$

$A \to A + \frac{2\hbar\chi}{q}$ $\quad q = 2e$

$= A + \frac{\hbar}{e}\chi$

How can we accomplish this?

Path integrals provide a way to accomplish this.

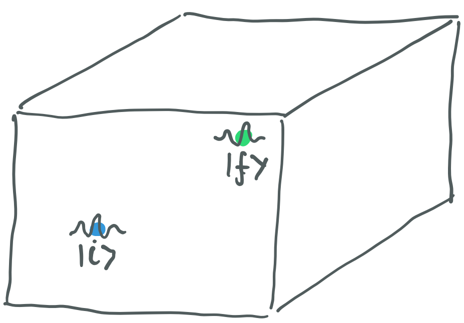


Basic idea: reformulate the quantum amplitude to go from $i \to f$ as a sum over all possible paths, in which the classical action plays the role of the phase

$$A_{PATH} = e^{i\phi_{PATH}}$$

$$\phi = S_{PATH}/\hbar .$$

Particle in a Box

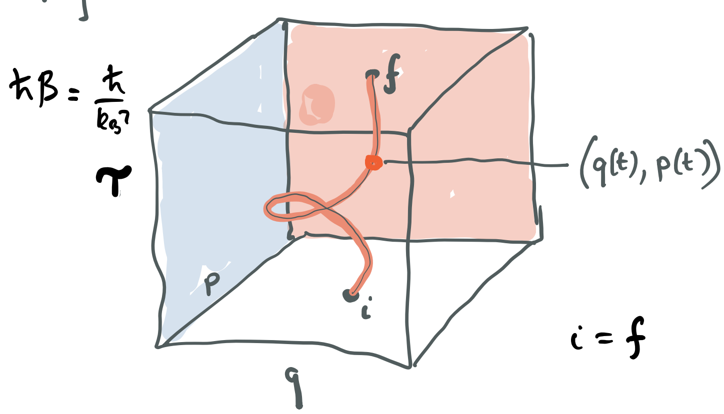


$$A_{i\to f} = \langle f | \overbrace{e^{-\frac{it\hat{H}}{\hbar}}}^{\left(e^{-\frac{i\Delta t H}{\hbar}}\right)^N} | i \rangle$$

$$= \sum_{\text{PATHS } i\to f} \exp\left[i \frac{S_{PATH}}{\hbar} \right] .$$

$$S_{PATH} = \int_0^t dt' \left[p\dot{q} - H[p,q] \right] .$$

This is a precise reformulation of Heisenberg's operator Q.M. Moreover $\hbar \to 0$ corresponds to the classical limit

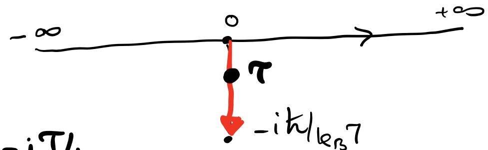


Feynman's idea can be extended to statistical mechanics by treating the Boltzmann density matrix as a time evolution operator in imaginary time

$\langle f | e^{-iHt/\hbar} | i \rangle \quad i = f$

$$e^{-F/k_BT} = Z = \sum_\lambda e^{-\beta E_\lambda} = \overbrace{\mathrm{Tr}\left[e^{-\beta \hat{H}}\right]}^{\text{Density Matrix}} = \sum_\lambda \langle \lambda | e^{-i\hat{H}t/\hbar} | \lambda \rangle \Big|_{t=-i\hbar\beta}$$

$\beta = \hbar / k_B T.$



$t = -i\tau/\hbar$

By changing variables $it/\hbar \to T$, so that $\frac{i\,dt}{\hbar} = dT$ and $p\dot{q}_t dt = p\dot{q}_\tau dT$ we can rewrite this quantity as

$$Z = \sum_{\substack{\text{periodic} \\ \text{paths}}} \exp\left[-S_E\right]$$

$$S_E = -\frac{iS_{cl}}{\hbar} = \int_0^{\hbar\beta = \text{PLANCK TIME}} \left[-\frac{i}{\hbar} p\dot{q}_\tau + H[p,q] \right] dT$$

