5. Fractional Quantum Hall Effect

- Early expts showed that for $\nu < 1$, σ_{xx} & σ_{xy} vanished, as expected from the perspective of the IQHE. However, with the invention of modulation doping in GaAs quantum wells, much higher quality samples were possible, with much less disorder.

- A Wigner crystal was expected for low enough e^- concentrations in pure samples. Pinned by disorder it would be insulating.

- 1982 Tsui, Störmer & Gossard discovered a QH plateau at $\nu = 1/3$, with $\sigma_{xx} \to 0$ & $\sigma_{xy} = \frac{1}{3} e^2/h$.

- Tsui joked that it might be quarks! The effect did not involve quarks, but incredibly, the electrons in the $\nu = 1/3$ FQH state have indeed condensed into a state with fractional charge excitations $q^* = \frac{1}{3}e$.

- Many more fractions observed

- Because $\sigma_{xx} \to 0$, this is a dissipationless state, with a gap, presumeably driven by the $e^- - e^-$ Coulomb interaction.

- The new ideas developed to explore & understand these new phases of matter & their topological properties have had profound implications for physics, both in the lab, and the cosmos!

5.1 PRELIMINARIES: Mechanical Momentum + Guiding Centers

Recall
$$H = \frac{(\vec{p} + e\vec{A})^2}{2m} = \frac{\vec{\Pi}^2}{2m}$$

where
$$\vec{\Pi} = (\Pi_x, \Pi_y) = \vec{p} + e\vec{A}(\vec{r})$$
MECHANICAL MOMENTUM

This is a gauge-invariant quantity, but unlike the canonical momentum, Π_x & Π_y do not commute

$$[\Pi_x, \Pi_y] = [\overset{-i\hbar\partial_x}{p_x}, eA_y] + [eA_x, \overset{-i\hbar\partial_y}{p_y}]$$
$$= e\left([-i\hbar\partial_x, A_y] - [-i\hbar\partial_y, A_x]\right)$$

$$\boxed{[\Pi_x, \Pi_y] = -i\hbar e \underbrace{(\partial_x A_y - \partial_y A_x)}_{B_z = -\mathbf{B} \text{ (for convenience)}} = i\hbar e B = \frac{i\hbar^2}{\ell^2} \quad \left(\frac{\hbar}{eB} = \ell^2\right)}$$

$$\boxed{a = \frac{\ell}{\sqrt{2}\hbar}\left(\Pi_x + i\Pi_y\right), \qquad a^\dagger = \frac{\ell}{\sqrt{2}\hbar}\left(\Pi_x - i\Pi_y\right)}$$

L.L operators.

$$\Rightarrow [a, a^\dagger] = \frac{\ell^2}{2\hbar^2}\left([\Pi_x, -i\Pi_y] + [i\Pi_y, \Pi_x]\right) = 1$$

$$H = \hbar\omega_c\left(a^\dagger a + \tfrac{1}{2}\right).$$

$$\Rightarrow E_n = \hbar\omega_c\left(n + \tfrac{1}{2}\right).$$

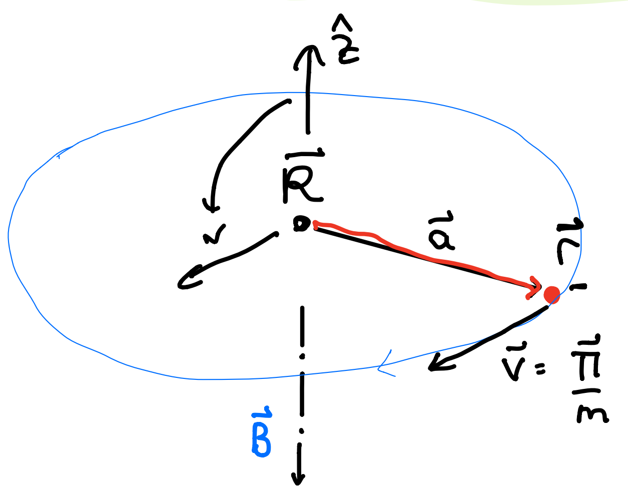
States in the lowest Landau level satisfy

$$a|\psi\rangle = 0 \quad \text{in any gauge.}$$

Now so far, we can not see the massive degeracy (νN_Φ).

