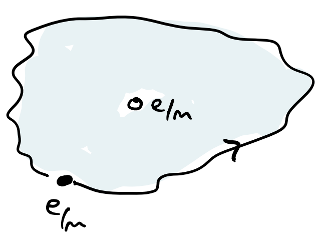
$$\left.\begin{aligned}
\Psi^{(+)}_{Z}(z) &= \left[\prod_{j=1}^{N}(z - Z)\right]\Psi_m(z) \\
\Psi^{-}_{\bar Z}(z) &= \left[\prod_{j}\left(\overbrace{2\frac{\partial}{\partial z_j}}^{P_L \bar z} - \bar Z\right)\right]\Psi_m(z)
\end{aligned}\right\} \quad \begin{aligned} q &= \pm\frac{e}{m} \\ \theta &= \pm \pi/m. \end{aligned}$$

$$P_{LLL}\left[\prod(\bar z_j - Z)\,\Psi_m(z)\right]$$

$$e^{i\theta}$$

AHARANOV BOHM PHASE

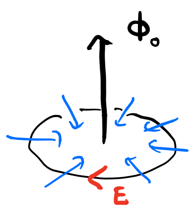
$$-\frac{2\pi}{m}\left(\frac{\Phi}{\Phi_0}\right)$$



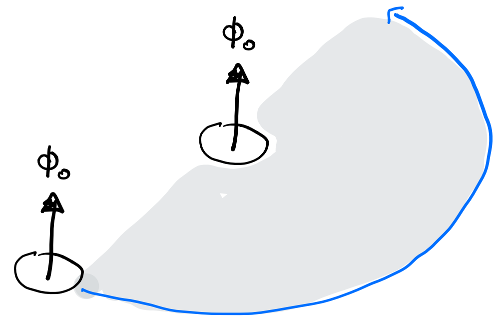
$$\underbrace{e^{2i\theta}}_{\text{Statistics}} \times e^{\frac{iq}{\hbar}\oint \vec{A}\cdot d\vec{\ell}}$$

16.8 Connection with Flux Insertion

The Anyon charge + statistics can be associated with a flux tube insertion into the Laughlin state. Recall the charge $q = e/m$ can be associated with the Inward Hall current induced by an added flux.



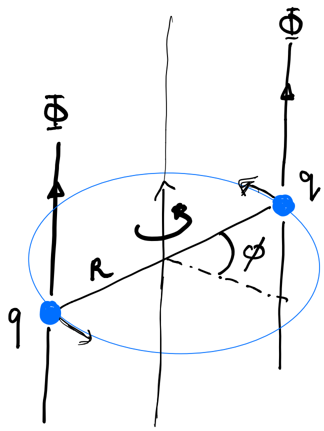
$$q = \sigma_{xy} \int \vec{E} \cdot d\vec{\ell} \, dt$$
$$= \frac{1}{m}\frac{e^2}{h} \int \frac{\partial \phi}{\partial t} dt = \frac{e}{m}$$



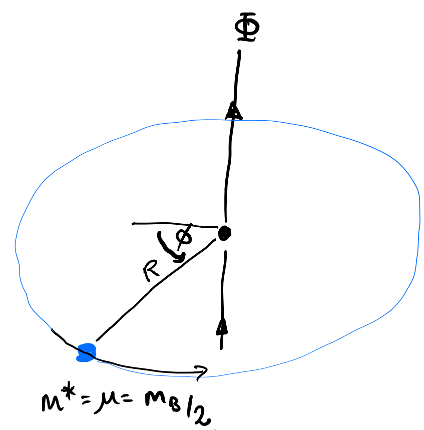
$$\theta_{ABohm} = \frac{q}{\hbar} \int \vec{A}_{flux} \cdot d\vec{x}$$
$$= \frac{q}{\hbar}\left(\frac{h}{e} \times \frac{1}{2} \right) = \pi \left(\frac{q}{e} \right)$$

$$\boxed{\text{ANYON} \equiv \text{FLUX TUBE}}$$

Toy Model for Anyons



$\equiv$



$$\Psi(\phi + \pi) = + \Psi(\phi)$$

$$\boxed{\begin{aligned} H &= \frac{L^2}{2I} = \frac{\hbar^2}{2I}\left(-i\frac{d}{d\phi}\right)^2 \\ \Psi(\phi) &= e^{im\phi}, \quad E_m = \frac{(m\hbar)^2}{2I} \end{aligned}}$$

m – even

$E_g = 0$

Attach the Fluxoids : each particle feels field of other.