Coherent States provide the key (sometimes "Glauber" states)

$$|\alpha\rangle = e^{\hat{b}^{+}\alpha} |0\rangle$$

$$\hat{H} = \epsilon \hat{b}^{+}\hat{b}$$

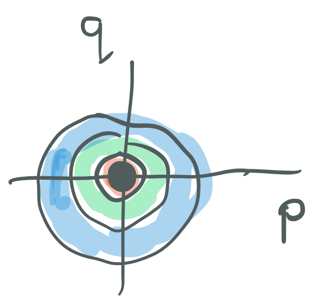
$$\hat{b} = \frac{q + ip}{\sqrt{2}} \quad \hbar = 1.$$

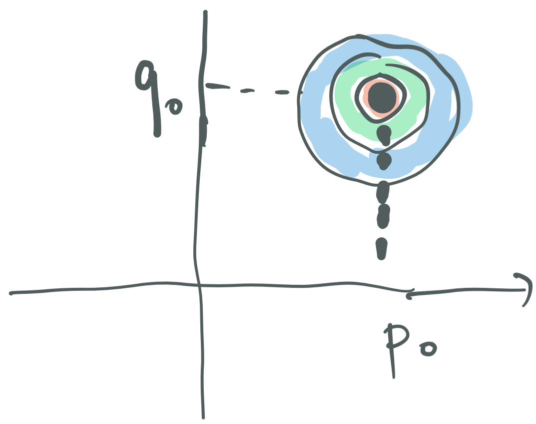
$$\hat{b}|\alpha\rangle = \alpha|\alpha\rangle$$

$$\alpha = (q_0 + ip_0)/\sqrt{2} \qquad \left(\hat{b} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}\right)$$

$$\hat{b}|\alpha\rangle = \frac{(q_0 + ip_0)}{\sqrt{2}} |\alpha\rangle$$

eigenstate of b, not b^+!





Ground-state

$$\psi \sim e^{-x^2/4\ell^2} \equiv \langle x|0\rangle$$

Coherent State

$$\langle x|\alpha\rangle = \langle x| e^{b^{+}\alpha} |0\rangle$$

$$= \exp\left[\left(\frac{q + i(-i\partial_q)}{\sqrt{2}}\right)\alpha\right] \langle x||0\rangle$$

$$\psi_\alpha(x) = \exp\left[\left(\frac{q + \partial_q}{\sqrt{2}}\right)\alpha\right] e^{-x^2/4\ell^2}$$

Many body physics $\quad \hat{\psi}(x)$ – field

$$\hat{\psi}(x)|\phi\rangle = \phi(x)|\phi\rangle$$

Eigenstate of the field op.

$$\hat{b}^{\dagger} = \int d^d x \, \psi^{\dagger}(x)\,\phi(x)$$

Coherently adds a boson to the condensate with wavefunction $\phi(x)$.

$$|\phi\rangle = \exp[b^{\dagger}]|0\rangle$$
$$= \exp\left[\int d^d x \, \psi^{\dagger}(x)\,\phi(x)\right]|0\rangle$$

We can use these as wavepackets for Many Body Physics.

$$\int_0^{\hbar\beta = \text{PLANCK TIME}} \left[-\frac{i}{\hbar} p \dot{q}_\tau + H[p,q] \right] d\tau$$

$$\psi \sim q(x) \qquad i\hbar \psi^\dagger \sim p(x)$$

$$[\psi(x), \psi^\dagger(y)] = \delta(x-y) \equiv \left[q(x), \frac{p(y)}{i\hbar} \right]$$

$$\phi(x) \sim q \qquad i\hbar \bar{\phi}(x) \sim \pi$$

$$-\frac{i}{\hbar} p \dot{q}_\tau \sim \bar{\phi} \; \partial_\tau \phi$$

$$S_E = \int_0^\beta d\tau\, d^3x \left[\bar\phi(x,\tau)\,\partial_\tau \phi(x,\tau) + H[\bar\phi,\phi] \right]$$

