7.1 Thermodynamics of Bose Einstein Condensation

$$F = -k_B T \ln \mathcal{Z} = -PV = +\sum_k k_B T \ln\left(1 - z e^{-\beta\epsilon_k}\right)$$

$$z = e^{\mu/k_B T}$$

$$N = \sum_k \frac{1}{z^{-1} e^{\beta\epsilon_k} - 1}$$

$$N_0 = \frac{1}{z^{-1} - 1}$$ needs special attention because if $\mu \to 0^-$ $z^{-1} \to 1^+$, $N_0 \to \infty$

Require $z^{-1} > 1 \Rightarrow \mu < 0$

We separate out the $\vec{k} = 0$ component for special attention, then let $V \to \infty$

$$N = V \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\epsilon_k} z^{-1} - 1} + \overbrace{\underbrace{\frac{1}{z^{-1} - 1}}_{\text{non-negligible}}}^{N_0}$$

Also

$$P = -\int \frac{d^3k}{(2\pi)^3} k_B T \ln\left(1 - z e^{-\beta\epsilon_k}\right) - \underbrace{\frac{1}{V} k_B T \ln(1 - z)}_{-\frac{1}{V} k_B T \ln \frac{1}{N_0} \sim O\left(\frac{1}{V}\right)}$$

negligible.

$$z = \frac{1}{1 + \frac{1}{N_0}} \simeq 1 - \frac{1}{N_0}$$

$$x = \beta\epsilon_k = \frac{\beta\hbar^2 k^2}{2m} \qquad dx = \frac{2\beta\hbar^2}{2m} k\,dk$$

$$\frac{d^3k}{(2\pi)^3} \longrightarrow \frac{4\pi k^2 dk}{(2\pi)^3} = \frac{1}{2\pi^2}\frac{1}{2}\left(\frac{2m}{\beta\hbar^2}\right)^{3/2}\sqrt{x}\,dx$$

$$= \frac{1}{4\pi^2}\left(\frac{2\pi m k_B T}{h^2\ \pi/(2\pi)^2}\right)^{3/2}\sqrt{x}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{\lambda_T^3}\sqrt{x}\,dx \qquad (4\pi)^{3/2} = 2^3\pi^{3/2}$$

$$\frac{N}{V} = \frac{1}{\lambda_T^3}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dx\,\sqrt{x}}{(e^x z^{-1} - 1)} + \frac{1}{V}\frac{1}{z^{-1}-1}$$

$$P = -\frac{2}{\sqrt{\pi}}\left(\frac{k_BT}{\lambda_T^3}\right)\int_0^\infty dx\,\sqrt{x}\,\ln(1 - ze^{-x}) = +\frac{2}{3}\frac{2}{\sqrt{\pi}}\left(\frac{k_BT}{\lambda_T^3}\right)\int \frac{dx\ x^{3/2}}{(e^x z^{-1} - 1)}$$

$$\int dx: \ \frac{2}{3}x^{3/2} \qquad \frac{\partial}{\partial x}: \ \frac{+ze^{-x}}{1 - ze^{-x}} = \frac{1}{e^x z^{-1} - 1}$$

Convenient to use

$$g_n(z) = \frac{1}{\Gamma(n)}\int \frac{x^{n-1}\,dx}{e^x z^{-1} - 1}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(5/2\right) = \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \frac{3}{2}\frac{\sqrt{\pi}}{2}$$

$$g_n(z) = z + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \ldots$$

$$\frac{N}{V} = \underbrace{\frac{1}{\lambda_T^3} g_{3/2}(z)}_{N_n/V} + \underbrace{\frac{1}{V(z^{-1}-1)}}_{N_0/V}$$

$$P = \frac{k_BT}{\lambda_T^3} g_{5/2}(z)$$

$T > T_{BE}$ $\quad z^{-1} \sim O(1)$

$$\left.\begin{aligned} \frac{N}{V} &= \frac{1}{\lambda_T^3}\, g_{3/2}(z) \\ \frac{P}{k_B T} &= \frac{1}{\lambda_T^3}\, g_{5/2}(z) \end{aligned}\right\}$$

parametric equations.

$$\frac{PV}{Nk_BT} = \frac{g_{5/2}(z)}{g_{3/2}(z)}$$

High T $\quad z \to 0 \quad g_n(z) \sim z \qquad \dfrac{P}{k_BT} = \dfrac{N}{V} = \dfrac{1}{\lambda_T^3}\, z$.

