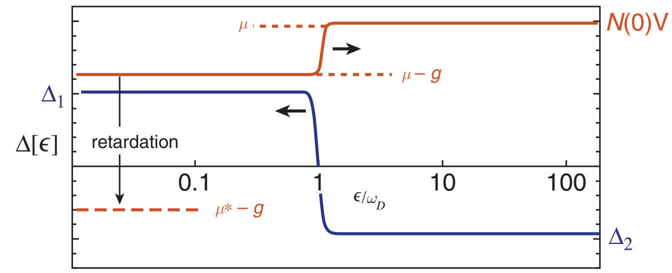


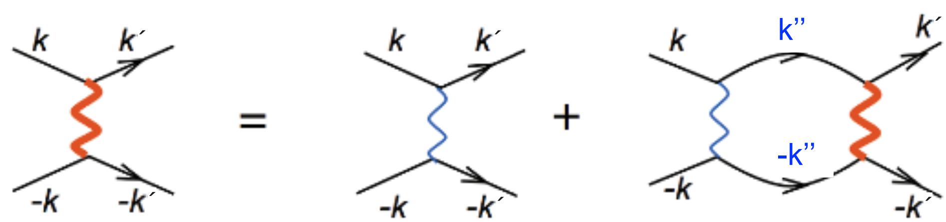
Fig. 15.1 Showing model interaction (orange line) and energy-dependent gap function that it gives rise to. The gap function changes sign around $\epsilon = \omega_D$ to minimize the effect of the Coulomb interaction. In this example $\mu - g > 0$, but because of renormalization the effective interaction at low energies becomes $\mu^* - g < 0$.

gap function contains an essentially instantaneous negative component and a retarded positive component, of the form

$$\Delta(t) = -|\Delta_2|\delta(t) + \Delta_1 \frac{\omega_D}{\pi}\left(\frac{\sin \omega_D t}{\omega_D t}\right).$$

It is thus adapted to the interaction so as to minimze the energy.

- The renormalization of the Coulomb interaction can be understood as the screening effect of high-frequency virtual pair fluctuations on energy scales between ω_D and the bandwidth D. Whereas pair fluctuations enhance an attractive interaction, they screen a repulsive interaction, driving it logarithmically towards weak coupling at low energies. The renormalized Coulomb interaction can be written self-consistently as follows:



$$-\frac{\mu^*}{N(0)} \qquad -\frac{\mu}{N(0)} \qquad +\left(-\frac{\mu}{N(0)}\right)\left(\sum_{|\epsilon_{k''}|>\omega_D} \frac{1}{2|\epsilon_{k''}|}\right)\left(-\frac{\mu^*}{N(0)}\right).$$

Carrying out the integral over the intermediate virtual pairs, we obtain

$$\mu^* = \mu - \mu\mu^* \left[\int_{-D}^{-\omega_D} + \int_{\omega_D}^{D}\right] \frac{d\epsilon}{2|\epsilon|} = \mu - \mu\mu^* \ln\left(\frac{D}{\omega_D}\right), \tag{15.30}$$

from which we obtain the result $\mu^* = \mu/[1 + \mu \ln(D/\omega_D)]$.

15.3 Anisotropic pairing

At the turn of the 1960s, physicists on both sides of the Iron Curtain independently predicted that fermions interacting via repulsive interactions can develop pair condensates with pairs which carry finite orbital and spin angular momentum, giving rise to gap

Table 15.1 Pair potentials.

Interaction	Singlet $V^S_{\mathbf{k},\mathbf{k}'}$	Triplet $V^T_{\mathbf{k},\mathbf{k}'}$
$\frac{1}{2}V_\mathbf{q} : \rho_{-\mathbf{q}}\rho_\mathbf{q} :$	$\frac{1}{2}(V_{\mathbf{k}-\mathbf{k}'} + V_{\mathbf{k}+\mathbf{k}'})$	$\frac{1}{2}(V_{\mathbf{k}-\mathbf{k}'} - V_{\mathbf{k}+\mathbf{k}'})$
e.g. $V(\mathbf{q}) = -g$	$-g$	0
$\frac{1}{2}J_\mathbf{q} : \vec{S}_{-\mathbf{q}} \cdot \vec{S}_\mathbf{q} :$	$-\frac{3}{4}\left(\frac{J_{\mathbf{k}-\mathbf{k}'} + J_{\mathbf{k}+\mathbf{k}'}}{2}\right)$	$\frac{1}{4}\left(\frac{J_{\mathbf{k}-\mathbf{k}'} - J_{\mathbf{k}+\mathbf{k}'}}{2}\right)$
e.g. $J_\mathbf{q} = 2J(\cos q_x + \cos q_y)$	$-\frac{3}{2}J\left(\cos k_x \cos k'_x + \cos k_y \cos k'_y\right)$	$\frac{1}{2}J\left(\sin k_x \sin k'_x + \sin k_y \sin k'_y\right)$