$$\boxed{H = \frac{1}{2\mu}\Big(\underbrace{-\frac{i\hbar}{R}\frac{\partial}{\partial\phi}}_{P_r} + \frac{q}{\hbar}A_r\Big)^2}$$

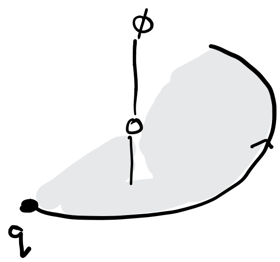
$$A_r = \frac{\Phi}{2\pi R}$$

$I = \mu R^2$

$$H = \frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\phi} + \frac{\theta}{\pi}\right)^2 \qquad \theta = \frac{\pi q}{\hbar}\frac{\Phi}{2\pi} = \pi \cdot \frac{q}{e}\cdot\frac{e}{h}\cdot\Phi$$

$$\boxed{\theta = \pi\left(\frac{q}{e}\right)\left(\frac{\Phi}{\Phi_0}\right)} \qquad \frac{q}{\hbar}\int \vec{A}\cdot d\vec{\ell} = \frac{q}{\hbar}\frac{\Phi}{2} = \theta$$

$$\eta = e^{i\theta}$$



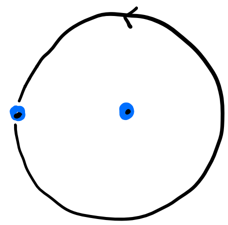
$$E_m = \frac{\hbar^2}{2I}\left(\underbrace{m}_{\text{even}} + \theta/\pi\right)^2$$

e.g if $q = e$, $\Phi = \Phi_0$, then $\theta/\pi = 1$ $\Rightarrow$ $\boxed{E_m = \frac{\hbar^2}{2I}(\overbrace{m+1}^{\text{odd}})^2 \quad \eta = e^{i\pi}}$

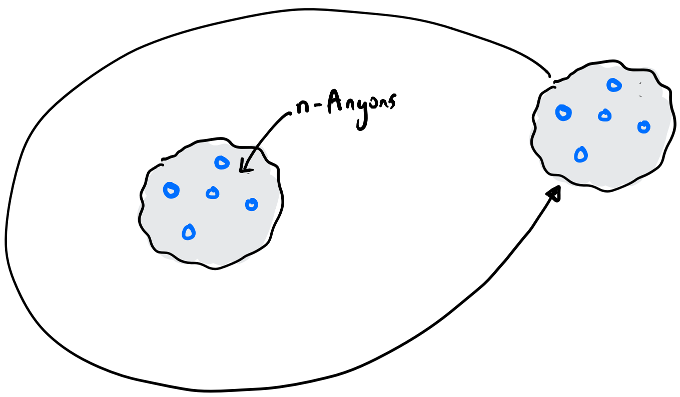
FERMIONS!

if $q = \frac{e}{2}$ $\Phi = \Phi_0 \Rightarrow \theta = \frac{\pi}{2}$ "semion"

Combinations of Anyons



$$e^{2i\theta} \qquad \theta = \pi/m.$$



$$e^{2i\theta_{TOT}}$$

$$\boxed{\theta_{TOT} = \frac{\pi}{m} \times n^2.}$$

m : Anyons

$$\theta_{TOT} = \frac{\pi}{m} \times m^2 = \pi m$$

e.g

	m	ν	Hole	θ	θ_{TOT}	m-Anyons.
e^-	1	1	Fermion	π	π	FERMION
boson	2	$1/2$	Semion	$\pi/2$	2π	BOSON
e^-	3	$1/3$	Anyon.	$\pi/3$	3π	FERMION
boson	4	$1/4$	"			BOSON.