As T reduces $z \to 1$. $\quad \mu \to 0$

T_{BE}:

$$\frac{P}{k_B T_0} = \frac{1}{\lambda_T^3}\, g_{5/2}(1) \qquad \frac{N}{V} = \frac{1}{\lambda_{T_{BE}}^3}\, g_{3/2}(1)$$

$$g_{3/2}(1) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots = \zeta\left(\tfrac{3}{2}\right) = 2.612.$$

$$\frac{1}{\lambda_T^2} = \frac{2\pi m k_B T_{BE}}{h^2} = \left[\frac{N}{V\, g_{3/2}(1)}\right]^{2/3}$$

$$\boxed{k_B T_{BE} = \frac{\hbar^2}{ma^2} \times \frac{2\pi}{\left[g_{3/2}(1)\right]^{2/3}} = 3.31\left(\frac{\hbar^2}{ma^2}\right)}$$

$$P_{BE} = \frac{Nk_BT_{BE}}{V} \times \frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{N}{V}\, k_B T_{BE} \overbrace{\left[\frac{\zeta(5/2)}{\zeta(3/2)}\right]}^{0.513}$$

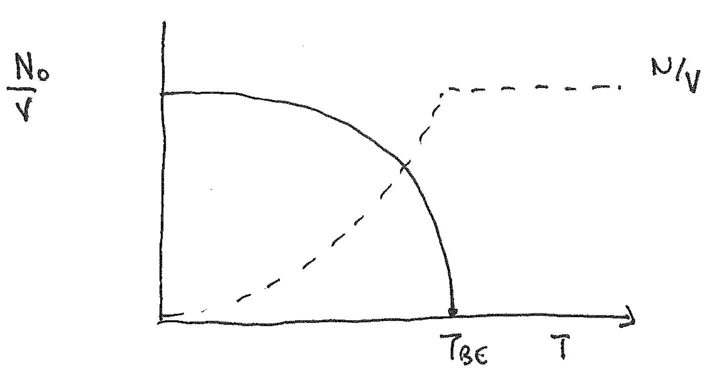
For $T < T_{BE}$

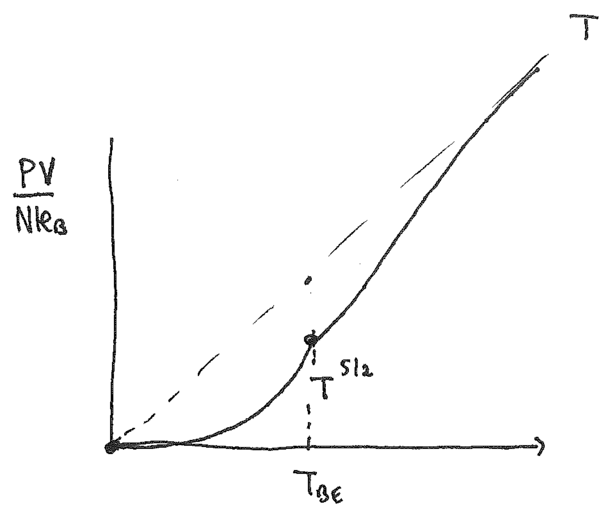
$$\frac{N}{V} = T^{3/2}\left(\frac{2\pi m k_B}{h^2}\right)^{3/2} g_{3/2}(1) + \frac{N_0}{V}$$

$$= \left(\frac{T}{T_{BE}}\right)^{3/2}\left(\frac{N}{V}\right) + \frac{N_0(T)}{V}$$

$$\Rightarrow \qquad N_0(T) = N\left(1 - \left(\frac{T}{T_{BE}}\right)^{3/2}\right)$$

$$P(T) = P_{BE}\left(\frac{T}{T_{BE}}\right)^{5/2}$$





Parametric Plot of P (T) for a Bose Gas

The parametric equations for the pressure and temperature are given by
(1) Above the BEC transition ($T > T_{BE}$)

$$PV/(Nk_B T_{BE}) = (T/T_{BEC})\, g_{5/2}(z)/g_{3/2}(z), \quad (T/T_{BE}) = (g_{3/2}(1)/g_{3/2}(z))^{2/3},$$

