5.7 Fractional Statistics

Conventional argument

$$P_{12}\,\psi(1,2\ldots n) = \psi(2,1,\ldots n)$$

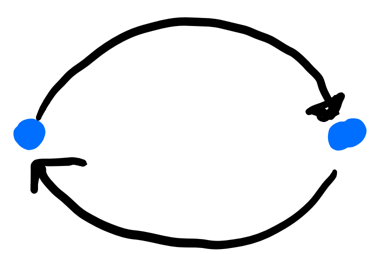
$$\psi(1,2,\ldots n) = p\,\psi(2,1\ldots n)$$

p = eigenvalue of exchange operator

$$P_{12}^2 = p^2 = 1 \implies p = \pm 1 \begin{cases} \text{BOSONS} \\ \text{FERMIONS.} \end{cases}$$

Physical exchange : adiabatic exchange of two particles.

$$e^{i\phi} : \begin{cases} \text{wavefn} \\ \text{Hamiltonian} \end{cases}$$

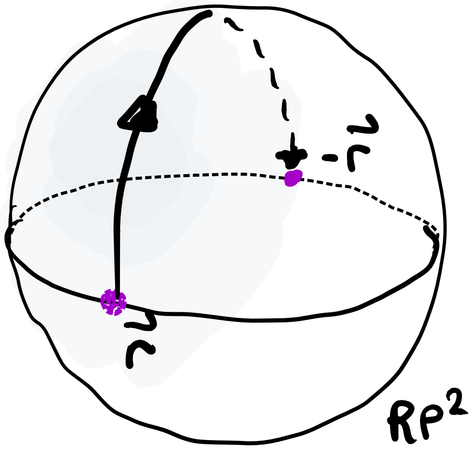


$$(r_1, r_2) \longrightarrow \begin{cases} r = (r_1 - r_2) & \text{relative} \\ \tilde{R} = (r_1 + r_2)/2 & \text{CM} \end{cases}$$

$$\left.\begin{matrix} \tilde{r} \\ -\tilde{r} \end{matrix}\right\}$$ SAME POINT IN CONFIGURATION SPACE

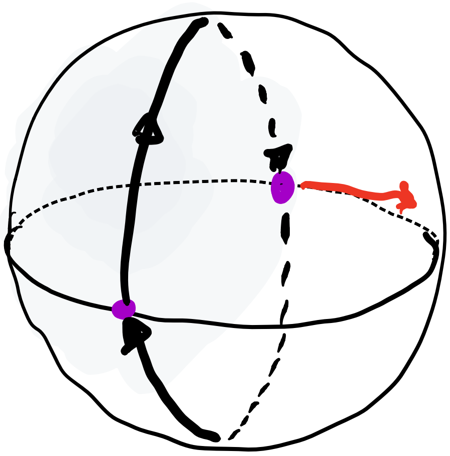
$\boxed{D=3}$

a.



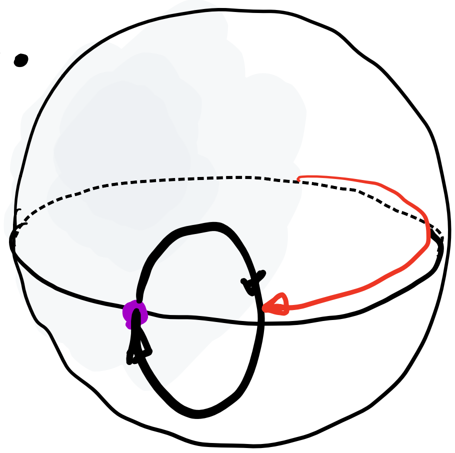
One exchange.

b.



DEFORM $\longrightarrow$

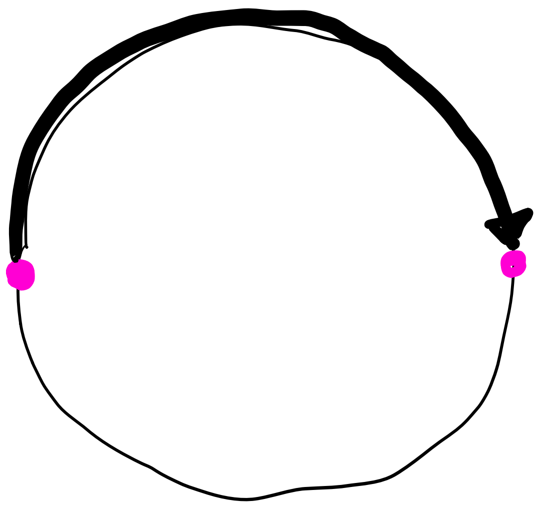
c.