functions that are anisotropic in momentum space. This idea was to have a wide applicability, accounting for superfluidity in ^{3}He and in certain kinds of nuclear matter; it also holds the key to high-temperature superconductivity.

Let us now explore the relationship between interactions and the pair potential $V_{\mathbf{k},\mathbf{k}'}$. We will examine two important examples, a repulsive potential,

$$V = \frac{1}{2}\sum V_\mathbf{q}\left[: \rho_{-\mathbf{q}}\rho_\mathbf{q} :\right] = \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} V_\mathbf{q} c^\dagger_{\mathbf{k}_1+\mathbf{q}\sigma} c^\dagger_{\mathbf{k}_2-\mathbf{q}\sigma'} c_{\mathbf{k}_2\sigma'} c_{\mathbf{k}_1\sigma}, \tag{15.31}$$

and a "magnetic" interaction,

$$V_{mag} = \frac{1}{2}\sum_\mathbf{q} J_\mathbf{q}\left[\vec{S}_{-\mathbf{q}} \cdot \vec{S}_\mathbf{q}\right], \tag{15.32}$$

where $J_\mathbf{q}$ is an effective interaction between quasiparticles on the Fermi surface. For example, in the *spin fluctuation* model discussed in Section 12.6, this interaction is given by $J_\mathbf{q} \sim -I/(1 - I\chi_0(\mathbf{q}))$, where $\chi_0(q)$ is the momentum-dependent spin susceptibility and $I = U/3$ is the local repulsive Hubbard interaction. Such magnetic interactions are enhanced in the vicinity of a magnetic instability or *quantum critical point*. A summary of the pair potentials associated with these interactions, which we shall now derive, is given in Table 15.1.

Let's first consider the potential interaction (15.31). The pairing potential $V_{\mathbf{k},\mathbf{k}'}$ it gives rise to is determined by the influence of the interaction on Cooper pairs, which have zero total momentum. The BCS interaction is thus a projection of those terms in the interaction in which the incoming and outgoing Cooper pairs have zero momentum, so that $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}'$ and $\mathbf{k}_1 + \mathbf{q} = -(\mathbf{k}_2 - \mathbf{q}) = \mathbf{k}$, hence $\mathbf{q} = \mathbf{k} - \mathbf{k}'$. The resulting interaction is

$$V_{BCS} = \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} V_{\mathbf{k}-\mathbf{k}'} c^\dagger_{\mathbf{k}\sigma} c^\dagger_{-\mathbf{k}\sigma'} c_{-\mathbf{k}'\sigma'} c_{\mathbf{k}'\sigma} = V^{\uparrow\uparrow}_{BCS} + V^{\downarrow\downarrow}_{BCS} + V^{\uparrow\downarrow}_{BCS}, \tag{15.33}$$

which we have split up into terms according to the spin of the fermions in the pair. Let us first focus on the opposite-spin pairing term $V^{\uparrow\downarrow}_{BCS}$,

$$V^{\uparrow\downarrow}_{BCS} = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k}-\mathbf{k}'}(c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})(c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}) = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k}-\mathbf{k}'} \Psi^\dagger_\mathbf{k} \Psi_{\mathbf{k}'}, \tag{15.34}$$