This requires the introduction of **guiding center** coordinates.

$$\boxed{\vec{R} = (R_x, R_y) = \vec{r} - \ell^2(\hat{z}\times\vec{\Pi})/\hbar}$$



$$V = \omega_c a$$

$$\Rightarrow a = \frac{V}{\omega_c} = \frac{\Pi}{eB} = \frac{\ell^2\Pi}{\hbar}$$

$$\vec{a} = \frac{\ell^2}{\hbar}\hat{z}\times\vec{\Pi}$$

$$\Rightarrow \vec{R} = \vec{r} - \frac{\ell^2}{\hbar}(\hat{z}\times\vec{\Pi})$$

$$[R_x, R_y] = \left[r_x + \frac{\ell^2}{\hbar}\Pi_y,\ r_y - \frac{\ell^2}{\hbar}\Pi_x\right]$$

$$= \frac{\ell^2}{\hbar}\Big(\underbrace{[\Pi_y, r_y] + [\Pi_x, r_x]}_{-2i\hbar}\Big) + \left(\frac{\ell^2}{\hbar}\right)^2 \overbrace{[\Pi_x, \Pi_y]}^{i\hbar^2/\ell^2} = -i\ell^2$$

"QUANTUM GEOMETRY"

$\Delta R_x \Delta R_y \gtrsim \ell^2/2$

$$\boxed{b = \frac{1}{\sqrt{2}\,\ell}(R_x - iR_y), \quad b^+ = \frac{1}{\sqrt{2}\,\ell}(R_x + iR_y)}$$

$$[b, b^+] = \frac{1}{2\ell^2}\Big(\overbrace{[-iR_y, R_x]}^{\ell^2} + \overbrace{[R_x, iR_y]}^{\ell^2}\Big) = 1$$

$$[R_x, \Pi_x] = \left[r_x + \frac{\ell^2\Pi_y}{\hbar},\ \Pi_x\right] = \overbrace{[r_x, p_x]}^{i\hbar} - \frac{\ell^2}{\hbar}\overbrace{[\Pi_x, \Pi_y]}^{i\hbar^2/\ell^2} = 0$$

$$[R_x, \Pi_y] = \left[r_x + \frac{\ell^2\Pi_y}{\hbar},\ \Pi_y\right] = 0$$

$$\Rightarrow [R_i, \Pi_j] = [R_i, H] = 0$$

Commute w/ Hamiltonian

$$\boxed{\begin{aligned} [a, b] &= [a, b^+] = 0 \\ [b, H] &= 0 \end{aligned}}$$

$|n, m\rangle$

$$= \frac{(a^+)^n (b^+)^m}{\sqrt{n!}\sqrt{m!}}|0\rangle$$

Guiding c. coords ≡ projection of position into LL

$$P_n r_x P_n = P_n \left(\frac{\ell (b+b^\dagger)}{\sqrt{2}} + \cancel{\frac{i\ell}{\sqrt{2}t} (a - a^\dagger)} \right) P_n$$

$$= R_x P_n \equiv R_x$$

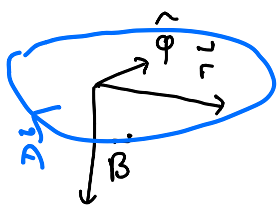
$$P_n = \sum_m |n,m\rangle\langle n,m|$$

So $\boxed{\vec{R} = P_n \vec{r} P_n}$ is the projection of the position co-ordinate into the nth Landau level.

