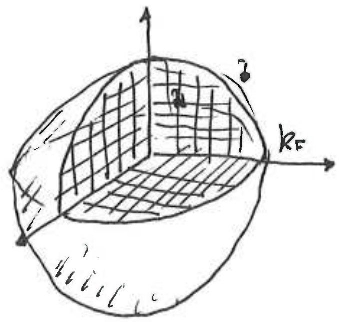
The Fermi Gas



Fermi sphere

$$\boxed{\begin{aligned} \frac{N}{V} &= g \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\epsilon\beta} z^{-1}+1} \\ \frac{F}{V} = -P &= -g k_B T \int \frac{d^3k}{(2\pi)^3} \ln\left(1+ z e^{-\epsilon\beta}\right) \end{aligned}}$$

$z = e^{\beta\mu}$

As in the case of the Bose gas, by making the substitution

$$x = \frac{\hbar^2 k^2}{2m k_B T} = \epsilon_k \beta$$

we can replace

$$\int \frac{d^3k}{(2\pi)^3} \to \int \frac{k^2 dk}{2\pi^2} \longrightarrow \frac{1}{\lambda_T^3} \frac{2}{\sqrt{\pi}} \int dx \sqrt{x}$$

Thus the density of particles is given by

$$\boxed{\frac{N}{V} = g \frac{1}{\lambda_T^3} f_{3/2}(z)}$$

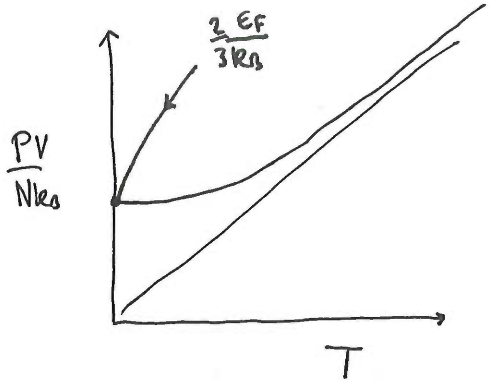
$$\Gamma(3/2) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$f_{3/2}(z) = \frac{1}{\Gamma(3/2)} \int dx \sqrt{x} \frac{1}{e^x z^{-1}+1}$$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int \frac{dx\, x^{\nu-1}}{e^x z^{-1}+1}.$$

Similarly

$$P = g \frac{k_B T}{\lambda_T^3 \Gamma(3/2)} \int dx \sqrt{x} \ln\left(1+z^{-1}e^{-x}\right) = g \frac{k_B T}{\lambda_T^3 \underbrace{\tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right)}_{\frac{3}{4}\sqrt{\pi}}} \overbrace{\int \frac{dx\, x^{3/2}}{e^x z^{-1}+1}}^{f_{5/2}(z)} = \boxed{g \frac{k_B T}{\lambda_T^3} f_{5/2}(z)}$$

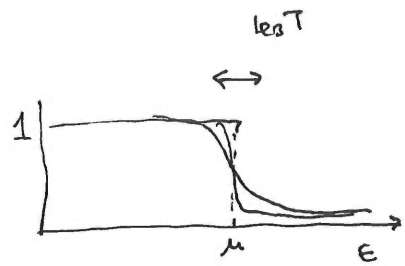


$$\boxed{\frac{PV}{Nk_BT} = \frac{f_{5/2}(z)}{f_{3/2}(z)}}$$

Note also $P = \frac{2U}{3V}$ as in classical gas!

High T $\quad z \to 0 \qquad PV = Nk_BT$

$f_\nu(z) \to z$



Low T

$$f_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}(\ln z)^\nu$$

$$n_F(\epsilon) = \frac{1}{e^{(\epsilon-\mu)\beta}+1}$$

$\mu(T\to 0) = \epsilon_F \qquad$ "Fermi energy"

$$\frac{N}{V} = g\,\frac{\frac{4\pi}{3}k_F^3}{(2\pi)^3} = \frac{g}{6\pi^2}k_F^3$$

$$\Rightarrow \quad k_F = \left(\frac{6\pi^2 n}{g}\right)^{1/3}$$ Fermi wavevector

$$p_F = \hbar\left(\frac{6\pi^2 n}{g}\right)^{1/3}$$ Fermi momentum.

$$\epsilon_F = \frac{(\hbar k_F)^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$ Fermi energy.

$$\frac{P}{n} = \frac{k_BT\,\Gamma(5/2)}{\Gamma(7/2)}\ln z = \frac{2}{5}\epsilon_F$$

<u>Summarising</u>

$$\frac{N}{V} = \frac{g}{\lambda_T^3} f_{3/2}(z)$$

$$P = \frac{2}{3}\frac{U}{V} = \frac{g}{\lambda_T^3} k_B T \, f_{5/2}(z)$$

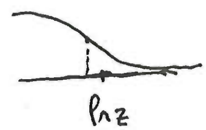
$$k_F = \left(\frac{6\pi^2 n}{g}\right)^{\frac{1}{3}}$$

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2ma^2}\left(\frac{6\pi^2}{g}\right)^{2/3}$$

$$\frac{U}{V} = \frac{3}{2} P = \frac{3}{5} \epsilon_F . \left(\frac{N}{V}\right)$$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)}\int \frac{dx\, x^{\nu-1}}{e^{x}z^{-1}+1}$$