where $z \in [0, 1]$ and

$$g_n[z] = (1/\Gamma[n]) \int_0^\infty \frac{x^{n-1}}{(e^x z^{-1} - 1)}\, dx,$$

(2) $T < T_{BEC}$

$$P/P_{BEC} = (T/T_{BEC})^{5/2}.$$

```
(* Define g_n[z] *)
g[n_, z_] := (1 / Gamma[n]) NIntegrate[x^(n - 1) / ((Exp[x] / z) - 1), {x, 0, ∞}];
(* Carry out parametric plot *)
p1 = ParametricPlot[{(g[3 / 2, 1] / g[3 / 2, z])^(2 / 3),
    (g[5 / 2, z] / g[3 / 2, z]) (g[3 / 2, 1] / g[3 / 2, z])^(2 / 3)}, {z, 0.1, 0.999},
   PlotRange → {{0, 6}, {0, 6}}, Frame → True, AxesStyle → Black] ;
p2 = Plot[x, {x, 0, 6}, PlotStyle → Dashed];
p3 = Plot[x^(5 / 2) (g[5 / 2, 1] / g[3 / 2, 1]), {x, 0, 1}, PlotStyle → Red];
Show[p1, p2, p3, FrameLabel → {HoldForm[T / T_BEC], HoldForm[P V / (Nk_B T_BE)]},
 PlotLabel → None, LabelStyle → {GrayLevel[0]}]
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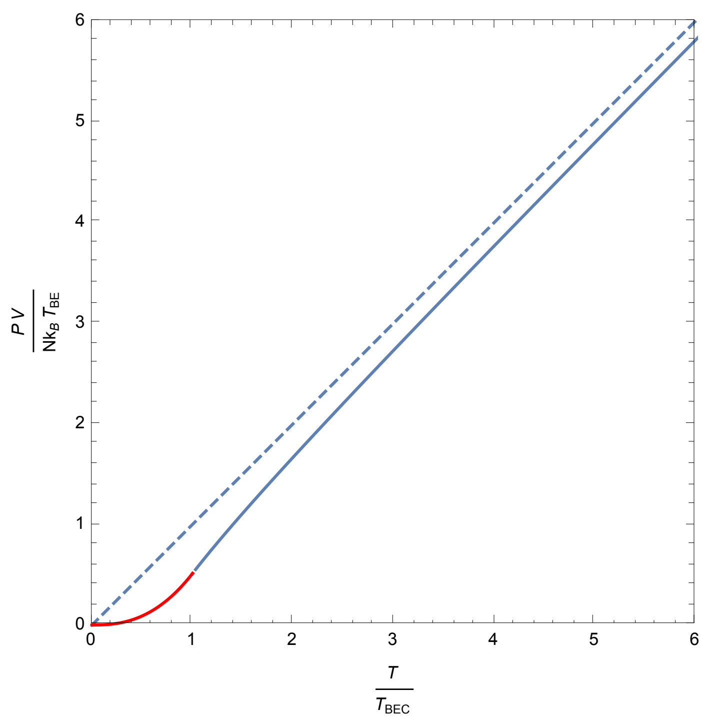
Out[26]=



Specific Heat & Energy

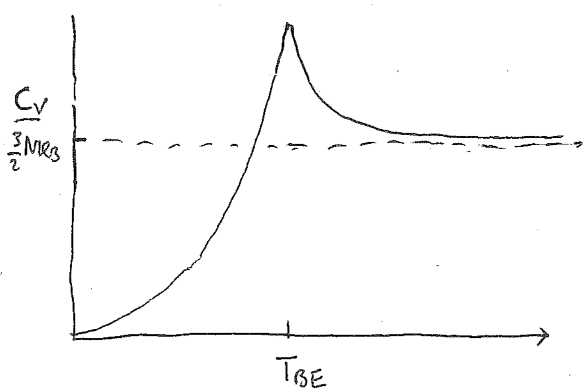
$$U = -\frac{\partial}{\partial \beta} \ln Z(\beta, z) = k_B T^2 \left(\frac{\partial}{\partial T} \ln Z\right)_z$$

$$\ln Z = \frac{V}{\lambda_T^3} g_{5/2}(z)$$

$$\Rightarrow \quad U = k_B T^2 \frac{\partial}{\partial T}\left[\frac{V}{\lambda_T^3} g_{5/2}(z)\right] = \frac{3}{2} k_B T \frac{V}{\lambda_T^3}$$

$$P = \frac{k_B T}{\lambda_T^3} g_{5/2}(z)$$

$$\Longrightarrow \quad \boxed{U = \frac{3}{2} PV}$$ at all temperatures.



$$C_V = \frac{3}{2}\left(\frac{\partial P}{\partial T}\right) V.$$

$T < T_{BE}$

$$P = \frac{N k_B T_{BE}}{V}\left(\frac{T}{T_{BE}}\right)^{5/2} \frac{\zeta(\frac{5}{2})}{\zeta(3/2)}$$

$$C_V = N k_B \left[\underbrace{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\frac{\zeta(5/2)}{\zeta(3/2)}}_{1.91}\right]\left(\frac{T}{T_{BE}}\right)^{3/2} = 1.91\, N k_B \left(\frac{T}{T_{BE}}\right)^{3/2}$$

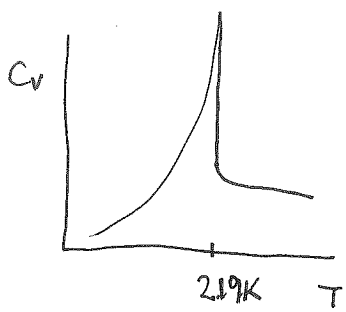
Note finally that

$$S(T) = \int^{T} \frac{C_V}{T'} dT' = N k_B \left[\left(\frac{5}{2}\right) \frac{\zeta(5/2)}{\zeta(3/2)} \right] \left(\frac{T}{T_{BE}} \right)^{3/2}$$

$$= N_{normal} \, k_B \, \frac{5}{2} \left(\frac{\zeta(5/2)}{\zeta(3/2)} \right)$$

Entropy per particle is in normal fluid $\frac{5}{2} \left(\frac{\zeta(5/2)}{\zeta(3/2)} \right) \frac{1}{\zeta\left(\frac{3}{2}\right)} = 0.86 \, k_B$

Close analogy with He-4 T_λ.



Keesom, 1928 – λ point