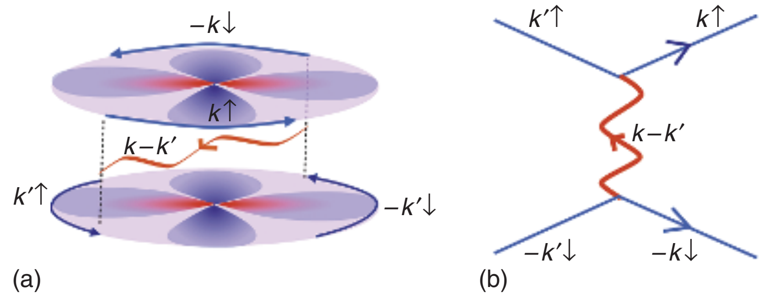


Fig. 15.2 (a) Illustrating transfer of momentum between Cooper pairs in condensate. (b) Feynman diagram representation of the transfer of momentum $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ between Cooper pairs.

as shown in Figure 15.2, where we have lumped $V^{\uparrow\downarrow}$ and $V^{\downarrow\uparrow}$ together, absorbing the factor of $\frac{1}{2}$. Now despite appearances, an opposite-spin pair is not a well-defined singlet *or triplet*, since this requires appropriately symmetrized wavefunctions. If $F(\mathbf{k})_{\alpha\beta} = \langle \mathbf{k}\alpha, -\mathbf{k}\beta | \mathbf{k}_P \rangle$ is the pair wavefunction, then we can define a spatial parity P by $F(-\mathbf{k})_{\alpha\beta} = PF(\mathbf{k})_{\alpha\beta}$ and a spin exchange parity X by $F(\mathbf{k})_{\beta\alpha} = XF(\mathbf{k})_{\alpha\beta}$. Since this interaction is inversion-symmetric, $V_{\mathbf{q}} = V_{-\mathbf{q}}$, it preserves the parity of the Cooper pairs. The spin-exchange parity discriminates between spin singlets with $X = -1$ and spin triplets with $X = +1$. The joint process of spin and momentum exchange exchanges two fermions and must give an exchange eigenvalue of -1, $F(-\mathbf{k})_{\beta\alpha} = XPF(\mathbf{k})_{\alpha\beta} = -F(\mathbf{k})_{\alpha\beta}$, so that $XP = -1$. Even-parity pairs are thus singlets, $(P, X) = (+, -)$, whereas odd-parity pairs are triplets, $(P, X) = (-, +)$. To display the singlet and triplet pair scattering, we divide the interaction into symmetric and antisymmetric parts:

$$V_{BCS}^{\uparrow\downarrow} = \sum_{\mathbf{k},\mathbf{k}'} \Bigg[\overbrace{\left(\frac{V_{\mathbf{k}-\mathbf{k}'} + V_{\mathbf{k}+\mathbf{k}'}}{2} \right)}^{V^S_{\mathbf{k},\mathbf{k}'}} + \overbrace{\left(\frac{V_{\mathbf{k}-\mathbf{k}'} - V_{\mathbf{k}+\mathbf{k}'}}{2} \right)}^{V^T_{\mathbf{k},\mathbf{k}'}} \Bigg] \Psi^\dagger_{\mathbf{k}} \Psi_{\mathbf{k}'}, \tag{15.35}$$

where

$$V^{S,T}_{\mathbf{k},\mathbf{k}'} = \frac{1}{2} \left(V_{\mathbf{k}-\mathbf{k}'} \pm V_{\mathbf{k}+\mathbf{k}'} \right) \tag{15.36}$$

are the BCS pairing interactions in the singlet and triplet channels, respectively. Now because the first term is even in $\mathbf{k}$ and $\mathbf{k}'$ while the second is odd, the summations over momenta will project out pairs of definite parity: the first term scatters even-parity singlets while the second scatters odd-parity triplets, represented as

$$\begin{aligned} \Psi^{S\dagger}_{\mathbf{k}} &= (c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + c^\dagger_{-\mathbf{k}\uparrow} c^\dagger_{\mathbf{k}\downarrow}), \qquad & \Psi^{S\dagger}_{\mathbf{k}} &= +\Psi^{S\dagger}_{-\mathbf{k}} \\ \Psi^{T\dagger}_{\mathbf{k}} &= (c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - c^\dagger_{-\mathbf{k}\uparrow} c^\dagger_{\mathbf{k}\downarrow}), \qquad & \Psi^{T\dagger}_{\mathbf{k}} &= -\Psi^{T\dagger}_{-\mathbf{k}}. \end{aligned} \tag{15.37}$$