$z \longrightarrow \infty$ $\qquad \dfrac{1}{e^{x}z^{-1}+1}$



$$f_\nu(z) \sim \frac{1}{\Gamma(\nu)}\int^{\ln z} dx\, x^{\nu-1}$$

$$= \frac{1}{\Gamma(\nu)}\,\frac{1}{(\nu)}\,(\ln z)^{\nu} = \frac{(\ln z)^{\nu}}{\Gamma(\nu+1)}$$

$$\frac{1}{e^{\alpha}+1} - 1 = \frac{-e^{\alpha}}{e^{\alpha}+1} = \frac{-1}{1+e^{-\alpha}}$$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)}\left[\int^{\xi} dx\, \frac{x^{\nu-1}}{e^{(x-\xi)}+1} + \int_{\xi}^{\infty} dx\, \frac{x^{\nu-1}}{e^{x-\xi}+1}\right]$$

$$= \frac{1}{\Gamma(\nu)}\int^{\xi} dx\, x^{\nu-1}\left[1 - \frac{1}{e^{-(x-\xi)}+1}\right] + \int_{\xi}^{\infty} dx\, \frac{x^{\nu-1}}{e^{x-\xi}+1}$$

$r = x-\xi \quad x = r+\xi$

$$= \frac{\xi^{\nu}}{\Gamma(\nu+1)} - \frac{1}{\Gamma(\nu)}\left[\int_0^{\xi} \frac{(\xi-r)^{\nu-1}\,dr}{e^{r}+1} - \int_{\xi} \frac{dr\,(r+\xi)^{\nu-1}}{e^{r}+1}\right]$$

$$\approx \frac{\xi^{\nu}}{\Gamma(\nu+1)} - \frac{1}{\Gamma(\nu)}\int_0^{\infty} \frac{dr}{e^{r}+1}\left[(\xi-r)^{\nu-1} - (\xi+r)^{\nu-1}\right]$$

$$= \frac{\xi^{\nu}}{\Gamma(\nu+1)} + \frac{2}{\Gamma(\nu)}\sum_{j=\mathrm{odd}}\binom{\nu-1}{j}\xi^{\nu-1-j}\int_0^{\infty}\frac{dn\, n^{j}}{e^{n}+1}$$

$$\int \frac{dn\, n^{j}}{(e^{n}+1)} = \int dn\, n^{j} e^{-n}\left(1-e^{-n}+e^{-2n}-\dots\right) = \Gamma(j+1)\left[1-\frac{1}{2^{j+1}}+\frac{1}{3^{j+1}}\dots\right]$$

$$\Gamma(j+1)\,\zeta(j+1)\left(1-\frac{1}{2^{j}}\right)$$

$$\frac{1}{\Gamma(\nu)}\int dn\, \frac{n^j}{e^{n}+1} = \frac{\xi^\nu}{\Gamma(\nu+1)} + \frac{2}{\Gamma(\nu+1)} \sum_{j=\text{odd}} \nu(\nu-1)\dots(\nu-j)\left(1-\frac{1}{2^j}\right)\frac{\zeta(j+1)}{\xi^{j+1}}$$

$$f_\nu(e^\xi) = \frac{\xi^\nu}{\Gamma(\nu+1)}\left[1 + \nu(\nu-1)\frac{\pi^2}{6\xi^2} + \nu(\nu-1)(\nu-2)(\nu-3)\frac{7\pi^4}{360}\frac{1}{\xi^4} + \dots\right]$$

$\ln z = \xi = (\beta\mu)$.

$$f_{1/2}(z) = \frac{2}{\sqrt{\pi}}(\ln z)^{1/2}\left[1 - \frac{\pi^2}{24}\frac{1}{\ln z^2} + \dots\right]$$

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}}(\ln z)^{3/2}\left[1 + \frac{\pi^2}{8}(\ln z)^{-2} + \dots\right]$$

$$f_{5/2} = \frac{8}{15\pi^{1/2}}(\ln z)^{5/2}\left[1 + \frac{5\pi^2}{8}(\ln z)^{-2} + \dots\right]$$

$\Gamma(5/2) = \frac{3}{2}\Gamma(\frac{3}{2})$
$= \frac{3}{4}\sqrt{\pi}$
$\Gamma(3/2) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

$$\frac{N}{V} = \frac{g}{\lambda_T^3} f_{3/2}(z) = \frac{4\pi g}{3}\left(\frac{2m}{h^2}\right)^{3/2}(k_B T \ln z)^{3/2}\left[1 + \frac{\pi^2}{8}\frac{1}{\ln z^2} + \dots\right]$$

Lowest approx $\quad k_B T \ln z = \mu \simeq \left(\frac{3N}{4\pi g V}\right)^{2/3}\frac{h^2}{2m} = \epsilon_F$

Next approx $\quad \frac{N}{V} = [\quad]\left[1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \dots\right]$

$$\Rightarrow \mu = k_B T \ln z = \epsilon_F\left(1 - \frac{\pi^2}{8}\left(\frac{k_B T}{\epsilon_F}\right)^2\right)^{2/3} \simeq \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]$$

$$\mu(T) = \epsilon_F + O(T^2)$$

$$\frac{g}{2\pi^2} k^2\, dk = \frac{g}{2\pi^2}\sqrt{\frac{2m\epsilon}{\hbar^2}}\left(\frac{m}{\hbar^2}\right)d\epsilon$$

$$= \frac{g}{\sqrt{2}\pi^2}\left(\frac{m}{\hbar^2}\right)^{3/2}\epsilon^{\frac{1}{2}}d\epsilon$$

$$\int N(\epsilon)\, d\epsilon = N$$

$$\int A\epsilon^{1/2} d\epsilon = \frac{2A\epsilon_F^{3/2}}{3} = N$$

$$A = \frac{3}{2} N_0 \frac{1}{\epsilon_F}.$$

$$N(\epsilon) = \frac{3}{2}\left(\frac{\epsilon}{\epsilon_F}\right)^{1/2}\frac{N}{\epsilon_F}$$

$$C_V = \left.\frac{d(U-\mu N)}{dT}\right|_{T\to 0} = k_B\int N(\epsilon)\frac{(\epsilon-\mu)^2}{(k_BT)^2}\frac{d\epsilon}{\left(e^{\beta(\epsilon-\mu)}+1\right)\left(e^{-\beta(\epsilon-\mu)}+1\right)}$$

$$= N(\epsilon_F)\, k_B^2\, T \underbrace{\int dx \frac{x^2}{(e^x+1)(e^{-x}+1)}}_{\pi^2/3}$$