1937 $\begin{cases} \text{Kapitza} \\ \text{Allen + Misener} \end{cases}$ Superfluid

$$N = N_0(T) + \underbrace{N_n(T)}_{\text{normal fluid}}$$

$N_0(T)$: condensate

"Two fluid"

He-4 ~ BEC

$a \sim 3.76 \times 10^{-10}$

$m = 4 \times 10^{-27}$ kg

$T_{BE} = \hbar^2 / (k_B m a^2) \times 3.31$

$= \underline{2.71 \text{ K}}$

Superfluidity also requires an <u>interaction</u> between particles. $\underline{\underline{T_\lambda = 2.19\text{K}}}$.

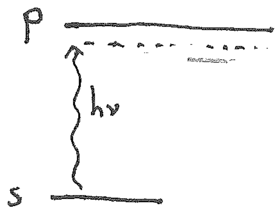
7.2 BEC in ultracold gases

^{87}Rb Wieman & Cornell (1995)

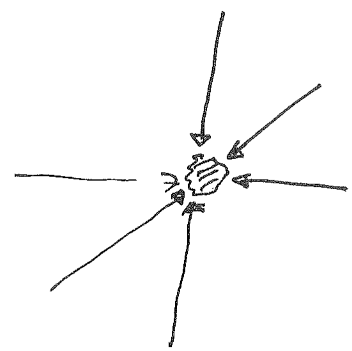
^{23}Na Ketterle et al (1995)

1. LASER COOLING

"Doppler cooling"

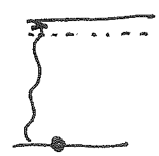


↕ detuned

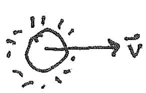


stationary - detuned

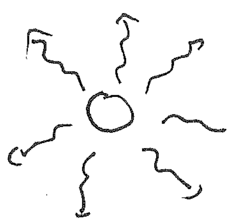
blue shifted ∴ absorbs

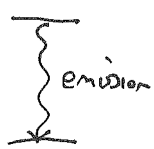


$\updownarrow \frac{v}{c}\omega_0$



excited state





emits photon in random direct

$$\langle \Delta p \rangle = -\frac{h\nu}{c}\hat{v}$$

Momentum of absorbed photon

$$kT_0 \sim \frac{(h\nu)^2}{2mc^2} \sim 1\mu K$$

2. Magnetic trap

$$V(r) = \frac{1}{2}m\omega_0^2(x^2+y^2+z^2)$$

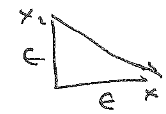
$$\epsilon(n_1,n_2,n_3) = \hbar\omega_0\left(\frac{3}{2} + n_1+n_2+n_3\right)$$

$$a(\epsilon) = \int dn_1\, dn_2\, dn_3 \;\; \delta\left(\epsilon - \hbar\omega_0(n_1+n_2+n_3)\right)$$

$$= \frac{1}{(\hbar\omega_0)^3}\int dx_1\, dx_2\, dx_3 \;\; \delta\left(\epsilon - (x_1+x_2+x_3)\right)$$

$x_3 = \epsilon - (x_1+x_2)$

$$= \frac{1}{(\hbar\omega_0)^3}\int_{x_1+x_2<\epsilon} dx_1\, dx_2$$



$$= \frac{\epsilon^2}{2(\hbar\omega_0)^3}$$

$$-k_BT \ln Z = \int d\epsilon\; a(\epsilon)\; k_BT \ln\left(1 - e^{\beta\mu}e^{-\beta\epsilon}\right)$$

$x = \beta\epsilon$

$k_BT\, dx = d\epsilon$

$$= \frac{(k_BT)^4}{2(\hbar\omega_0)^3}\int_0^\infty dx\; \underset{\downarrow \; \frac{x^3}{3}}{x^2}\; \ln\underset{\downarrow\; \frac{e^{\beta\mu}e^{-x}}{(\;-\;)} = \frac{1}{e^x e^{-\beta\mu}-1}}{\left(1-e^{\beta\mu}e^{-x}\right)}$$

$$= -\frac{(k_BT)^4}{(\hbar\omega_0)^3}\,\frac{1}{6}\int_0^\infty dx\,\frac{x^3}{e^x e^{-\beta\mu}-1}$$

$6 = 3! = \Gamma(4)$

$$F = -\frac{(k_BT)^4}{(\hbar\omega_0)^3}\, g_4(z)$$

$z = e^{\beta\mu}$

$$-\frac{\partial}{\partial x}\frac{1}{(e^x z^{-1}-1)}$$