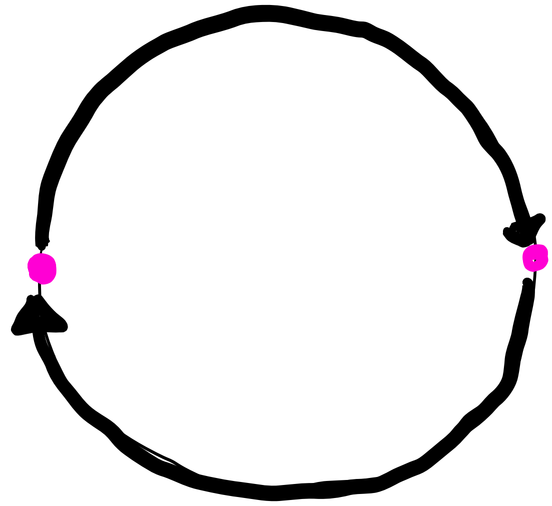
Two exchanges $\equiv$ No exchange.

$$\pi_1(RP^2) = \mathbb{Z}_2$$

$\boxed{D=2}$



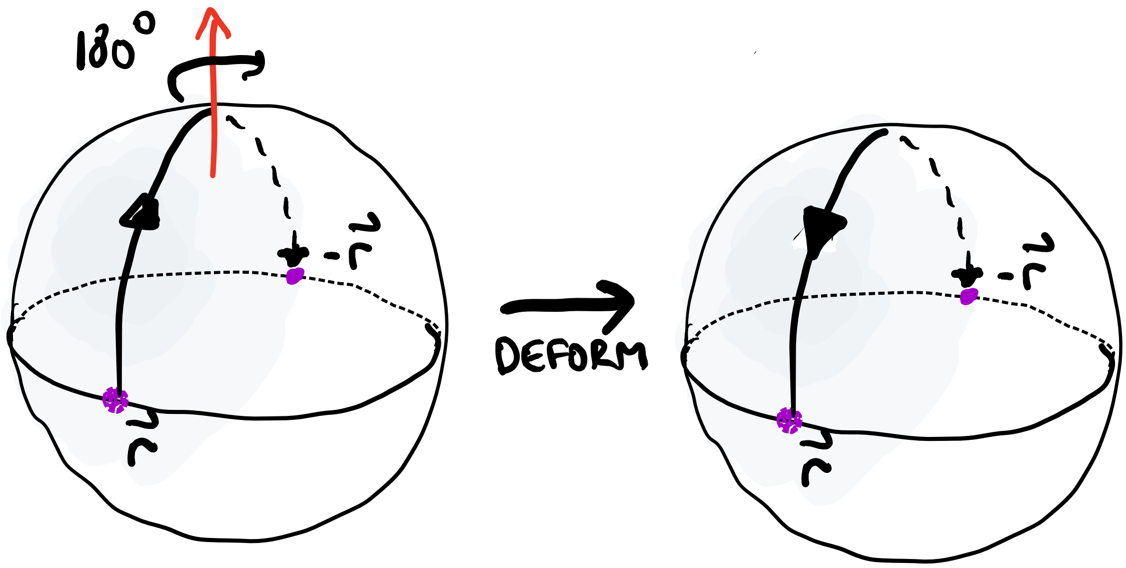
One exchange



Two exchanges are not equivalent to no exchange.

$$\boxed{\pi_1(\mathbb{RP}^1) = \mathbb{Z}}$$

3D



Time reversal

$$\eta \to \eta^*$$

$$\Rightarrow \eta = \eta^* \Rightarrow \eta = \pm 1.$$

2D: NOT POSSIBLE.

Fractional Statistics and the Quantum Hall Effect

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(Received 18 May 1984)

The statistics of quasiparticles entering the quantum Hall effect are deduced from the adiabatic theorem. These excitations are found to obey fractional statistics, a result closely related to their fractional charge.