In terms of these operators, the unequal spin pairing interaction takes the form

$$\begin{aligned} V_{BCS}^{\uparrow\downarrow} &= \frac{1}{4} \sum_{\mathbf{k},\mathbf{k}'} \left[V^S_{\mathbf{k},\mathbf{k}'} \Psi^{S\dagger}_{\mathbf{k}} \Psi^S_{\mathbf{k}'} + V^T_{\mathbf{k},\mathbf{k}'} \Psi^{T\dagger}_{\mathbf{k}} \Psi^T_{\mathbf{k}'} \right] \\ &= \sum_{\mathbf{k},\mathbf{k}' \in \frac{1}{2}BZ} \left[V^S_{\mathbf{k},\mathbf{k}'} \Psi^{S\dagger}_{\mathbf{k}} \Psi^S_{\mathbf{k}'} + V^T_{\mathbf{k},\mathbf{k}'} \Psi^{T\dagger}_{\mathbf{k}} \Psi^T_{\mathbf{k}'} \right], \end{aligned} \tag{15.38}$$

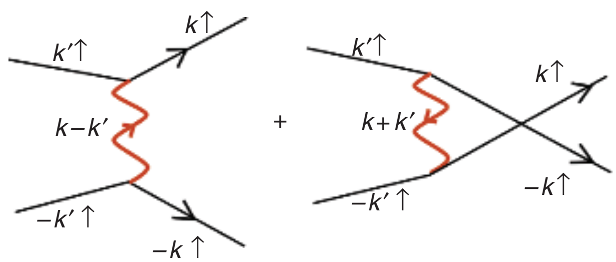


Fig. 15.3 Direct and exchange scattering of a triplet pair.

where we have restricted the sum over momentum space to one-half of momentum space, reflecting the fact that singlet and triplet pairs are only independently defined in half of momentum space ($\mathbf{k} \in \frac{1}{2}BZ$). Now the equal spin pairing terms also involve triplet pairs, which also interact via the triplet interaction $V^T_{\mathbf{k},\mathbf{k}'}$. We find

$$V^{\uparrow\uparrow}_{BCS} + V^{\downarrow\downarrow}_{BCS} = \sum_{\mathbf{k},\mathbf{k}'\in\frac{1}{2}BZ} (V_{\mathbf{k}-\mathbf{k}'} - V_{\mathbf{k}+\mathbf{k}'}) \left[(c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\uparrow})(c_{-\mathbf{k}'\uparrow} c_{\mathbf{k}'\uparrow}) + (c^\dagger_{\mathbf{k}\downarrow} c^\dagger_{-\mathbf{k}\downarrow})(c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\downarrow})\right]. \tag{15.39}$$

The appearance of scattering of amplitude $V_{\mathbf{k}-\mathbf{k}'}$ and amplitude $V_{\mathbf{k}+\mathbf{k}'}$ can be understood as a result of the exchange scattering term shown in Figure 15.3. A compact way to represent both parallel and unequal spin pair operators is to use the vector of $S = 1$ triplet pair operators:

$$\vec{\Psi}^{T\dagger}_{\mathbf{k}} = c^\dagger_{\mathbf{k}\alpha} \left(\vec{\sigma} i\sigma_2\right)_{\alpha\beta} c^\dagger_{-\mathbf{k}\beta} = \begin{cases} c^\dagger_{\mathbf{k}\downarrow} c^\dagger_{-\mathbf{k}\downarrow} - c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\uparrow}, & \text{x} \\ i(c^\dagger_{\mathbf{k}\downarrow} c^\dagger_{-\mathbf{k}\downarrow} + c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\uparrow}), & \text{y} \\ c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}, & \text{z} \end{cases} \tag{15.40}$$

refering to the x, y, and z components of the pair operator. The z component describes unequal spin pairing, while the x and y components describe linear combinations of equal spin pairing. Under a rotation, the triplet creation operator $\vec{\Psi}^{T\dagger}_{\mathbf{k}}$ transforms as a vector. In this notation, the BCS interaction is written

$$\hat{V}_{BCS} = \sum_{\mathbf{k},\mathbf{k}'\in\frac{1}{2}BZ} \left(V^S_{\mathbf{k},\mathbf{k}'} \Psi^{S\dagger}_{\mathbf{k}} \Psi^S_{\mathbf{k}'} + V^T_{\mathbf{k},\mathbf{k}'} \vec{\Psi}^{T\dagger}_{\mathbf{k}} \cdot \vec{\Psi}^T_{\mathbf{k}'}\right). \tag{15.41}$$