$$\frac{1}{6}\int dx\; x^3\,\frac{e^{-\beta\mu}e^x}{(e^x e^{-\beta\mu}-1)^2}\;\beta$$

$$N = \frac{-\partial F}{\partial\mu} = \left(\frac{k_BT}{\hbar\omega_0}\right)^3 g_3(z)$$

$$= \frac{1}{2}\int\frac{x^2}{e^x z^{-1}-1} = g_3(z)$$

$$\frac{k_BT_c}{\hbar\omega_0} = \left(\frac{N}{\zeta_3(3)}\right)^{\frac{1}{3}} = \left(\frac{N}{1.202}\right)^{1/3}$$

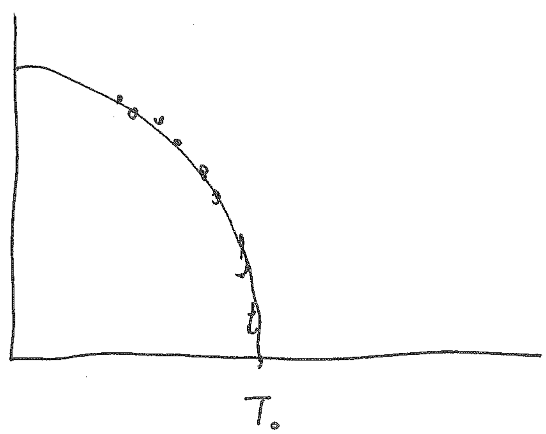
Mag trap $f = 100\,\text{Hz}$

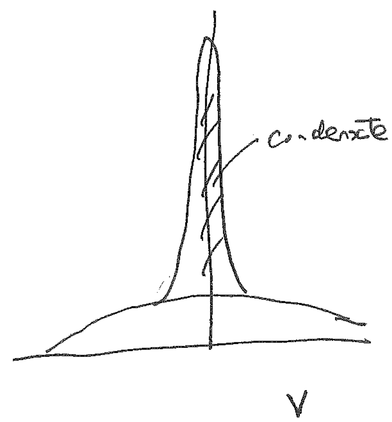
$N = 2\times10^4$ $\quad \frac{k_BT_c}{\hbar\omega_0} = 25.5 \sim \underline{\underline{200\,\text{nK}}}$

$T < T_c$

$$N = N_0(T) + \left(\frac{k_B T}{k_B T_c}\right)^3 N$$

$$\Rightarrow \quad N_0(T) = \left[1 - \left(\frac{T}{T_c}\right)^3\right] N$$

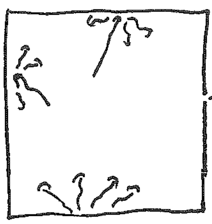




7.3 BLACKBODY RADIATION

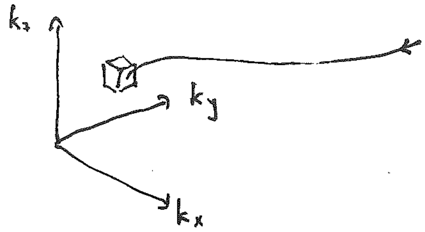
The radiation from our sun, from the surface of a black-top limousine, from our bodies, from outer space: all of it is a form of "black-body radiation".

Let us think of a cavity containing radiation in equilibrium with the walls.



$$I(\omega) = \frac{u(\omega)\,c}{4}$$

- Photons: quanta with energy $hf = \hbar\omega$
- # photons not conserved



2 polarizations

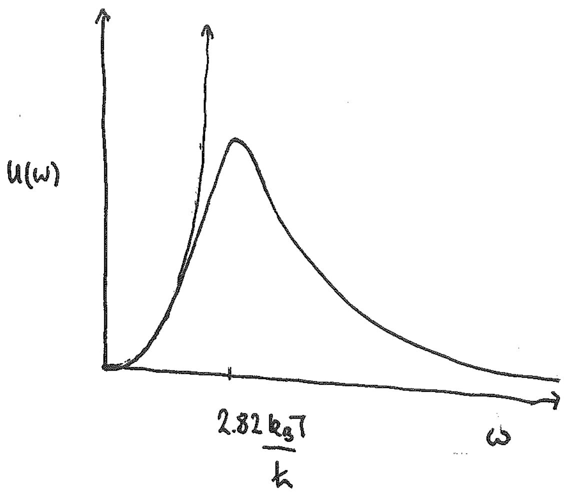
$$\#\text{ normal modes} = V \times 2 \times \frac{d^3k}{(2\pi)^3} = 2V\,\frac{4\pi k^2\,dk}{(2\pi)^3} = V\,\frac{k^2\,dk}{\pi^2}$$

$$\text{Number of photons/mode} = \langle n_{\vec{k}} \rangle = \frac{1}{e^{\beta\hbar\omega_k} - 1}$$

$$\text{Energy/unit vol/unit frequency} = u(\omega) = \frac{U(\omega)}{V}$$

$$\frac{k^2\,dk}{\pi^2} \equiv \frac{\omega^2\,d\omega}{\pi^2 c^3}$$

$$\boxed{u(\omega) = \left(\frac{\hbar\omega_k}{e^{\beta\hbar\omega_k} - 1}\right) \times \frac{\omega^2}{\pi^2 c^3}}$$



Frequency @ peak

$$hf_{max} = 2.82 k_B T$$

$$\Rightarrow f_{max} = \left(\frac{2.82 k_B}{h}\right) T = 58.8 \text{ GHz/K}.$$

"Wien's law"
(also λmax T=2.89 x 10-3 Km)

$$u(\omega) = \begin{cases} k_B T \left(\dfrac{\omega^2}{\pi^2 c^3}\right) \\ \dfrac{\hbar}{\pi^2 c^3} \omega^3 e^{-\beta\hbar\omega} \end{cases}$$

$k_B T \gg \hbar\omega$
CLASSICAL RAYLEIGH-JEANS

Quantum mechanical cutoff when $\hbar\omega \gg k_B T$.

Can also write $x = \beta\hbar\omega$. $\quad dx = \beta\hbar\, d\omega \quad \omega^3 d\omega = \left(\frac{k_B T}{\hbar}\right)^4 x^3 dx$