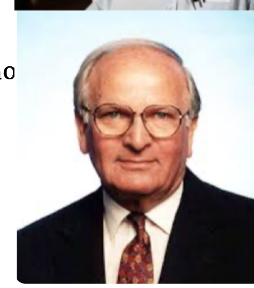
PACS numbers: 73.40.Lq, 05.30.–d, 72.20.My

Extensive experimental studies have been carried out[1] on semiconducting heterostructures in the quantum limit $\omega_0\tau >> 1$, where $\omega_0 = eB_0/m$ is the cyclotron frequency and τ is the electronic scattering time. It is found that as the chemical potential μ is varied, the Hall conductance $\sigma_{xy} = I_x/E_y = \nu e^2/h$ shows plateaus at $\nu = n/m$, where n and m are integers with m being odd. The ground state and excitations of a two-dimensional electron gas in a strong magnetic field B_0 have been studied[2–4] in relation to these experiments and it has been found that the free energy shows cusps at filling factors $\nu = n/m$ of the Landau levels. These cusps correspond to the existence of an "incompressible quantum fluid" for given n/m and an energy gap for adding quasiparticles which form an interpenetrating fluid. This quasiparticle fluid in turn condenses to make a new incompressible fluid at the next larger value of n/m, etc.

The charge of the quasiparticles was discussed by Laughlin[2] by using an argument analogous to that used in deducing the fractional charge of solitons in one-dimensional conductors.[5] He concluded for $\nu = 1/m$ that quasiholes and quasiparticles have charges $\pm e^* = \pm e/m$. For example, a quasihole is formed in the incompressible fluid by a two-dimensional bubble of a size such that $1/m$ of an electron is removed. Less clear, however, is the statistics which the quasiparticles satisfy; Fermi, Bose, and fractional statistics having all been proposed. In this Letter, we give a direct method for determining the charge and statistics of the quasiparticles.

In the symmetric gauge $\vec{A}(\vec{r}) = \frac{1}{2}\vec{B}_0 \times \vec{r}$ we consider the Laughlin ground state with filling factor $\nu = 1/m$,

$$\psi_m = \prod_{j<k} (z_j - z_k)^m \exp(-\tfrac{1}{4}\textstyle\sum_l |z_l|^2), \qquad (1)$$



where $z_j = x_j + iy_j$. A state having a quasiho

ized at z_0 is given by

$$\psi_m^{+z_0} = N_+ \textstyle\prod_i (z_i - z_0)\psi_m,$$

while a quasiparticle at z_0 is described by

$$\psi_m^{-z_0} = N_- \textstyle\prod_i (\partial/\partial z_i - z_0/a_0^2)\psi_m,$$

where $2\pi a_0^2 B_0 = \phi_0 = hc/e$ is the flux quantum and $N_\pm$ are normalizing factors.

To determine the quasiparticle charge e^*, we calculate the change of phase γ of $\psi_m^{+z_0}$ as z_0 adiabatically moves around a circle of radius R enclosing flux ϕ. To determine e^*, γ is set equal to the change of phase,

$$(e^*/\hbar c)\oint \vec{A} \cdot d\vec{l} = 2\pi(e^*/e)\phi/\phi_0, \qquad (4)$$

that a quasiparticle of charge e^* would gain in moving around this loop. As emphasized recently by Berry[6] and by Simon[7] (see also Wilczek and Zee[8] and Schiff[9]), given a Hamiltonian $H(z_0)$ which depends on a parameter z_0, if z_0 slowly transverses a loop, then in addition to the usual phase $\int^t E(t')\,dt'$, where $E(t')$ is the adiabatic energy, an extra phase γ occurs in $\psi(t)$ which is independent of how slowly the path is traversed. $\gamma(t)$ satisfies

$$d\gamma(t)/dt = i\langle\psi(t)|d\psi(t)/dt\rangle \quad . \qquad (5)$$

From Eq. (2),

$$\frac{d\psi_m^{+z_0}}{dt} = N_+ \sum_i \frac{d}{dt}\ln[z_i - z_0(t)]\psi_m^{+z_0}, \qquad (6)$$

so that

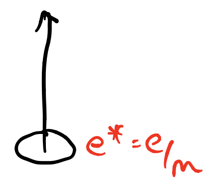
$$\frac{d\gamma}{dt} = iN_+^2 \left\langle \psi_m^{+z_0} \middle| \frac{d}{dt} \sum_i \ln(z_i - z_0) \middle| \psi_m^{+z_0} \right\rangle. \qquad (7)$$