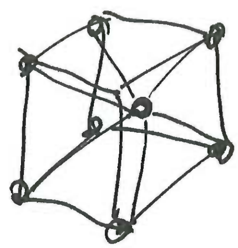
$$C_V = \frac{\pi^2}{3} k_B^2\, N(\epsilon_F)\; T$$

$$= \gamma T.$$

ELECTRON GAS IN METALS

$$\epsilon_F = \left(\frac{3\pi^2 N}{V}\right)^{2/3}\left(\frac{\hbar^2}{2m}\right) = (3\pi^2)^{2/3}\frac{\hbar^2}{2ma^2}(n_e n_a)^{2/3}$$

$$\frac{N}{V} = n_e \frac{n_a}{a^3}$$



e.g. Na $\quad Na^+ \quad$ B.C.C.

$n_e = 1 \quad n_a = 2$

$a = 4.29\,\mathring{A} \quad m^* = 0.98 m_e$

$$\epsilon_F = (3.1.2\,\pi^2)^{2/3}\left(\frac{1.04\times10^{-34}}{2\times(0.98)\times 9.1\times10^{-31}\times(4.29\times10^{-10})^2}\right)$$

$$\simeq 5\times10^{-19}\,J$$

$$\simeq 3.12\,eV$$

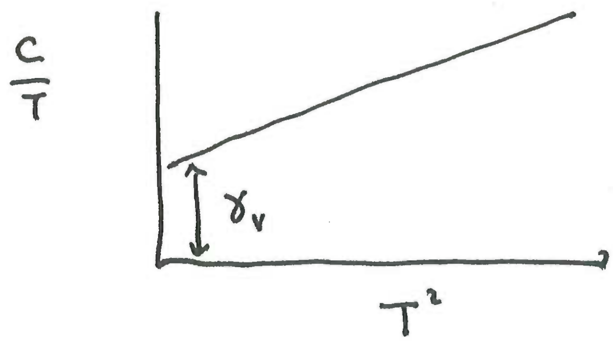
$$\simeq 36\times10^3\,K = T_F \gg 300K\,!$$

$$C_V = \frac{\pi^2}{3}N(0)\,k_B^2\,T = \gamma_v T$$

$$\boxed{\frac{k_B T}{\epsilon_F}\sim 10^{-2}}$$

$$\gamma T = \frac{\pi^2}{2}N k_B\left(\frac{k_B T}{\epsilon_F}\right) \ll \frac{3Nk_B}{2}$$

$$C_v = \gamma_v T + \delta T^3$$



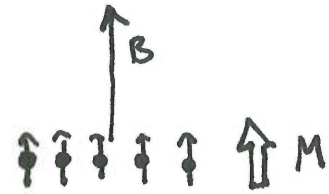
$$\gamma_v = \frac{\pi^2 N(0) k_B^2}{3} = \frac{\pi^2 N k_B^2 m^*}{\hbar^2 k_F^2} \qquad N = \frac{V k_F^3}{3\pi^2}$$

$$\gamma_v = V\left(\frac{m^* k_F}{\hbar^2}\right)\frac{k_B^2}{3}. \qquad \propto m^*$$

$\gamma_v \sim 1\,mJ/mol/K^2$ Cu $m^* \sim m_e$

$\sim 1600\,mJ$ $CeCu_6$ $m^* \sim 1000\,m_e$ "heavy fermion"

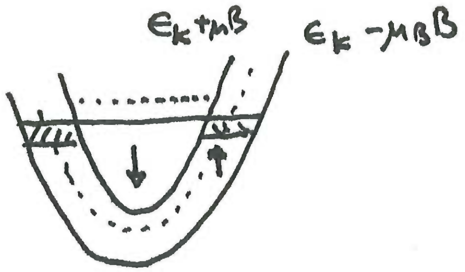
PAULI PARAMAGNETISM

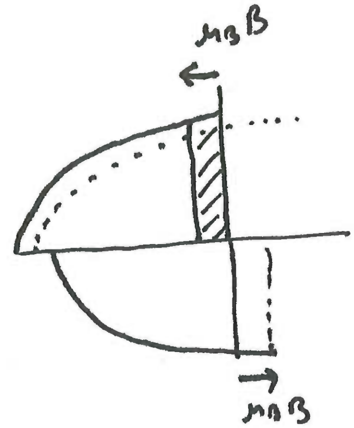


$$\vec{\mu} = g_r \left(\frac{e\hbar}{2m}\right)\left(\frac{\vec{\sigma}}{2}\right) = \vec{\mu}^*$$

$$\hat{H} = -\vec{\mu}^* \cdot B \equiv \mp \mu_B B \qquad \mu^* = \left(\frac{g_r}{2}\right)\left(\frac{e\hbar}{2m}\right)$$

$$= \mu_B \qquad \text{For electrons.}$$





$$n_\sigma(k) = \frac{1}{e^{\beta(\epsilon_k - \mu_B^* B\sigma - \mu)} + 1} = \theta\left(\mu - (\epsilon_k - \sigma\mu^* B)\right)$$

$$= \theta\left(\mu + \sigma\mu^* B - \epsilon_k\right)$$

$$N_\sigma = \frac{V\, 4\pi k_\sigma^3}{3(2\pi)^2} = V\frac{4\pi}{3\hbar^2}\left(2m(\epsilon_F + \sigma\mu^* B)\right)^{3/2}$$

$$\boxed{M = \mu^*(N_\uparrow - N_\downarrow) = \mu^*\left(\frac{N_0(\epsilon_F)}{2}\right)(2\mu^* B) = (\mu^*)^2 N(\epsilon_F) B}$$