$$u(\omega)d\omega = \tilde{u}(x)dx$$

$$\tilde{u}(x) = \frac{1}{\pi^2 (c\hbar)^3} (k_B T)^4 \left(\frac{x^3}{e^x - 1}\right)$$

Maximum @ $\frac{\partial u}{\partial x} = 0 \Rightarrow \underline{x = 2.82}$

$$\frac{U}{V} = \int_0^\infty u(x)dx = \frac{(k_B T)^4}{\pi(c\hbar)^3} \overbrace{\int dx \frac{x^3}{e^x - 1}}^{\pi^4/15} = \left(\frac{\pi^2 k_B^4}{15(\hbar c)^3}\right) T^4.$$

$$= \left(7.56 \times 10^{-16} \text{ J/K}^4/\text{m}^3\right) T^4$$

Classically: $\frac{U}{V} = \int k_B T \left(\frac{\omega^2}{\pi^2 c^3}\right) d\omega = \infty$

ULTRAVIOLET CATASTROPHY

What about intensity $\equiv$ energy flux

$$I(\omega) = u(\omega) \times \langle \cos\theta \rangle \times c$$

$$\langle \cos\theta \rangle = \frac{\int_0^1 \cos\theta \, d\cos\theta \, d\phi}{\int_{-1}^{1} d\cos\theta \, d\phi} = \frac{1}{4}$$

$$I = \frac{c}{4} u = \left(\frac{\pi^2 k_B^4}{60 \hbar^3 c^2} \right) T^4$$

$$= 5.67 \times 10^{-8} \, (W/K^4) \times T^4$$

$$= \sigma T^4$$ Stefan-Boltzmann law

a) Calculate energy flux on earth due to sunlight

b) Calculate wavelength of light at peak intensity.

$$I_{EARTH} \, 4\pi R_{EARTH}^2 = I_{SUN} \, 4\pi R_{sun}^2$$

$$\Rightarrow I_{EARTH} = \sigma T_s^4 \left(\frac{R_s}{R_E} \right)^2 = 5.67 \times 10^{-8} \times (5778K)^4 \times \left(\frac{6.96 \times 10^2}{150 \times 10^9} \right)^2$$

$$= \underline{1360 \, W/m^2}$$

b) $I(\lambda) d\lambda = I(\omega) d\omega \quad \Rightarrow \quad I(\lambda) d\lambda = \frac{4\pi^2 c \hbar}{\lambda^5} \frac{1}{e^{\hbar c / \lambda k_B T} - 1}$

Write $\tilde{\lambda} = \frac{\lambda k_B T}{hc}$

$$I(\tilde{\lambda})\,d\tilde{\lambda} = 2\pi \frac{(k_B T)^4}{(hc)^3} \frac{1}{\tilde{\lambda}^5\left(\exp\left[\frac{hc}{\lambda k_B T}\right] - 1\right)}$$

$$I(\tilde{\lambda}) = 2\pi \frac{(k_B T)^4}{(hc)^3} \frac{1}{\tilde{\lambda}^5} \frac{1}{\left(e^{\frac{1}{\tilde{\lambda}}} - 1\right)}$$

Maximum of $I(\tilde{\lambda})$ occurs @ $\tilde{\lambda} = 0.2$

$$\Rightarrow \frac{\lambda k_B T}{hc} = 0.2$$

$$\Rightarrow \lambda = \left(\frac{hc}{k_B T}\right) 0.2 = \frac{2.89 \times 10^{-3}\,\mathrm{m}}{T\,(K)}$$

Wien's law

For sun $\lambda = \frac{2.89 \times 10^{-3}}{5778} = 5 \times 10^{-7}\,\mathrm{m} = \underline{\underline{500\,\mathrm{nm}}}.$