Since the one-electron density in the presence of

15.7b STATISTICS ANGLE OF FQHE ANYONS.

$$\eta = e^{i\theta} \qquad \theta = \pm \pi/m.$$

Require a normalized hole wavefunction



$$\psi_Z[z] = e^{-\frac{\bar{Z} Z}{4m}} \prod_{j=1}^{N} (z_j - Z)\, \psi_m(z)$$

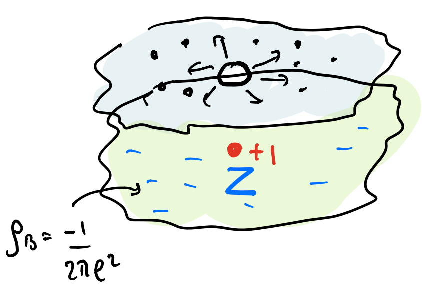
$$\boxed{\frac{\partial \psi}{\partial \bar{Z}} = -\frac{Z}{4m}\psi}$$

The prefactor makes the norm of the wavefunction Z independent of Z. $\langle \bar{Z} | Z \rangle$ = constant.

$$|\psi_Z(z)|^2 = e^{-2/m\, V_{eff}}$$

$$V_{eff} = -\frac{m}{2} \ln |\psi_Z(z)|^2$$

$$= \frac{|Z|^2}{4} - \sum m \ln |z_j - Z| - \sum_{i<j} m^2 \ln |z_i - z_j| + \frac{m}{4} \sum |z_j|^2$$



When we add the hole, the density of charge is locally reduced in the plasma by

$$\Delta\rho = -\delta(z - Z)\,\delta(\bar{z} - \bar{Z}) = -\delta^{(2)}(\vec{x})$$

$$\Delta\left[\frac{m}{4}\sum |z_j|^2\right] = \frac{1}{4}\int d^2x \,\Delta\rho(x)(\vec{x}^{\,2}) = -\frac{1}{4}|Z|^2$$

The term $+|Z|^2/4$ is needed to compensate for the Z dependence of the potential energy.

The calculation of the statistical phase will involve

- Calculation of the Berry phase associated with moving the holes.
- Calculating the additional phase due to moving one hole past the other.

Let $Z = X + iY$, $\bar{Z} = X - iY$. Now we want

$X = x/\ell$
$Y = y/\ell$

$$\vec{A} = i\langle \psi_Z | \vec{\nabla} | \psi_Z \rangle$$

We will write

$$\nabla_x = \nabla_Z + \nabla_{\bar{Z}}, \text{ so}$$
$$\nabla_y = i(\nabla_Z - \nabla_{\bar{Z}})$$

$$\left.\begin{aligned} A_x &= A_Z + A_{\bar{Z}} \\ A_y &= i(A_Z - A_{\bar{Z}}) \end{aligned}\right\} \qquad \begin{aligned} A_Z &= i\langle \psi_Z | \partial_Z | \psi_Z \rangle \\ A_{\bar{Z}} &= i\langle \psi_Z | \partial_{\bar{Z}} | \psi_Z \rangle \end{aligned}$$

Now, with the normalization, since $\partial_{\bar z}|\psi_z\rangle = -\frac{z}{4}|\psi_z\rangle$

$$\boxed{A_{\bar z} = -i\frac{z}{4m}}$$

Calculating A_z directly is complicated, easier to use

$$i\partial_z\langle\psi_z|\psi_z\rangle = 0 = A_z + i\left(\partial_z\langle\psi|\right)|\psi_z\rangle$$

$$\Rightarrow \quad A_z = -i\left(\partial_z\langle\psi|\right)|\psi\rangle$$

note
$$A_z^* = \langle\psi|\left(i\partial_{\bar z}|\psi\rangle\right)$$
$$= A_{\bar z}$$

$$A_z = \left(A_{\bar z}\right)^*$$

$$\partial_z\langle\psi| = \left(\partial_{\bar z}|\psi\rangle\right)^\dagger = -\bar z/_{4m}\langle\psi|$$