16.2 THE SYMMETRIC GAUGE

$$\boxed{\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B} = -\frac{rB}{2}\hat{\varphi}}$$

$\vec{B} = -B\hat{z}$



Convenient to introduce $z = (x+iy)/\ell$, $\bar{z} = (x-iy)/\ell$

$$a = \frac{\ell}{\sqrt{2}\hbar}\left[\overbrace{p_x + ip_y}^{-\frac{i\hbar 2}{\ell}\frac{\partial}{\partial z^*}} - \frac{ieB}{2}(x+iy)\right]$$

$$= \frac{-i}{\sqrt{2}}\left[\frac{z}{2} + 2\frac{\partial}{\partial z^*}\right]$$

$$y = \frac{z - z^*}{2i}\ell$$

$$x = \frac{z + z^*}{2}\ell$$

$$\Downarrow$$

$$\frac{\partial}{\partial z^*} = \frac{\partial x}{\partial z^*}\partial_x + \frac{\partial y}{\partial z^*}\partial_y$$

$$= \frac{\ell}{2}(\partial_x + i\partial_y)$$

$$\boxed{a = \frac{-i}{\sqrt{2}}\left[\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right]}$$

$$\boxed{a^{+} = \frac{i}{\sqrt{2}}\left[\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right]}$$

$$\boxed{b = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right)}$$

$$\boxed{b^{+} = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial \bar{z}}\right)}$$

"Guiding center"

$$\boxed{\psi(x,y) = f(z)e^{-\bar{z}z/4}}$$

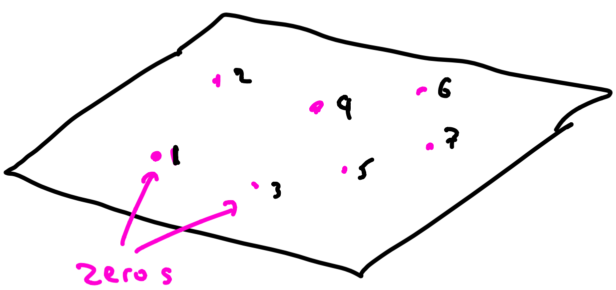
State in the lowest Landau level.

$$a = \frac{-i}{\sqrt{2}}\left[\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right]$$

$$2\frac{\partial \psi}{\partial \bar{z}} = -\frac{1}{2} z \psi \quad , \Rightarrow \quad \underline{a\psi = 0}$$

General analytic form for $f(z)$

$$f(z) = \prod_{j=1}^{N} (z - Z_j)$$



Another useful form is

$$f_\lambda(z) = \frac{1}{\sqrt{2\pi \ell^2}}\, e^{\frac{1}{2}\bar{\lambda} z}\, e^{-\frac{1}{4}\bar{\lambda}\lambda} \qquad \boxed{\text{COHERENT STATE}}$$

$$|\psi_\lambda|^2 = |f_\lambda|^2 e^{-\bar{z}z/2}$$

$$= \frac{1}{2\pi \ell^2} e^{\frac{1}{2}(\bar{\lambda} z + \bar{z}\lambda) - \frac{\bar{z}z}{2} - \frac{\bar{\lambda}\lambda}{2}}$$

$$= \frac{1}{2\pi \ell^2} \exp\left[-\frac{1}{2}(\bar{z}-\bar{\lambda})(z-\lambda)\right] \sim P_0\, \delta^2(\vec{r}-\vec{\lambda}) P_0$$

Normalized ψ

$$\varphi_m(x,y) = \varphi_m(z,\bar{z}) = \frac{1}{\sqrt{2\pi \ell^2\, 2^m\, m!}}\, z^m e^{-|z|^2/4}.$$