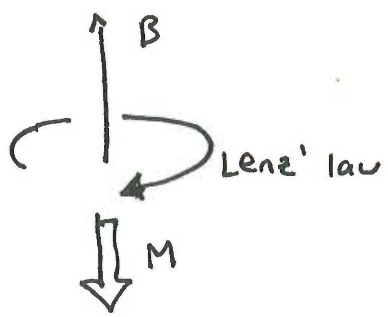
$$N(\epsilon_F) = V\left(\frac{m p_F}{\pi^2\hbar^3}\right) \qquad \chi_0^{PARA} = \frac{M}{VB} = (\mu^*)^2\left(\frac{m p_F}{\pi^2\hbar^3}\right).$$

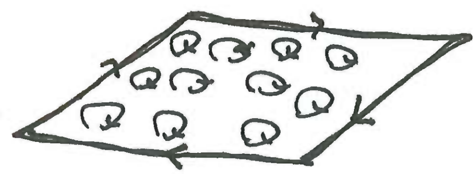
$$\chi_0 = (\mu^*)^2 \left(\frac{N_0}{V}\right) \frac{3}{2\epsilon_F}$$

$$\chi_{\text{HIGH T}} = (\mu^*)^2 \left(\frac{N_0}{V}\right) \frac{1}{k_B T} \qquad k_B T \gg \epsilon_F$$

$$\frac{\chi_0}{\chi_T} = O\left(\frac{k_B T}{\epsilon_F}\right)$$

LANDAU DIAMAGNETISM





Electron in a magnetic field

Classically

$$F = q(E + v \times B)$$

$$\frac{mv^2}{r} = qvB \Rightarrow \frac{v}{r} = \omega_c = (qB/m).$$

Quantum Mechanically

$$\hat{H} = \frac{(\vec{p} - eA)^2}{2m} \qquad \vec{B} = \nabla \times A$$

$$\vec{p} = -i\hbar\vec{\nabla} \qquad \vec{A} = Bx\,\hat{y}$$

$$A = (\;\; , Bx \; 0)$$

$$\vec{B} = (\nabla \times A) = \left(0, 0, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

2D

$$\psi(x,y) = e^{ik_y y}\,\phi(x)$$

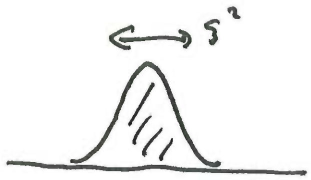
$$H\psi = \left[\left(\frac{\hbar k_y - eBx}{2m}\right)^2 - \frac{\hbar^2}{2m}\partial_x^2\right]\psi(y) = E\psi(y)$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega_c^2}{2}\left(x - \frac{\hbar k_y}{eB}\right)^2\right)\phi(y) = E\phi(y)$$

$$E = \hbar\omega_c\left(n+\tfrac{1}{2}\right)$$

$$\phi(x) = \frac{1}{2^{n/2}}\frac{1}{\sqrt{n!}}\left(\frac{x}{\xi} + \hbar\xi\frac{\partial}{\partial x}\right)^n \exp\left[-\frac{1}{2}\left(x - \frac{\hbar k_y}{eB}\right)^2\frac{1}{\xi^2}\right]$$

$$\xi = \sqrt{\frac{\hbar}{m\omega}}. \qquad \psi(x,y) = \phi(x)\, e^{ik_y y}$$



30.

$$\psi(x,y,z) = \phi(x)\, e^{i(k_y y + k_z z)}$$

$$\boxed{E = \hbar\omega_c\left(n+\tfrac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m}}$$

Electron DOS

2D $A\frac{d^2k}{(2\pi)^2} = A\frac{2\pi k\,dk}{(2\pi)^2} = A\frac{1}{2\pi}(k\,dk) = A\frac{m}{2\pi\hbar^2}\,d\epsilon$

$\frac{\hbar^2 k\,dk}{m} = d\epsilon$ $\qquad N(\epsilon) = A\left(\frac{m}{2\pi\hbar^2}\right)$ 2D

$$N(0) = \left(\frac{N_0}{\epsilon_F}\right)$$

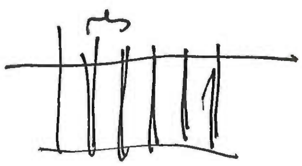
$$\left(N(0) = \left(\frac{N_0}{\epsilon_F}\right)\left(\frac{2}{d}\right)\left(\frac{\epsilon}{\epsilon_F}\right)^{\frac{(d-2)}{2}}\right)$$

$$\underbrace{A\left(\frac{m}{2\pi\hbar^2}\right)}\times(\hbar\omega) = \underline{A\left(\frac{m\omega}{2\pi\hbar}\right) = A\left(\frac{eB}{h}\right)}$$

No of electrons/ Landau Level

$$\Rightarrow \quad V\int\frac{d^3k}{(2\pi)^3} \xrightarrow{\text{Quantization of energy levels}} \underline{\frac{VeB}{h}\int\frac{dk_z}{(2\pi)}\sum_n}$$