Calculate Free energy & Pressure

$$A = \sum_k 2k_BT \ln\left(1-e^{-\hbar\omega\beta}\right)$$

$$2\sum_k = V\int \frac{k^2 dk}{\pi^2} = \frac{V}{\pi^2(\hbar c)^3}\int \epsilon^2 d\epsilon$$

$$A = \frac{Vk_BT}{\pi^2(\hbar c)^3}\int d\epsilon\, \epsilon^2 \ln\left(1-e^{-\epsilon\beta}\right)$$

$$= -\frac{V}{3\pi^2(\hbar c)^3}\int d\epsilon \frac{\epsilon^3}{(e^{\epsilon\beta}-1)} = -\frac{U}{3} = -PV$$

$$\Rightarrow \quad P = \left(\frac{U}{3V}\right).$$

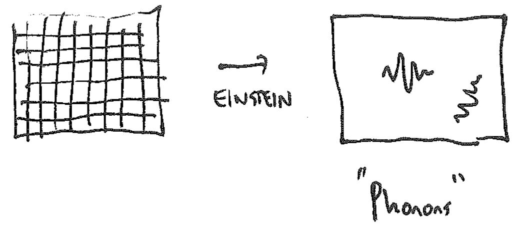
$$A = U - TS \Rightarrow \quad S = \frac{U-A}{T} = \frac{4}{3}\frac{U}{T} \propto VT^3.$$

In an adiabatic expansion $T \propto \frac{1}{V^{1/3}} \propto \frac{1}{\ell}$ Redshifted

COSMIC MICROWAVE BACKGROUND. $t_0 =$ 379k years $T_0 \sim 3000K$

$T_{now} \approx 2.73K$ Redshifted!

PHONONS



Normal modes of a crystal.

$$H = \Phi_0 + \left\{ \sum_j \frac{1}{2} m \dot{\xi}_e^2 + \frac{1}{2} \sum_{i,j} \alpha_{ij} \xi_i \xi_j \right\}$$

$$\alpha_{ij} = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}$$

Normal co-ords:

$$H = \Phi_0 + \sum_{j=1}^{3N} \left(\frac{p_j^2}{2m} + \frac{m\omega_j^2 q_j^2}{2} \right)$$

Quantum mechanically $\quad [p_i, q_j] = -i\hbar \delta_{ij}$

$$E\{n_j\} = \Phi_0 + \sum \left(n_j + \frac{1}{2} \right) \hbar\omega_j$$

$$\Rightarrow \quad U(T) = \underbrace{\left\{ \Phi_0 + \underbrace{\sum \frac{1}{2} \hbar\omega_j}_{\text{ZERO POINT}} \right\}}_{\text{GROUND STATE ENERGY}} + \underbrace{\sum_e \left(\frac{\hbar\omega_e}{e^{\hbar\omega_e / k_B T} - 1} \right)}_{\text{THERMAL ENERGY OF OSCILLATORS}}$$

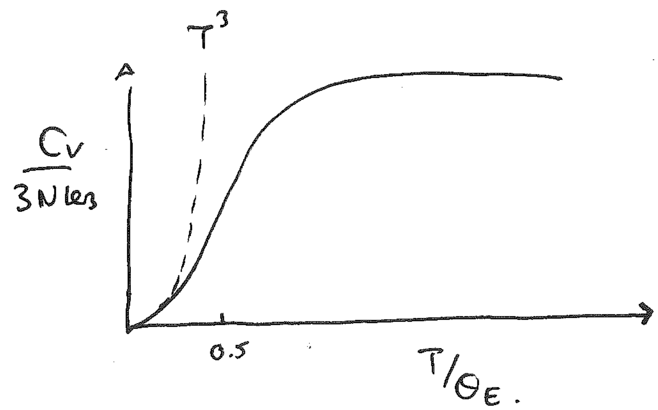
$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = k_B \sum_{i=1}^{3N} \frac{(\hbar\omega_i/k_BT)^2 e^{\hbar\omega/k_BT}}{(e^{\hbar\omega\beta}-1)^2}$$

If all the modes have equal frequency $\omega_i = \omega_E$

$$C_V(T) = 3Nk_B\, E(x)$$

$$x = \hbar\omega_E/k_BT = \Theta_E/T.$$

$$E(x) = \frac{x^2 e^x}{(e^x-1)^2}$$



More generally $g(\omega)$ where

$$\int_0^{\omega_D} d\omega\, g(\omega) = 3N$$

$$\omega = ck \Rightarrow V\frac{4\pi k^2 dk}{(2\pi)^3} = \frac{V\omega^2 d\omega}{2\pi^2 c^3} \Rightarrow g(\omega) = \frac{\omega^2}{2\pi^2 c^3}V$$

$$g(\omega) = V\left[\frac{\omega^2}{2\pi^2 c_L^3} + \frac{\omega^2}{\pi^2 c_T^3}\right]$$