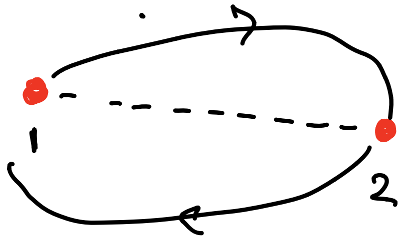
$$L_z = m\hbar \qquad n = \frac{1}{2\pi \ell^2} \qquad \frac{1}{\sqrt{n}}\left(\frac{z}{\sqrt{2}}\right)^m \frac{1}{\sqrt{m!}}\, e^{-\frac{|z|^2}{4}}$$

- ## One body potential

If $U(\vec{r}) = U(|\vec{r}|)$, then U conserves angular momentum & does not mix φ_m & $\varphi_{m'}$ $(m \neq m')$. In a strong field, φ_m is an eigenstate, & a potential just shifts the energies by

$$u_m = \langle \varphi_m | V | \varphi_m \rangle \qquad E_m = \frac{1}{2}\hbar\omega + u_m$$

5.3 Two body Problem + Haldane Pseudopotentials



$$T = \frac{\vec{\Pi}_1^{\,2} + \vec{\Pi}_2^{\,2}}{2m}$$

$$= \frac{\vec{\Pi}_{CM}^{\,2}}{2M} + \frac{\vec{\Pi}_r^{\,2}}{2\mu}$$

$$\vec{\Pi}_{CM} = \vec{\Pi}_1 + \vec{\Pi}_2 \qquad M = 2m$$

$$\vec{\Pi}_r = (\vec{\Pi}_1 - \vec{\Pi}_2)/2 \qquad \mu = \frac{1}{\frac{1}{m} + \frac{1}{m}} = \frac{m}{2}$$

$$[\Pi_{CM}, \Pi_r] = 0$$

$$\tilde{\Pi}_{cm} = \frac{\Pi_{cm}}{\sqrt{2}} \qquad \tilde{\Pi}_{rel} = \sqrt{2}\,\Pi_{rel}$$

$$[\tilde{\Pi}^x_{cm}, \tilde{\Pi}^y_{cm}] = \frac{i\hbar^2}{\ell^2} \qquad [\tilde{\Pi}^x_{rel}, \tilde{\Pi}^y_{rel}] = \frac{i\hbar^2}{\ell^2}$$

Both CM + relative motion involve the same ω_c & ℓ.

$$\boxed{\Psi_{mM} = (z_1 - z_2)^m (z_1 + z_2)^M \, e^{-\frac{1}{4}(\bar{z}_1 z_1 + \bar{z}_2 z_2)}}$$