$\left(\frac{AeB}{h}\right)$



$$\boxed{F = gV\frac{eB}{h}\sum_n\int\left(\frac{dk_z}{2\pi}\right)\left[-k_BT\ln\left(1+e^{-\frac{\epsilon(k_z,n)\beta}{z}}\right)\right]}$$

$$\frac{e^{-x/2}}{1-e^{-x}}$$

High Temperature

$$\frac{F}{-k_BT} = \frac{gVz\,eB}{h}\int\frac{dk_z}{2\pi}\,e^{-\frac{\hbar^2k^2}{2}\beta}\sum_n e^{-\frac{\hbar\omega}{k_BT}(n+\frac{1}{2})}$$

$$\frac{F}{-k_BT} = g\frac{V\,eB}{h^2}\sqrt{2\pi m k_BT}\;\frac{1}{2\sinh\left(\beta\frac{\hbar\omega}{2}\right)}\;e^{-\beta\mu}$$

$$N = \frac{-\partial F}{\partial\mu} = [\quad]$$

$F = -Nk_BT$

$$F = -k_BT\;gz\left(\frac{V}{\lambda^3}\right)\left(\frac{x}{\sinh x}\right)$$

$$N = -\frac{\partial F}{\partial\mu} = gz\left(\frac{V}{\lambda^3}\right)\frac{x}{\sinh x}$$

$$N = g\frac{VeB}{h}\frac{1}{\lambda_T}\frac{\left(\frac{\beta\hbar\omega}{2}\right)}{\cancel{x}\sinh\left(\frac{\beta\hbar\omega}{2}\right)}\frac{\cancel{2}}{\beta\hbar\omega}$$

$$= g\frac{VeB}{h}\frac{1}{\lambda_T}\left(\frac{k_BT}{\hbar\omega}\right)\left(\frac{x}{\sinh x}\right)$$

$$= g\frac{VeB}{h}\frac{1}{\lambda_T}\left(\frac{2\pi k_BT m}{h\omega m}\right)\left(\frac{x}{\sinh x}\right)$$

$$N = gz\left(\frac{V}{\lambda_T^3}\right)\left(\frac{x}{\sinh x}\right)$$

$$\lambda_T = \frac{h}{\sqrt{2\pi m k T}}$$

$$x = \frac{\hbar\omega\beta}{2}$$

$$M = -\frac{\partial F}{\partial B} \quad g\frac{V}{\lambda_T^3}(+\cancel{k_BT})\left[\frac{1}{\sinh x} - \frac{x\cosh x}{\sinh^2 x}\right]\left(\frac{\hbar e}{2m\cancel{k_BT}}\right) = \left(\frac{\hbar eB}{2mk_BT}\right)$$

$$\left(\frac{\hbar e}{2mk_BT}\right)\left(\frac{1}{\sinh x} - \frac{x\cosh x}{\sinh^2 x}\right)$$

$$M = -\mu_B N\left[\coth x - \frac{1}{x}\right]$$

$$\left(\frac{e\hbar}{2m}\right) \quad \frac{e\hbar}{2m} = \mu$$

$$= -\mu_B N\, L(x)$$ DIAMAGNETISM

$$x - \frac{x^3}{3\cdot 2}$$

$$\hbar\omega_c \ll k_BT$$

$$\frac{1+x^2/2}{x\left(1+\frac{x^2}{6}\right)} - \frac{1}{x}$$

$$M = \mu_B N\left(\frac{\hbar\omega_c}{2k_BT}\right)\frac{1}{3}$$

$$= \frac{1+x^2/2\left(1-\frac{x^2}{6}\right)}{x} - \frac{1}{x} = +\frac{x}{3}.$$

8.16

$$\frac{\hbar\omega_c}{2} = \left(\frac{\hbar e}{2m}\right) B = \mu_B B.$$

$$M = -N\frac{\mu_B^2}{3k_BT}$$

mass rel $\mu_B \to \mu_B'$

$$\chi_{TOT} = \frac{N}{k_BT}\left(\mu_B^2 - \frac{\mu_B'^2}{3}\right).$$

All temps.

$$\sum_{j=0}^{\infty} f(j+\tfrac{1}{2}) \simeq \int dx \left[f(x)\right] + \frac{f'(0)}{24}$$

j

$$f(x) = f(x_j) + f'(x_j)(x-x_j) + f''(x_j)\frac{(x-x_j)^2}{2}$$

$$\int dx\, f(x) = f(x_j) + f''(x_j)/24$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{2}\,dx$$

$$\frac{2}{6}\left[\frac{1}{2}\right]^3 = \frac{1}{24}$$

$$\int dx\, f(x) = \sum f(x_j) + \sum \frac{f''(x_j)}{24}$$

$$\int dx\, f(x) \simeq \sum f(x_j) + \frac{1}{24}\int dx\, f'(x_j)$$

$$= \sum f(x_j) - \frac{1}{24} f'(0).$$

$$dy = \left(\frac{\hbar^2 k\,dk\,\beta}{m}\right) \qquad k = \sqrt{\frac{2m}{\hbar^2\beta}\,y}$$

$$dk = \sqrt{\frac{2m}{\hbar^2\beta}}\,\frac{1}{2y^{1/2}}$$

$$\mu_B^* B = \left(\frac{e\hbar}{2m^*}B\right)$$

$$\frac{gVeB}{24h}\,\hbar\omega_c = \frac{gVeB}{(24)\,h}\left(\frac{eB}{m}\right)$$

$$= \frac{gV}{12h}\,eB\,(\mu_B B)$$

$$\frac{\hbar\omega_c}{24} = \left(\frac{\hbar eB}{2m}\right)\left(\frac{1}{12}\mu_B B\right)$$

$$F = F_0(T) + \frac{gVeB}{h}\int \frac{dk_z}{2\pi}\,\frac{\mu_B B}{12}\,\frac{1}{\left(e^{\epsilon\beta}z + 1\right)}$$