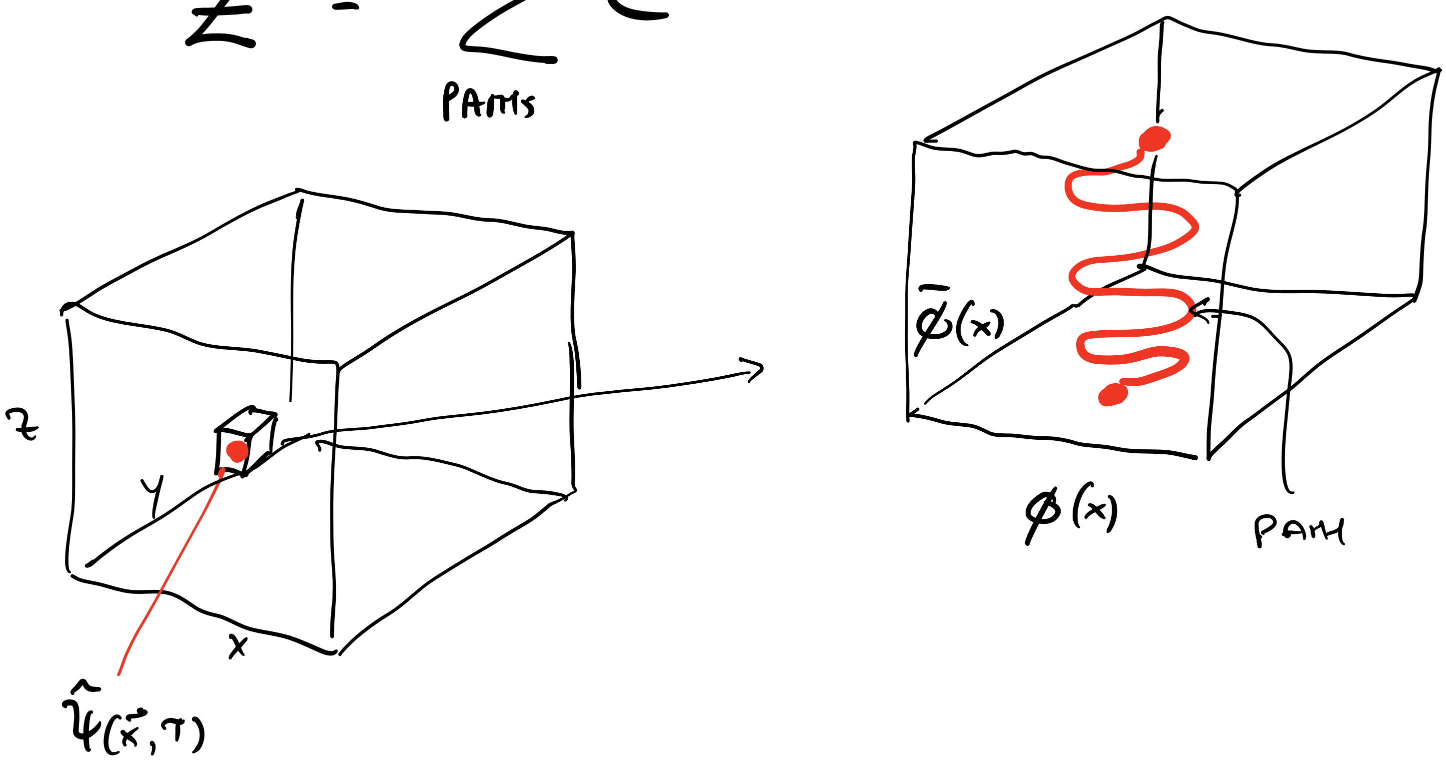
MB HAMILTONIAN

$\hat\psi \to \phi$

$\hat\psi^\dagger \to \bar\phi$

$$\langle T\,\psi(1)\,\psi^\dagger(2)\rangle = \frac{1}{Z}\sum_{\text{PATH}} \phi(1)\,\bar\phi(2)\, e^{-S_{\text{PATH}}}$$