$$\Rightarrow \quad \boxed{A_z = i\bar z/_{4m}}$$

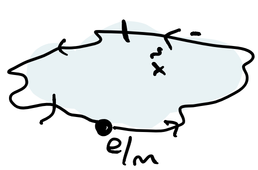
So $\quad A_x = A_z + A_{\bar z} = \frac{y}{2m} \qquad A_y = i\left(A_z - A_{\bar z}\right)$

$$= i\left(\frac{i\bar z}{4m} + \frac{iz}{4m}\right) = -\frac{x}{2m}$$

$$\gamma = \int A(\vec R)\,d\vec R = \frac{1}{2m}\int\frac{Y\,dX - X\,dY}{\ell^2} = -\frac{\text{Area}}{m\ell^2} = +\frac{\Phi}{n(\hbar/e)}$$

$$\text{Area} = \frac{1}{2}\int \vec x\times d\vec x \qquad \ell^2 = \frac{\hbar}{eB}$$

$$\boxed{\gamma = \frac{2\pi\,\Phi}{n\,\Phi_0}}$$



$$\boxed{\gamma = \oint \frac{e}{m\hbar}\vec A\cdot d\vec r}$$

$$|\Psi|^2 \sim \left(|z_1 - z_2|\right)^{\frac{2}{m}} \longrightarrow -\ln\left[z_1 - z_2\right]$$

$$\Psi_{Z_1 Z_2}(z) = \left[(\bar{Z}_1 - \bar{Z}_2)(Z_1 - Z_2)\right]^{1/2m} e^{-\left(\frac{|Z_1|^2 + |Z_2|^2}{4m}\right)}$$

$$\times \prod_{j=1}^{N} (z_j - Z_1)(z_j - Z_2)\ \Psi_m[z]$$

$$|\Psi|^2 = e^{-2/m\, V_{cl}} \qquad V_{cl} = -\frac{m}{2} \ln|\Psi|^2$$

$$V_{cl} = \frac{|Z_1|^2 + |Z_2|^2}{4} - \overbrace{\ln|Z_1 - Z_2| - m^2 \sum_{i<j} \ln|z_i - z_j|}^{\text{only dependent on relative positions}}$$

$$- m \sum_{i,\ell} \ln|z_i - Z_\ell|$$

$$+ \frac{m^2}{4} \sum |z_i|^2$$

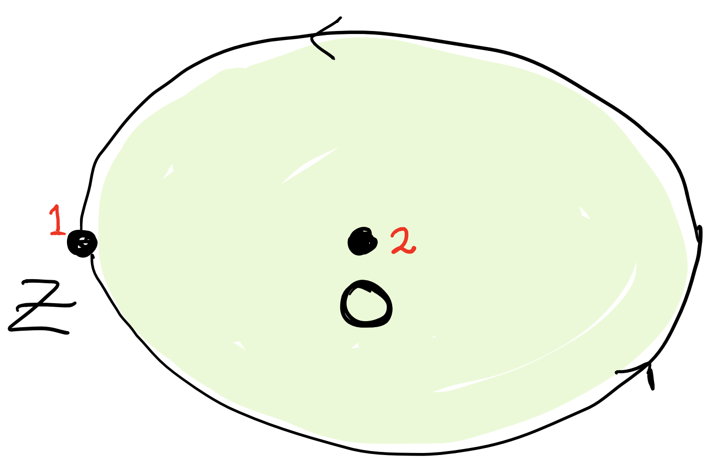
Z dependences

cancel.

$(\bar{Z}_1 Z_1 + \bar{Z}_2 Z_2)/4m$

$$\Psi_{Z_1 Z_2}(z) = \left[(\bar{Z}_1 - \bar{Z}_2)(Z_1 - Z_2)\right]^{1/2m} e^{-\left(\frac{|Z_1|^2 + |Z_2|^2}{4m}\right)}$$

$$\times \prod_{j=1}^{N} (z_j - Z_1)(z_j - Z_2) \; \Psi_m[z]$$



$Z_2 = 0$

$Z_1 = Z$

relative motion

$$A_{\bar{Z}} = i\langle \Psi | \partial_{\bar{Z}} | \Psi \rangle = A_{\bar{Z}} + a_{\bar{Z}}$$

$$\Psi_Z(z) = \underbrace{\left[\bar{Z} Z\right]^{1/2m}}_{\text{Relative motion of anyons}} \underbrace{e^{-\left(\frac{\bar{Z}Z}{4m}\right)}}_{\text{Background}} \prod_{j=1}^{N} (z_j - Z)\, z_j \, \Psi_m[z]$$