Topology + Fractional Statistics

g = genus = 0 $\qquad$ $g = 1$ $\qquad$ $g = 2$

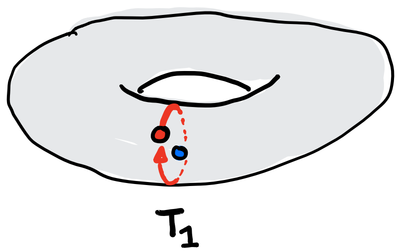
D = degeneracy of ground-state = $(m)^g$

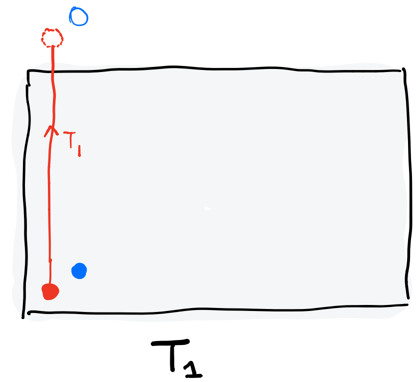
"Topologically ordered" (Wen).

- create particle-antiparticle pair
- carry +ve particle around path on torus.
- reannihilate pair.

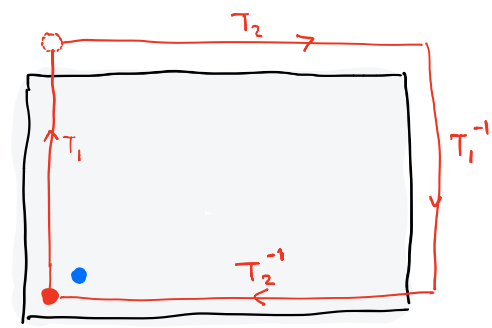
T_1 $\qquad$ T_2

Consider the process $T_2^{-1} T_1^{-1} T_2 T_1$





$$T_2^{-1} T_1^{-1} T_2 T_1 = e^{-2\pi i/m}$$



But if the ground-state $|\Omega_g\rangle$ is unique, then

$$T_1 |\Omega_g\rangle = e^{i\phi_1} |\Omega_g\rangle \qquad T_2 |\Omega_g\rangle = e^{i\phi_2} |\Omega_g\rangle$$

$$\Rightarrow \quad (T_2^{-1} T_1^{-1} T_2 T_1) |\Omega_g\rangle = |\Omega_g\rangle \neq e^{-2\pi i/m} \cdot |\Omega_g\rangle$$

We conclude that the g.s is degenerate.

e.g $\quad T_1 |n\rangle = e^{i\phi_n} |n\rangle \qquad T_2 |n\rangle = e^{i\alpha_n} |n+1\rangle$

$$T_2^{-1} T_1^{-1} T_2 T_1 |n\rangle = T_2^{-1} T_1^{-1} e^{i(\alpha_n + \phi_n)} |n+1\rangle$$
$$= T_2^{-1} e^{i[(\phi_n - \phi_{n+1}) + \alpha_n]} |n+1\rangle$$
$$= e^{i(\phi_n - \phi_{n+1})} |n\rangle$$

$\Rightarrow \quad \phi_n = 2\pi i n / m$ & can choose $\alpha_n = 0$.

$\therefore$ gs is m fold degenerate.

On a manifold of genus g $D = m^g$ fold degenerate.

Composite Fermions + Heirachy States

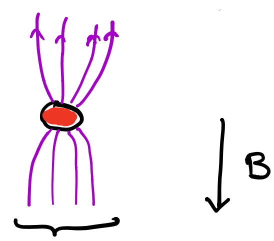
$$2p = m-1$$

$$f_N^m(z) = \left[\prod_{i'<j'}^{N} (z_{i'} - z_{j'})^{2p}\right] \underbrace{\left[\prod_{i<j}^{N} (z_i - z_j)\right]}_{\text{Vandermonde Determinant}}$$

Vandermonde Determinant
$\equiv \nu = 1$ Landau Level

c.f $\prod(z_i - Z)$ one hole wavefn, but now a vortex of vorticity $2p$ is <u>bound</u> to each electron.