$$Z = \sum_{\text{PATHS}} e^{-S_{\text{PATH}}}$$



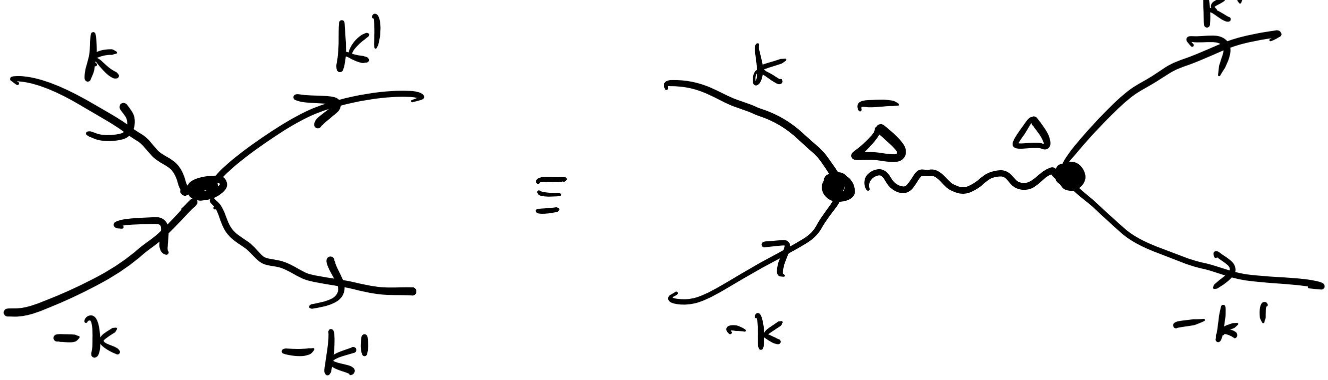
Fermions $\phi(x)$ will have be a new kind of number.

$$\hat{c}\,\hat{c}^{\dagger} = -\hat{c}^{\dagger}\hat{c}$$ EXCLUSION PRINCIPLE

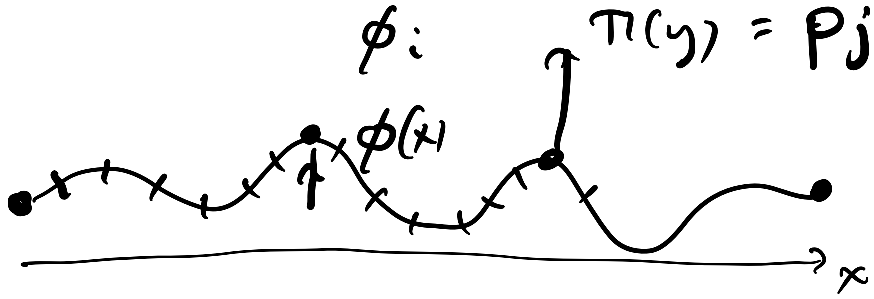
$$\varphi\,\bar{\varphi} = -\bar{\varphi}\,\varphi$$

"Grassmann number"

$$Z_{\text{interacting}} = \sum_{\text{PAIRS}} \int \left[\text{fermions interacting fluctuating gap function}\right]$$



"unfolding of interactions in terms of a fluctuating order parameter field



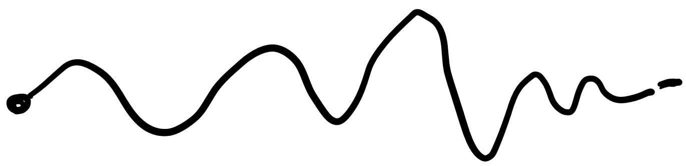
$$[\phi_i, p_j] = i\hbar\,\delta_{ij}$$

$$\downarrow$$

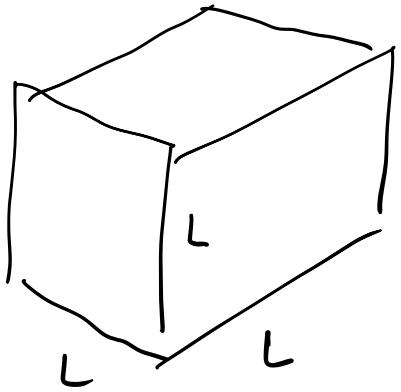
$$[\phi(x), \pi(y)] = i\hbar\,\delta(x-y)$$