$$= F_0(T) + \left(\frac{gVeB}{h}\right)\frac{\mu_B B}{(12)(2\pi)}\,\frac{\sqrt{\pi}\,m}{\lambda_T}\left(\int \frac{dy\,y^{-1/2}}{e^y z^{-1} + 1}\right)$$

$$= F_0(T) + \frac{m\,g\,V\mu_B B}{h^2}\left(\frac{\mu_B B}{6}\right)\frac{\sqrt{\pi}}{\lambda_T}\int \frac{dy\,y^{-1/2}}{e^y z^{-1}+1}$$

$$\frac{eB}{h} = \frac{e\hbar B}{2m}\left(\frac{2m}{\hbar^2}\right)\frac{1}{\hbar}$$

$$f_{\frac{1}{2}}(z)$$

$$\frac{\sqrt{2m\,\cancel{\beta}\,(2\pi)^2}}{h}\,\frac{1}{2y^{1/2}} = \frac{\sqrt{2\pi k_B T m}\;\sqrt{\pi}}{h\,y^{1/2}}$$

$$\frac{\sqrt{2}\,\pi}{6} \qquad \frac{\sqrt{\pi}\,\sqrt{2\pi}}{(12)\,2\pi}$$

$$= \frac{1}{\sqrt{2}(12)} = \qquad \frac{\sqrt{\pi}\,\sqrt{2\pi}}{6} = \frac{\sqrt{2}\,\pi}{6}$$

$$\frac{(2\pi m k_B T)}{h^2} \times \frac{1}{2\pi k_B T}$$

$$F_0(T) + g\frac{V(\mu_B B)^2}{k_B T\ \lambda_T^3}\frac{\sqrt{\pi}}{6(2\pi)} \times \text{Integral}$$

$$\frac{(2\pi m)^{3/2}\sqrt{\pi}}{6(2\pi)}$$

$$F(T) = F_0(T) + \frac{g}{12}\frac{V}{\lambda_T^3}\frac{(\mu_B B)^2}{k_B T} \qquad f_{\frac{1}{2}}(z)$$

$$F_0(T) = -gV\frac{(k_B T)}{(\lambda_T^3)} f_{5/2}(z)$$

$$= \frac{1}{6}\frac{\pi^2}{2\pi} 2^{3/2}$$

$$= \frac{2^{1/2}\pi}{6}$$

$$\frac{1}{6\ 2\sqrt{\pi}}$$

$$\left(\tfrac{1}{2}-1\right)$$

$$-\frac{\partial F}{\partial B} = M = -\left(\frac{g\ V\ \mu_B^2 B}{6\ \lambda_T^3 (k_B T)}\right) f_{\frac{1}{2}}(z) = -\frac{V}{3\lambda_T^3}\frac{\mu_B^2}{(k_B T)} f_{\frac{1}{2}}(z) B \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\chi_{DIA}(T) = \frac{-gV}{6(\lambda_T)^3}\frac{\mu_B^2}{(k_B T)} f_{\frac{1}{2}}(z)$$

$$N = \frac{V}{\lambda_T^3} f_{3/2}(z)$$

$$\chi_{DIA} = -\frac{g}{6}\frac{N}{V}\left(\frac{f_{1/2}(z)}{f_{3/2}(z)}\right)\frac{\mu_B^2}{(k_B T)}$$

$$z \to \infty \qquad f_{1/2}(z) \sim \frac{(\ln z)^{1/2}}{\Gamma(3/2)}$$

$$f_{3/2}(z) \sim \frac{(\ln z)^{3/2}}{\Gamma(5/2)} = \frac{(\epsilon_F \beta)^{3/2}}{3/2\ \Gamma(3/2)}$$

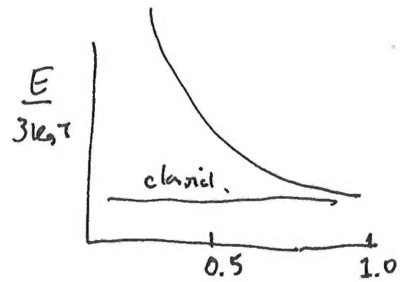
$$\chi_{DIA} = -\frac{1}{2}\left(\frac{n\,\mu_B^2}{\epsilon_F}\right)$$

$$N^*(0) = n\left(\frac{3}{2}\right)\frac{1}{\epsilon_F}$$

$$\boxed{\chi_{TOT} = N(0)\left[\mu_B^2 - \frac{\mu_B'^2}{3}\right]}$$

$$\chi_{DIA} = -\frac{N^*(0)\ \mu_B'^2}{3}$$

8.4 ULTRACOLD ATOMIC GASES



$$E = \hbar\omega_0\left(n_x+n_y+n_z+\tfrac{3}{2}\right) \quad \text{or} \quad \hbar\omega_x\left(n_x+\tfrac{1}{2}\right) + \hbar\omega_y\left(n_y+\tfrac{1}{2}\right) + \hbar\omega_z\left(n_z+\tfrac{1}{2}\right)$$

$$g(\epsilon) = \frac{\epsilon^2}{2(\hbar\omega_0)^3} \qquad \omega_0 = (\omega_x\omega_y\omega_z)^{1/3} \qquad /\text{spin}$$

$$g(\epsilon) = \int dn_x\, dn_y\, dn_z\ \delta(\hbar\omega_e n_e - \epsilon)$$

$$= \int \frac{d\epsilon_x\, d\epsilon_y\, d\epsilon_z}{(\hbar\omega_x)(\hbar\omega_y)(\hbar\omega_z)}\ \delta(\epsilon_x+\epsilon_y+\epsilon_z-\epsilon)$$

$$= \int^{\epsilon_x+\epsilon_y<\epsilon} \frac{d\epsilon_x\, d\epsilon_y}{(\hbar\omega_0)^3} = \frac{\epsilon^2}{2(\hbar\omega_0)^3}$$