"Composite Fermion" =
(Jain)



$$2p\,\Phi_0 = \Phi_V$$

The Aharanov-Bohm phase derived from this vorticity partially cancels the A-B phase from the external field.

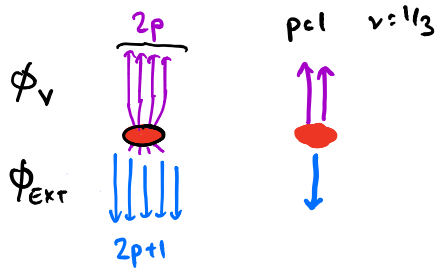
$$\theta = 2p\pi \Rightarrow \eta = e^{i\theta} = 1$$ $\therefore$ no effect on statistics.

But

$$N_\phi' = \underset{\text{external field}}{N_\phi} - \underset{\text{bound flux tubes}}{2pN} = (2p+1)N - 2pN = N$$

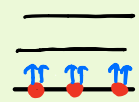
$\left(N/N_\Phi = \nu = \frac{1}{2p+1} \Rightarrow N_\phi = (2p+1)N \right)$

So the ratio of composite femions to flux is one



$$\Rightarrow \quad \nu^* = N / N'_\phi = 1$$

FILLED LANDAU LEVEL OF COMPOSITE FERMIONS

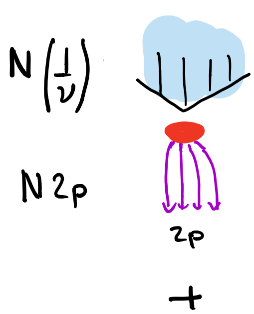


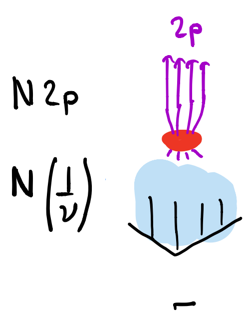
$\nu^* = 1$

$\nu_e = 1/3$

JAIN HEIRACHY

Consider ν filled Landau levels. Attach $2p$ flux tubes to electrons (+) parallel to field (−) antiparallel to field.





$$N'_\phi = N_\phi \pm 2pN = N\left(\frac{1}{\nu} \pm 2p\right) = \frac{N}{\nu^*}$$

$$\underset{\Uparrow \text{ INTEGER}}{\nu^*} = \frac{N}{N'_\phi} = \frac{1}{\underset{\uparrow \text{ FRACTIONAL}}{\frac{1}{\nu}} \pm 2p}$$

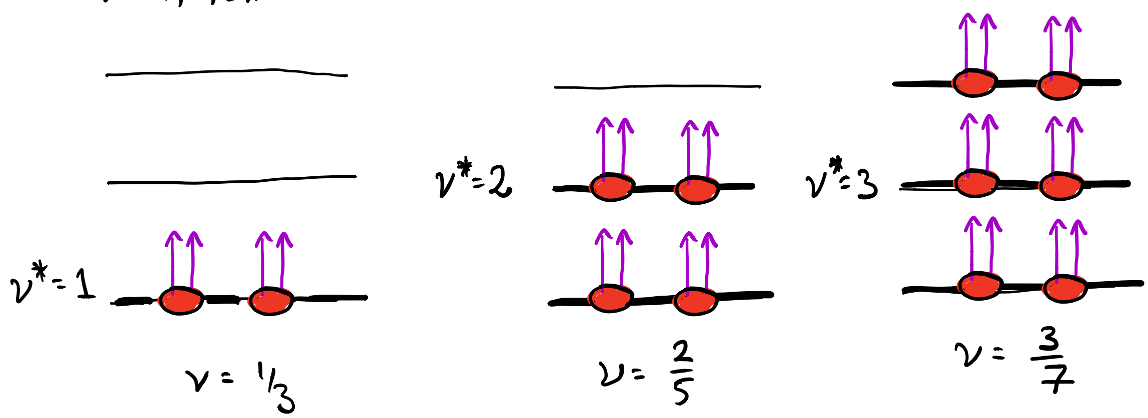
Filling factor for composite Fermions.