Note the following:

– If one is only interested in singlet pairing, one can drop the triplet pairing terms and consider the interaction

$$V_{BCS} = \sum V^S_{\mathbf{k},\mathbf{k}'} (c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})(c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}). \tag{15.42}$$

– One can decompose pairs into their orbital angular momentum components. Since the parity of a pair is related to its orbital angular momentum quantum number L by $P = (-1)^L$, even-parity superconductors involve even L (s, d, ... wave), while odd-parity triplet pairs involve L odd (p, f, ... wave).

Let us now return to consider the pair potential induced by the magnetic interaction

$$\begin{aligned} V_{mag} &= \frac{1}{2}\sum_{\mathbf{q}} J_{\mathbf{q}}\left[\vec{S}_{-\mathbf{q}}\cdot\vec{S}_{\mathbf{q}}\right] \\ &= \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} J_{\mathbf{q}} c^{\dagger}_{\mathbf{k}_1+\mathbf{q}\alpha} c^{\dagger}_{\mathbf{k}_2-\mathbf{q}\gamma}\left(\frac{\vec{\sigma}}{2}\right)_{\alpha\beta}\cdot\left(\frac{\vec{\sigma}}{2}\right)_{\gamma\delta} c_{\mathbf{k}_2\delta} c_{\mathbf{k}_1\beta}, \end{aligned} \tag{15.43}$$

where the $J_{\mathbf{q}}$ is an effective renormalized interaction between the quasiparticles. For example, in the cuprate superconductors, nearest-neighbor antiferromagnetic interactions derive from the vicinity to a Mott transition; these give rise to an antiferromagnetic interaction of the form $J_{\mathbf{q}} = 2J(\cos q_x a + \cos q_y a)$, where a is the separation of Cu atoms in a two-dimensional square lattice.

Now we need to consider the spin dependence of the interaction, determined by the matrices $\left(\frac{\vec{\sigma}}{2}\right)_{\alpha\beta}\cdot\left(\frac{\vec{\sigma}}{2}\right)_{\gamma\delta} \equiv \vec{S}_1\cdot\vec{S}_2$. We note that the eigenvalue of $\vec{S}_1\cdot\vec{S}_2$ is different for singlet and triplet states:

$$\vec{S}_1\cdot\vec{S}_2 = \begin{cases} +\frac{1}{4} & \text{(triplet)} \\ -\frac{3}{4} & \text{(singlet).} \end{cases} \tag{15.44}$$

Since the symmetric and antisymmetric parts of the interaction filter out the singlet and triplet pairs, respectively, these eigenvalues must now enter as prefactors into the pairing potentials, giving

$$\begin{aligned} V^S_{\mathbf{k},\mathbf{k}'} &= -\frac{3}{4}\left(\frac{J_{\mathbf{k}-\mathbf{k}'} + J_{\mathbf{k}+\mathbf{k}'}}{2}\right) \\ V^T_{\mathbf{k},\mathbf{k}'} &= \frac{1}{4}\left(\frac{J_{\mathbf{k}-\mathbf{k}'} - J_{\mathbf{k}+\mathbf{k}'}}{2}\right). \end{aligned} \tag{15.45}$$

Remarks

- Antiferromagnetic interactions ($J_{\mathbf{k}-\mathbf{k}'} > 0 \Rightarrow V^S_{\mathbf{k},\mathbf{k}'} < 0$) attract in the anisotropic singlet channel, whereas ferromagnetic interactions ($J_{\mathbf{k}-\mathbf{k}'} < 0 \Rightarrow V^T_{\mathbf{k},\mathbf{k}'} < 0$) attract in the triplet channel:

antiferromagnetic interaction ↔ singlet (mainly d-wave) anisotropic pairing
ferromagnetic interaction ↔ triplet (mainly p-wave) anisotropic pairing.