$$\psi = \frac{\phi + i\pi}{\sqrt{2}} \qquad \psi^\dagger = \frac{\phi - i\pi}{\sqrt{2}}$$

$$[\psi, \psi^\dagger] = \delta(x-y)$$



Coherent states for bosons



$b_{\vec{q}}$

$\vec{q} = \frac{2\pi}{L}(n, m, l)$

$$\hat{b}|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = e^{\hat{b}^\dagger \alpha}|0\rangle$$

$$e^{\hat{b}^\dagger \alpha} = \sum_{n=0}^{\infty} \frac{(b^\dagger \alpha)^n}{n!}$$

$$\hat{b}|\alpha\rangle = \hat{b}\left(1 + b^\dagger \alpha + \frac{(b^\dagger \alpha)^2}{2!} + \dots\right)|0\rangle$$

$$[b, b^\dagger] = 1 \qquad [\hat{b}, (b^\dagger)^n] = n(b^\dagger)^{n-1}$$

$$b(b^\dagger)^n|0\rangle = \left[\overbrace{(b^\dagger)^n \hat{b}}^{0} + n(b^\dagger)^{n-1}\right]|0\rangle$$

$$= n(b^\dagger)^{n-1}|0\rangle$$

$$[b, e^{\hat{b}^+\alpha}] = [b, \sum \frac{\alpha^n (b^+)^n}{n!}]$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \underbrace{[b, (b^+)^n]}_{n(b^+)^{n-1}} = \sum \frac{\alpha^n b^{n-1}}{(n-1)!}$$

$$= \alpha e^{b^+\alpha}$$

$$b e^{b^+\alpha}|0\rangle = (e^{b^+\alpha} \underbrace{b}_{=0} + \alpha e^{b^+\alpha})|0\rangle$$

$$= \alpha|\alpha\rangle$$

$$\langle\bar{\alpha}| = \langle 0| e^{\bar{\alpha}\hat{b}} = (|\alpha\rangle)^+$$

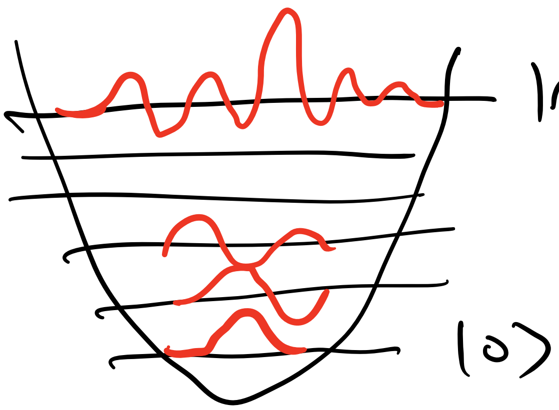
$$\langle\bar{\alpha}|b^+ = \langle\bar{\alpha}|\bar{\alpha}$$

Eigenstate of $\hat{b}^+$ acting left.

$$|\alpha\rangle = \sum \frac{\alpha^n (b^+)^n}{n!}|0\rangle$$

$$|n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle$$

n quanta in it



$$\hat{n}|n\rangle = b^\dagger b|n\rangle = n|n\rangle$$

$$\boxed{|\alpha\rangle = \sum \frac{\alpha^n}{\sqrt{n!}} \underbrace{|n\rangle}_{\text{eigenstate of number operator}}}$$