$$\Rightarrow \quad \frac{1}{\nu} \pm 2p = \frac{1}{\nu^*} \quad \Rightarrow \quad \frac{1}{\nu} = \frac{1}{\nu^*} \mp 2p$$

$$\Rightarrow \boxed{|\nu| = \frac{\nu^*}{2p\nu^* \mp 1}}$$ Jain Sequence

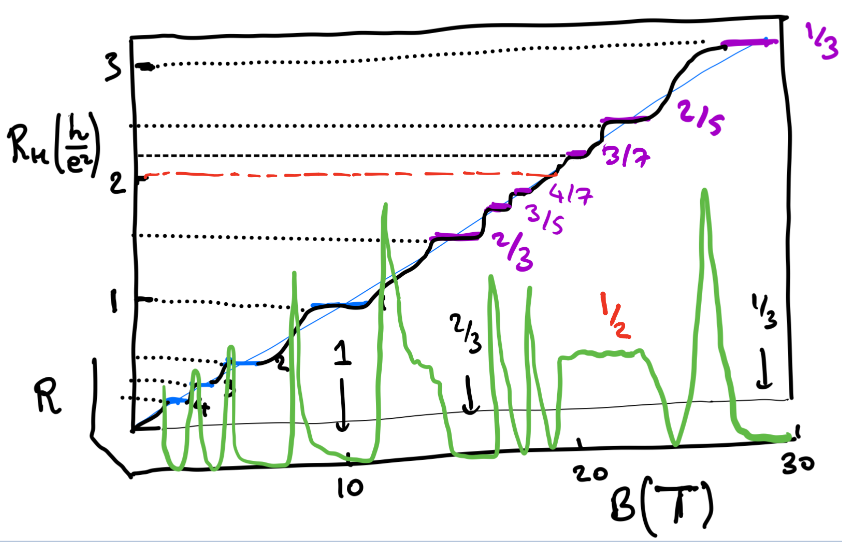
e.g. $p = 1$ $(+)$ $\quad |\nu| = \frac{\nu^*}{2\nu^*+1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \dots \dots \left(\frac{1}{2}\right)$ $\nu^* = \infty$

$\nu^* = 1, 2, 3, \dots \infty$



$$\boxed{\rho_{xy} = \frac{h}{e^2 \nu}}$$

Similarly $(-)$ $\quad |\nu| = \frac{\nu^*}{2\nu^* - 1} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \dots$



$$\Psi = P_{LLL}\left[\prod (z_i - z_j)^{2p}\ \Psi_{\nu^*}(\bar{z}, z)\right]$$

P_{LLL} = move all $\bar{z}$ to left, then replace $\bar{z}_i \to \frac{\partial}{\partial z_i}$

$$\boxed{\Psi = \Psi_{\nu^*}\left(\frac{\partial}{\partial z_i}, z_i\right) \prod_{i<j}^{N} (z_i - z_j)^{2p}}$$

Special case $\nu^* \to \infty$, $\nu = \frac{\nu^*}{2\nu^* \pm 1} \to \frac{1}{2}$
$p = 1$

HALF FILLED LL $\equiv$ Fermi sea of composite fermions.

$$B^* = B - 2n\Phi_0 = 0$$

$n = \frac{1}{2}\frac{N\Phi}{A} = \frac{1}{2}\frac{B}{\Phi_0}$

$$\Psi = P_{LLL}\left[\prod_{i<j} (z_i - z_j)^2\ \mathrm{Det}\left[e^{i\vec{k}_i \cdot \vec{x}_j}\right]\right]$$

Halperin, Lee, Read.