$\Gamma(2) = 1$
$\Gamma(1) = 0!$
$\Gamma(3) = 2\Gamma(2) = 2$

$$\frac{1}{\Gamma(\nu)} \quad x^{\nu-1}$$

$$N(\mu,T) = \frac{1}{2(\hbar\omega)^3}\int \frac{\epsilon^2 d\epsilon}{e^{\beta(\epsilon-\mu)}+1} = \frac{f_3(z)\,(k_BT)^3}{(\hbar\omega_0)^3} \quad \underline{\underline{= 3N\left(\frac{T}{T_F}\right)^3 \int \frac{dx\, x^2}{e^x z^{-1}+1}}}$$

$$\frac{\epsilon_F^3}{6(\hbar\omega_0)^3} = N \Rightarrow \epsilon_F = \left(6(\hbar\omega_0)^3 N\right)^{1/3} = \hbar\omega_0\,(6N)^{1/3}$$

$$U_0 = \frac{1}{2(\hbar\omega)^3}\int^{\epsilon<\epsilon_F} \epsilon^3 d\epsilon = \frac{\epsilon_F^4}{8(\hbar\omega)^3} = \frac{3}{4} N\epsilon_F.$$

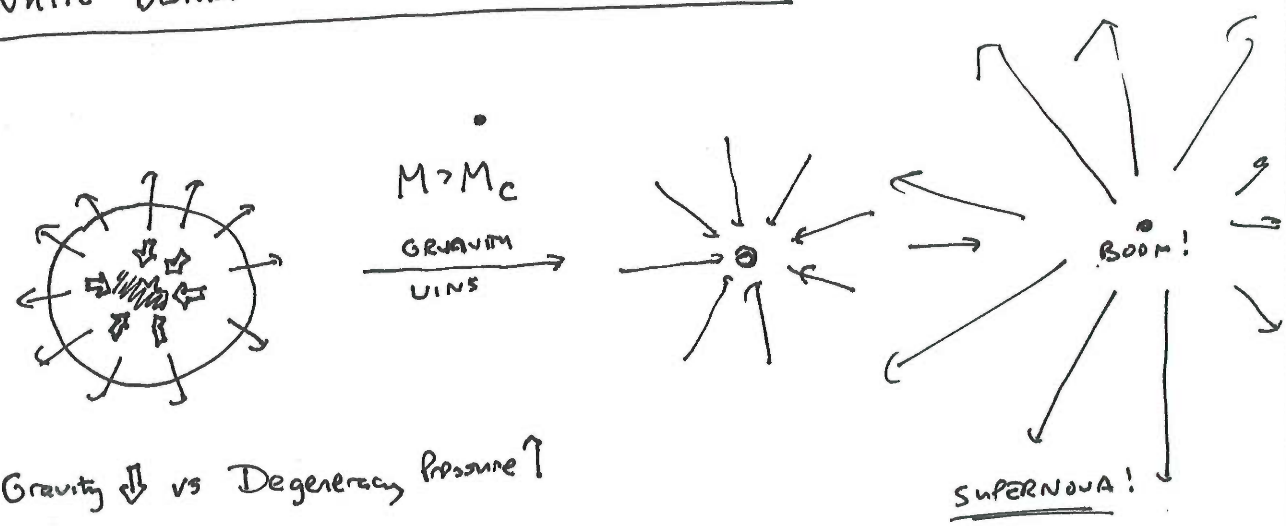
$$U(T) = \frac{1}{2(\hbar\omega)^3}\int \epsilon^3 d\epsilon\, \frac{1}{e^{(\beta\epsilon-\mu)}+1} = \frac{(k_BT)^4}{2(\hbar\omega)^3}\int \frac{x^3 dx}{(e^x z^{-1}-1)}$$

$$\frac{U}{U_0} = 4\left(\frac{T}{T_F}\right)^4 \int \frac{x^3 dx}{e^x z^{-1}-1} = 24\left(\frac{T}{T_F}\right)^4 f_4(z)$$

$$3\left(\frac{T}{T_F}\right)^3 \int \frac{dx\, x^2}{e^x z^{-1}+1} = 1$$

$$\frac{U}{U_0} = 4\left(\frac{T}{T_F}\right)^4 \int \frac{dx\, x^3}{e^x z^{-1}+1}.$$

8.5 WHITE DWARFS + GRAVITATIONAL COLLAPSE



Gravity ⇩ vs Degeneracy Pressure ↑

Everything on earth, our bodies, photosynthesis, breathing, depends on heavy elements, and these are all the product of supernovae that occur when Gravity overcomes the degeneracy pressure of the electrons inside a collapsing star.

$$He \rightleftharpoons He^{2+} + 2e^- \qquad 4 \text{ nucleons} + 2e^-$$

$$n \equiv n_e = \frac{N_e}{V} = \frac{1}{2}\frac{N_n}{V} = \frac{M}{2Vm_p} .$$

$$p_F = \hbar\left(3\pi^2 n\right)^{1/3} \quad \longleftrightarrow \quad \frac{k_F^3}{3\pi^2} = n$$

$n \sim 10^{36}\ e^-/m^3$ $\qquad p_F \sim 10^{-22}\ kg\,m/s$.

$M \sim 10^{30}$ kg He. $\qquad k_F \sim 10^{12}\ m^{-1}$

$\rho \sim 10^{10}\ kg/m^3$

$T \sim 10^7\ K \sim 10^3\ eV$ / particle $\gg$ ionization energy of He.