Linear combination of states with different particle number

amplitude to be in state $|n\rangle$

$$\phi_n(\alpha) = \langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}$$

$$\langle\bar{\alpha}| = \sum \langle m| \frac{\bar{\alpha}^m}{\sqrt{n!}}$$

$$\langle\bar{\alpha}|m\rangle = \frac{\bar{\alpha}^m}{\sqrt{m!}}$$

$$1 = \sum |n\rangle\langle n|$$

$$\langle\bar{\alpha}|\alpha\rangle = \sum \frac{\bar{\alpha}^m}{\sqrt{m!}} \overbrace{\langle m|n\rangle}^{\delta_{mn}} \frac{\alpha^n}{\sqrt{n!}} = \sum \frac{(\bar{\alpha}\alpha)^n}{n!} = e^{\bar{\alpha}\alpha}$$

$$\boxed{\langle\bar{\alpha}|\alpha\rangle = e^{\bar{\alpha}\alpha}}$$

exponential overlap

Glauber state $= \frac{1}{\sqrt{\bar{\alpha}\alpha}} |\alpha\rangle = e^{-\frac{\bar{\alpha}\alpha}{2}} e^{b^{+}\alpha} |0\rangle$

(Normalized)

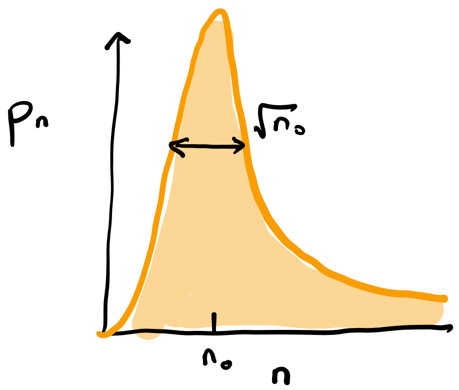
$$\boxed{p(n) = \frac{|\langle n|\alpha\rangle|^2}{\langle\bar{\alpha}|\alpha\rangle} = \frac{(\bar{\alpha}\alpha)^n}{n!} e^{-\bar{\alpha}\alpha} \quad \text{Poisson distn.}}$$

$\alpha = \sqrt{n_0}\, e^{i\theta}$ $\qquad p(n) = \left[\frac{(n_0)^n e^{-n_0}}{n!}\right]$

$$\langle\hat{n}\rangle = \sum n\, p(n) = \sum n \frac{(\bar{\alpha}\alpha)^n}{n!} e^{-\bar{\alpha}\alpha}$$

$$= \sum \frac{(\bar{\alpha}\alpha)^n}{(n-1)!} e^{-\bar{\alpha}\alpha} = \bar{\alpha}\alpha \sum \frac{(\bar{\alpha}\alpha)^{n-1}}{(n-1)!} e^{-\bar{\alpha}\alpha}$$

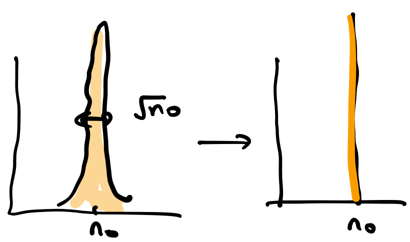
$$= \bar{\alpha}\alpha = n_0$$



$$\langle n^2\rangle - \langle n\rangle^2 = n_0$$

$n_0 \to \infty$

$n_0 \to \infty$



e.g. $\quad n_0 \sim 10^{23} \qquad \sqrt{\delta n^2} \sim 10^{11}$

$$\frac{\sqrt{\delta n^2}}{\langle n\rangle} \sim \frac{1}{10^{11}} \sim 10^{-11}$$

$$\langle\bar{\alpha}| :A[b^{+}, b]: |\alpha\rangle = A[\bar{\alpha}, \alpha] \langle\bar{\alpha}|\alpha\rangle$$

$$= A[\bar{\alpha}, \alpha]\, e^{\bar{\alpha}\alpha}$$

$$\frac{\langle\bar{\alpha}| :A[b^+,b]: |\alpha\rangle}{\langle\bar{\alpha}|\alpha\rangle} = A[\bar{\alpha},\alpha]$$

Operator wit fields replaced by c-numbers

$$:H[b^+,b]: \longrightarrow :H[\bar{\alpha},\alpha]:$$

operator number

Amplitudes $\longrightarrow$ Path integrals