$$\int_0^{\omega_D} g(\omega)d\omega = 3N$$

$$\Rightarrow \omega_D^3 = 18\pi^2\frac{N}{V}\left(\frac{1}{c_L^3} + \frac{2}{c_T^3}\right)^{-1}$$

$$g(\omega) = \begin{cases} \dfrac{9N}{\omega_0^3}\,\omega^2 & 0<\omega<\omega_0 \\ & \text{otherwise.} \end{cases}$$

DEBYE

More sophisticated treatment:

$$\int^{\omega_{D,L}} \frac{\omega^2\,d\omega}{2\pi^2 c_L^3} = N/V \qquad \int^{\omega_{D,T}} \frac{\omega^2\,d\omega}{2\pi^2 c_T^3} = (N/V).$$

corresponds to a common wavelength $\lambda_{min} = \left(\dfrac{4\pi V}{3N}\right)^{1/3}.$

$$\left(\frac{\omega}{c_L}\right)^3 = \frac{6N\,2\pi^2}{V} \qquad \left(\frac{2\pi}{\lambda}\right)^3 = \frac{6N\,2\pi^2}{V} \Rightarrow \lambda = 2\pi\left(\frac{V}{6N\,2\pi^2}\right)^{1/3} = \left(\frac{4\pi V}{3N}\right)^{1/3}.$$

$$C_V = k_B \int^{\omega_D} d\omega\, g(\omega)\, E\left(\frac{\hbar\omega}{k_B T}\right)$$

$$= 3Nk_B \int^{\omega_D} d\omega\, \frac{3\omega^2}{\omega_D^3}\, E\left(\frac{\hbar\omega}{k_B T}\right)$$

$$= \frac{3Nk_B}{\omega_D^3}\left(\frac{k_B T}{\hbar}\right)^3 \int_0^{x_0} dx\, \frac{3x^4 e^x}{(e^x-1)^2}$$

$$= 3Nk_B\, D(x_0)$$

$$D(x_0) = \frac{3}{x_0^3}\int_0^{x_0} dx\, \frac{x^4 e^x}{(e^x-1)^2}$$

$$x_0 = \left(\frac{\hbar\omega_D}{k_B T}\right) = \frac{\Theta_D}{T}.$$

7.18.

$$D(x_0) = \frac{-3}{x_0^3}\int_0^{x_0} x^4 \frac{d}{dx}\frac{1}{(e^x-1)}\,dx$$

$$= \frac{-3}{x_0^3}\int_0^{x_0}\left[\frac{d}{dx}\left(\frac{x^4}{e^x-1}\right) - \frac{4x^3}{e^x-1}\right]dx$$

$$= \frac{-3\,x_0}{e^x-1} + \frac{12}{x_0^3}\int_0^{x_0} dx\,\frac{x^3}{(e^x-1)} \qquad x_0 = \Theta_0/T.$$

$$T\to\infty \qquad D(x_0) \simeq 1 - \frac{x_0^2}{30}.$$

$$C_V = 3Nk_B\left(1 - \frac{(\Theta_0/T)^2}{30} + \ldots\right) \qquad \text{Dulong + Petit @ high } T.$$

$$\begin{matrix} T\to 0 \\ x_0\to\infty \end{matrix} \qquad D(x_0) \to \frac{12}{x_0^3}\int_0^\infty \underbrace{\frac{dx\,x^3}{e^x-1}}_{\pi^4/15} \simeq \frac{4\pi^4}{5}\left(\frac{T}{\Theta_0}\right)^3.$$

$$C_V \sim 3\cdot\frac{4\pi^5}{5} Nk_B\left(\frac{T}{\Theta_0}\right)^3 = n\left(\frac{12R\pi^5}{5}\right)\left(\frac{T}{\Theta_0}\right)^3$$

$$\boxed{C_V = 6106 \text{ J/K/mol} \times \left(\frac{T}{\Theta_0}\right)^3.}$$

	Pb	Ag	Cu	Al	MgB_2*
$\Theta_D(C_v)$	88	215	345	398 K	~800K.
Θ_D (elastic)	73	214	332	402K	

* 40K superconductor.