- Lies in the LLL
- $L_{rel} = \hbar m \qquad L_{CM} = \hbar M$

EXACT SOLUTION FOR RELATIVE POTENTIAL

$$V(|\vec{r}_1 - \vec{r}_2|)$$

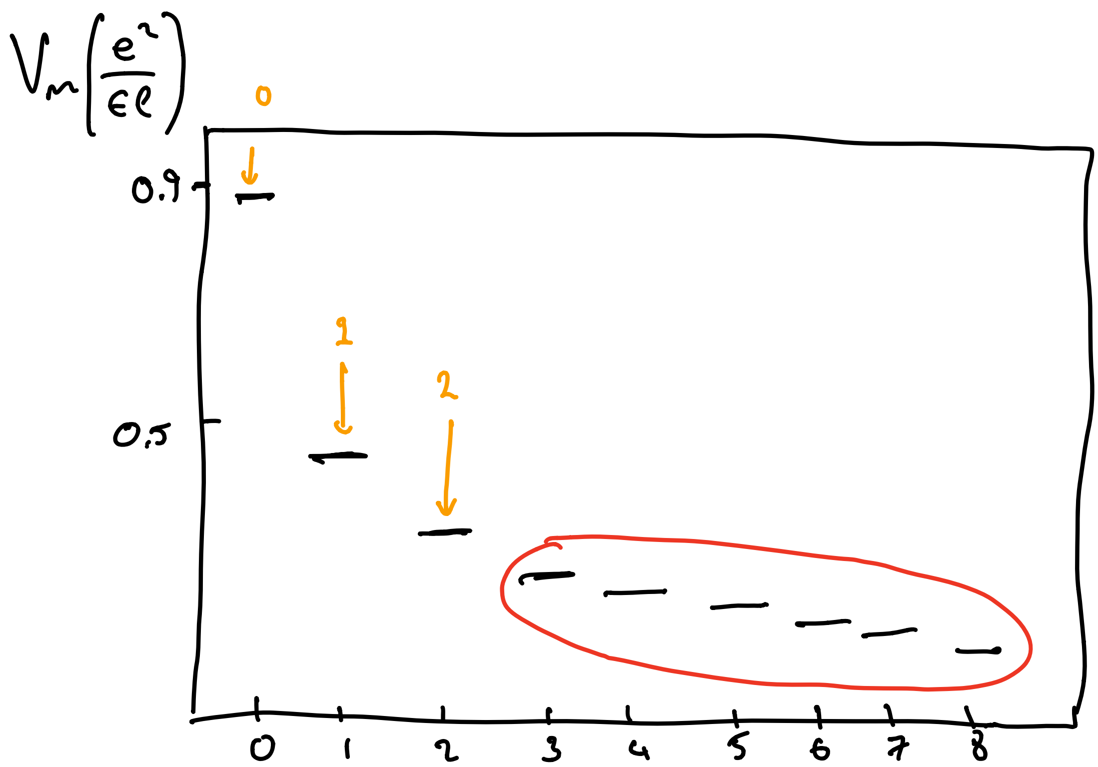
Corresponding energy eigenvalue is simply

$$V_m = \frac{\langle mM | V | mM \rangle}{\langle mM | mM \rangle}$$

$\boxed{\text{Haldane Pseudopotential}}$

e.g. Coulomb interaction

$$V = \frac{e^2}{\epsilon r} \qquad V_m = \frac{e^2}{2\epsilon \ell} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}$$



This provides an important clue to the Full many body problem.

e.g. Calculate V_m for Coulomb Interaction $V = e^2/\epsilon r$.

$$|z_1|^2 + |z_2|^2 = \left(\frac{z_r}{2} + \frac{z_{cm}}{2}\right)^2 + \left(\frac{-z_r + z_{cm}}{2}\right)^2 = \frac{|z_r|^2}{2} + \frac{|z_{cm}|^2}{2}$$

$$dz_1 dz_2 = dz_{cm}\, dz_{rel}$$

$$V_m = \frac{\int d\bar{z}_r dz_r \left(\frac{e^2}{\ell\sqrt{\bar{z}_r z_r}}\right) (\bar{z}_r z_r)^m e^{-\frac{1}{4}|z_r|^2}}{\int d\bar{z}_r dz_r\, e^{-\frac{1}{4}|z_r|^2} (\bar{z}_r z_r)^m}$$

$$= \frac{\frac{e^2}{\ell}\int r\,dr\, r^{(2m-1)} e^{-r^2/4}}{\int r\,dr\, r^{2m} e^{-r^2/4}} = \frac{\frac{e^2}{\ell}\int 2du\,(4u)^{\frac{2m-1}{2}} e^{-u}}{\int 2du\,(4u)^{\frac{2m}{2}} e^{-u}} = \frac{e^2}{\ell}\,\frac{1}{2}\,\frac{\left(\frac{2m-1}{2}\right)!}{\left(\frac{2m}{2}\right)!}$$

$$x! = \Gamma(x+1)$$

$$V_m = \frac{e^2}{\epsilon\ell}\boxed{\frac{\Gamma(m+\frac{1}{2})}{2\Gamma(m+1)}}$$

e.g. $B = 10T$ $\epsilon = 10$

$$\ell = \sqrt{\frac{\hbar}{eB}} = \sqrt{\frac{1.05\times10^{-34}}{1.6\times10^{-19}\times10}} = 8.1\,nm$$

$$\frac{e^2}{4\pi\epsilon\epsilon_0\ell} = \frac{(1.602\times10^{-19}C)^2}{4\pi\times(8.854\times10^{-12})\times10\times8.1\times10^{-9}m} = 2.84\times10^{-21}J \approx 206K$$