Relative motion of anyons $\downarrow$ a_Z

Background $\downarrow$ $A_{\bar{Z}}$

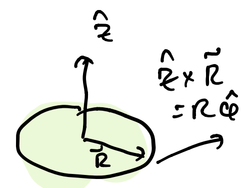
Set $z_2 = 0 \quad z_1 = z$

$$i \langle \psi | \partial_{\bar z} | \psi \rangle = \overbrace{i \frac{1}{2m \bar z}}^{a_{\bar z}} + A_{\bar z}$$

$$a_{\bar z} = i \frac{1}{2m \bar z} \,, \qquad a_z = -i \frac{1}{2m z}$$

$$\Rightarrow \quad a_x = a_z + a_{\bar z} = \frac{i}{2m}\left(\frac{1}{\bar z} - \frac{1}{z}\right) = \frac{i}{2m}\left(\frac{z - \bar z}{\bar z z}\right) = -\frac{Y}{mR^2}$$

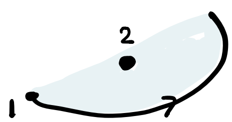
$$a_y = i\left(a_z - a_{\bar z}\right) = \frac{1}{2m}\left(\frac{1}{z} + \frac{1}{\bar z}\right) = \frac{X}{mR^2}$$



$$\vec a = (a_x, a_y) = \frac{1}{mR^2}\left(-Y, X\right) = -\frac{\vec R \times \hat z}{mR^2} = \frac{1}{mR} \hat\varphi$$

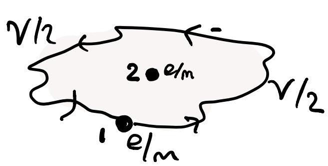
$$\therefore \quad \boxed{\theta = \frac{1}{2}\int d\vec R \cdot \vec a(r) = \frac{\pi}{m}.}$$

STATISTICAL PHASE



($\frac{1}{2}$ a circumference)

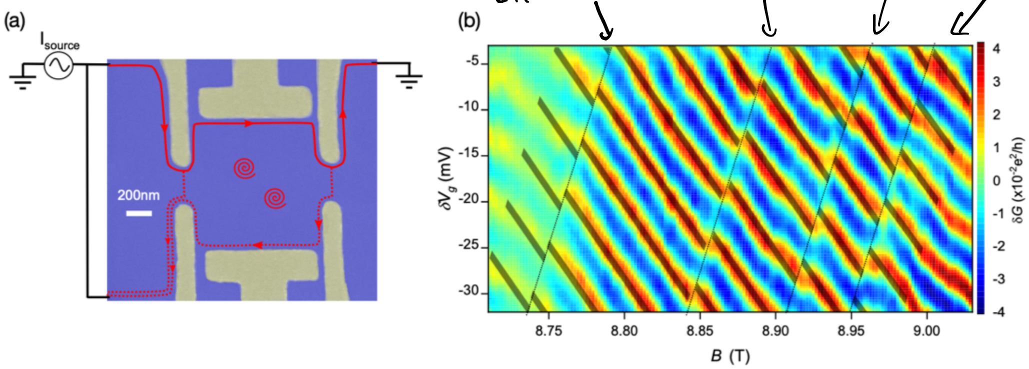
$$\boxed{\gamma_{total} = \gamma + 2\theta = \frac{2\pi}{m}\left(\underbrace{\frac{\text{Area}}{2\pi e^2}}_{\text{electromagnetic phase}} + \overset{\text{Statistical Contribution}}{1}\right)}$$



COMPLETE CIRCUIT OF SECOND PARTICLE

Arovas, Schrieffer + Wilczek (1984).

$\frac{\Delta\theta}{2\pi}$ -0.29 -0.28 -0.38 -0.32



changed by V_g

$$\Theta = 2\pi\left(\frac{e^*}{h}\right) BA + N\Theta^*$$

$$\Theta^* = \frac{2\pi}{m} \ (?)$$

$$\delta G = \delta G_0 \cos\left(\frac{2\pi AB}{\Phi_0} + \Theta_0\right)$$

Nakamura, Liang, Gardner + Manfra, ArXiv 2006 14115

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