- The idea that ferromagnetic interactions could drive triplet pairing in nearly ferromagnetic metals, such as palladium, was first proposed by Layzer and Fay in the early 1970s [2]. In 1986 the discovery of antiferromagnetic spin fluctuations in the heavy-fermion superconductor UPt_3 led three separate groups (Zazie Béal Monod, Claude Bourbonnais, and Victor Emery at Orsay, Sherbrooke, and Brookhaven National Laboratory [3]; Kazu Miyake, Stefan Schmitt-Rink, and Chandra Varma at Bell Laboratories [4]; and Douglas Scalapino, Eugene Loh, and Jorge Hirsh at the University of California, Santa Barbara [5]), to propose that antiferromagnetic fluctuations drive d-wave superconductivity.

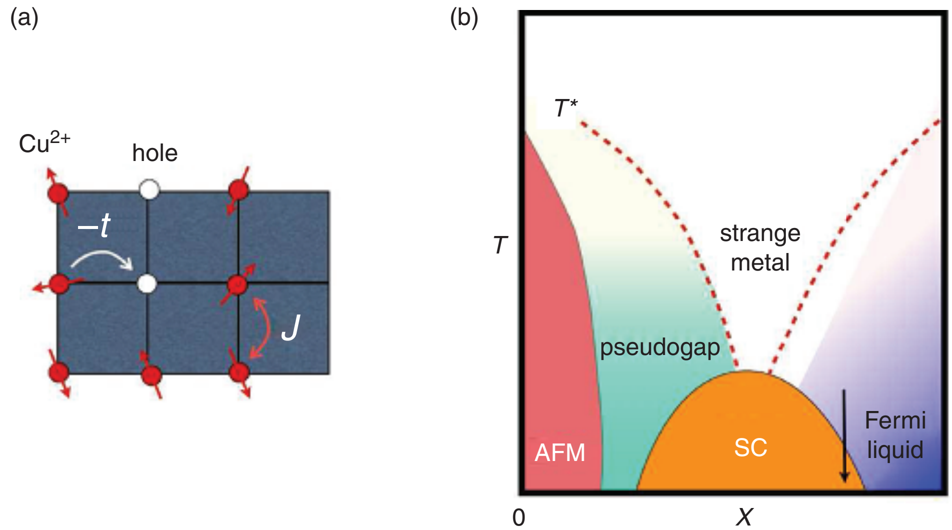


Fig. 15.4

(a) Superconductivity in cuprate superconductors involves the two-dimensional motion of electrons on a square lattice. The undoped material contains a square lattice of Cu^{2+} ions, each carrying a localized $S = \frac{1}{2}$ moment. When holes (or electrons) are introduced into the lattice via doping, the spins become mobile and the residual antiferromagnetic interactions drive d-wave pairing. A simplified model treats this as a single band of electrons of concentration $1 - x$, moving on a square lattice with hopping strength $-t$ and nearest-neighbor antiferromagnetic interaction J. (b) Schematic phase diagram of cuprate superconductors where x is the degree of hole doping. A commensurate antiferromagnetic insulator (pink) forms at small x, while at higher doping a superconducting dome develops. The normal state contains a pseudogap at low doping, forming a strange metal at optimal doping, with a linear resistivity. Fermi-liquid-like properties only develop at high doping, and it is only in this regime that the superconducting instability can be treated as a bona-fide Cooper pair instability of a Fermi liquid.

The basic connection between anisotropic singlet superconductivity and antiferromagnetic interactions is relevant to a wide variety of superconductors.

- If one is only interested in anisotropic singlet pairing, it is sufficient to work with an interaction of the form

$$V_{BCS} = -\frac{3}{4}\sum_{\mathbf{k},\mathbf{k}'}\left(\frac{J_{\mathbf{k}-\mathbf{k}'} + J_{\mathbf{k}+\mathbf{k}'}}{2}\right)(c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow})(c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow}).$$

15.4 d-wave pairing in two-dimensions

One of the most dramatic examples of anisotropic pairing is provided by the d-wave pairing in the copper oxide layers of the cuprate superconductors. These materials form antiferromagnetic Mott insulators, but when electrons or holes are introduced into the layers by doping, the magnetism is destroyed and the doped Mott insulator develops d-wave superconductivity. The normal state of these materials is not well understood, and for most of the phase diagram it cannot be treated as a Fermi liquid. For instance, at optimal doping these materials exhibit a linear resistivity $\rho(T) = AT + \rho_0$ due to electron–electron scattering that cannot be understood within the Fermi liquid framework. However, in the over-doped materials Fermi liquid behavior appears to recover and a BCS treatment is thought to be applicable.