- $p_F / m_e c = 10^{-22} / 9.1\times10^{-31} \times 3\times10^{8} \sim O(1)$
- $\epsilon_F \sim m_e c^2 \sim O(10^6 eV)$
- $T_F \sim \left(\frac{e}{k_B}\right)\times 10^6 = 10^{10} K. \gg T$

DEGENERATE $\boxed{\frac{T}{T_F} \sim 10^{-3}}$

Relativistic Fermi Gas.

Fowler

Anderson 1928

Stoner 1929, 30

Chandrasekhar 1931-1935.

- As before $N = 2V \int \frac{d^3k}{(2\pi)^3} = \frac{1}{3\pi^2} k_F^3 V$

$$\Rightarrow \quad k_F = (3\pi^2 n)^{1/3} = \left(\frac{9\pi M}{8 m_p R^3}\right)^{1/3}$$

- Total energy of relativistic electron.

$$\epsilon = \sqrt{(mc^2)^2 + (pc)^2}$$

- Thermodynamics

$$F = -PV = -2k_B T V \sum_k \ln\left(1 + e^{-\beta(\epsilon-\mu)}\right)$$

$T=0$

$$E = V \int^{k_F} \frac{k^2 dk}{\pi^2} \sqrt{(mc^2)^2 + (\hbar c k)^2}$$

$$E_g \approx V \frac{\hbar}{\pi^2} \int_0^{k_F} k^2 dk \; ck \left(1 + \frac{1}{2}\left(\frac{mc}{\hbar k}\right)^2 + \dots \right)$$

$$= V \frac{\hbar c}{\pi^2} \frac{k_F^4}{4} \left(1 + \left(\frac{mc}{\hbar k_F}\right)^2 + \dots \right)$$

$$= V \frac{mc^2}{4\pi^2} \left(\frac{mc}{\hbar}\right)^3 \left(\frac{\hbar k_F}{mc}\right)^4 \left(1 + \left(\frac{mc}{\hbar k_F}\right)^2 \right)$$

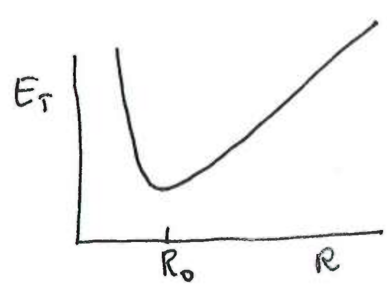
$$\hbar/mc = \lambda\!\!\!^{-}_{\text{COMPTON}}.$$

Now. $$\left(\frac{\hbar k_F}{mc}\right) = \frac{\hbar}{mc}\left(\frac{9\pi M}{8 m_p R^3}\right)^{1/3}$$

$$E_g = \frac{mc^2}{4\pi^2}\left(\frac{4\pi R^3}{3}\right)\left(\frac{\hbar}{mc}\right)\left[\left(\frac{9\pi M}{8 m_p R^3}\right)^{4/3} + \left(\frac{9\pi M}{8 m_p R^3}\right)^{2/3}\left(\frac{mc}{\hbar}\right)^2\right]$$

$$= \frac{mc^2}{3\pi}\left[\left(\frac{9\pi}{8 m_p}\right)^{4/3}\frac{M^{4/3}}{R} + \left(\frac{9\pi}{8 m_p}\right)^{2/3}\left(\frac{R}{\lambda\!\!\!^{-2}}\right)M^{2/3}\right]\lambda\!\!\!^{-}$$

$$E_{\text{gravitational}} \simeq -\frac{3}{5}\frac{GM^2}{R}$$



$$E_{TOT} = \frac{A-B}{R} + CR$$

$$A - B = \frac{mc^2}{3\pi}\left(\frac{9\pi}{8}\right)^{4/3}\left(\frac{M}{m_p}\right)^{4/3}\left(\frac{\hbar}{mc}\right) - \frac{3}{5}GM^2$$

No

$$C = \frac{mc^2}{3\pi}\left(\frac{9\pi}{8}\right)^{4/3}\left(\frac{1}{\lambda\!\!\!^{-}}\right)$$

Note: $(\hbar c/G)^{1/2}$ = Planck Mass = 21.8μg

$A - B > 0$ for stability.

$$\Rightarrow \quad M < M_c = \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2} \times \frac{15}{64}(5\pi)^{1/2}$$

$$= \frac{15}{64}(5\pi)^{1/2} \frac{\left(4.37 \times 10^{-16}\,kg^2\right)^{3/2}}{\left(1.67 \times 10^{-27}\right)^2}$$

$m_p = 1.67 \times 10^{-27}$

$M_\odot = 1.98 \times 10^{30}\,kg$

$$= 3.42 \times 10^{30}\,kg = 1.72\,M_\odot$$

Better calcn $\quad M_c = 1.44 M_\odot$ $\quad$ CHANDRASEKHAR LIMIT

R_{min} : $\quad \frac{\partial E}{\partial R} = 0 \Rightarrow -\frac{(A-B)}{R^2} + C = 0$

$$\Rightarrow R^2 = \frac{A-B}{C}. = \left(\frac{9\pi}{8}\right)^{2/3} \lambda^2 \left(\frac{M}{m_p}\right)^{2/3} \left(1 - \left(\frac{M}{M_c}\right)^{2/3}\right)$$

$$\Rightarrow \boxed{R = \left(\frac{9\pi}{8}\right)^{1/3} \lambda \left(\frac{M}{m_p}\right)^{1/3} \left(1 - \left(\frac{M}{M_c}\right)^{2/3}\right)^{1/2}}$$