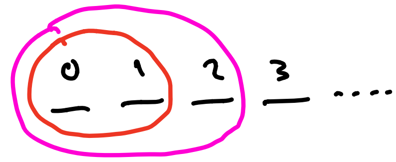
$$\hbar\omega_c = 13.4\,K$$

$$V_3 = 0.28\times206K = \underline{\underline{57K}}$$

5.4 The $\nu = 1$ STATE

$$\Psi_1[z] = f[z]\, e^{-\frac{1}{4}\sum_j |z_j|^2}$$

SLATER DET



Two particles

$c_1^\dagger c_0^\dagger |0\rangle$

$$f_2[z] = \begin{vmatrix} 1 & 1 \\ z_1 & z_2 \end{vmatrix} = (z_2 - z_1)$$

SLATER DETERMINANT

Three particles

$c_2^\dagger c_1^\dagger c_0^\dagger |0\rangle$

$$f_3(z) = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ (z_1)^2 & (z_2)^2 & (z_3)^2 \end{vmatrix}$$

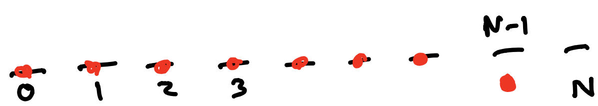
Zero when two columns are equal

$z_i = z_j$

$$= (z_3 - z_1)(z_3 - z_2)(z_2 - z_1)$$

$$= \prod_{i>j} (z_i - z_j)$$

$$f_N = \prod_{i>j} (z_i - z_j)$$



- Highest power of any z_i is z_i^{N-1}

- Antisymmetry guarantees single occupancy $\therefore$ states $m \in [0, N-1]$ are occupied.

$$\prod g(x_i - x_j) \to (x_i - x_j)^m$$

$$\Psi_1(z) = \left(\prod_{i>j} (z_i - z_j) \right) \exp\left[-\frac{1}{4} \sum_j |z_j|^2 \right]$$

Filled L.L.

Jastrow Fn.

$$|\Psi_1(z)|^2 = \prod_{i<j} |z_i - z_j|^2 \exp -\frac{1}{2} \sum_j |z_j|^2$$

$$= e^{-\beta U_{eff}} = e^{-\frac{2}{m}\left(\frac{m}{2} \ln |\Psi|^2\right)}$$

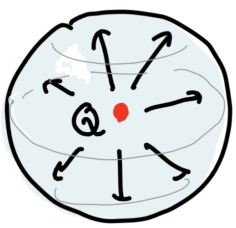
$$\beta = 2/m$$

$$\boxed{U_{eff} = m^2 \sum_{i<j} \ln |z_i - z_j| + \frac{m}{4} \sum_j |z_j|^2}$$

"Plasma Analogy"

Recall Electrostatics

3D



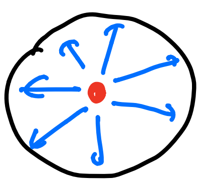
$$\int \vec{E} \cdot ds = 4\pi Q$$

$$E = Q \frac{\hat{r}}{r^2}$$

$$\phi = -\nabla E = Q/r$$

$$\nabla \cdot E = -\nabla^2 \phi = 4\pi Q \delta^3(r)$$

2D



$$\int \vec{E} \cdot ds = 2\pi Q$$

$$\vec{E} = (Q/r)\,\hat{r}$$

$$\phi = Q\left(-\ln r/r_0\right)$$

$$\nabla \cdot E = -\nabla^2 \phi = 2\pi Q \delta^2(r)$$

$$U_0 = m^2 \sum_{i>j} -\ln|z_i - z_j|$$

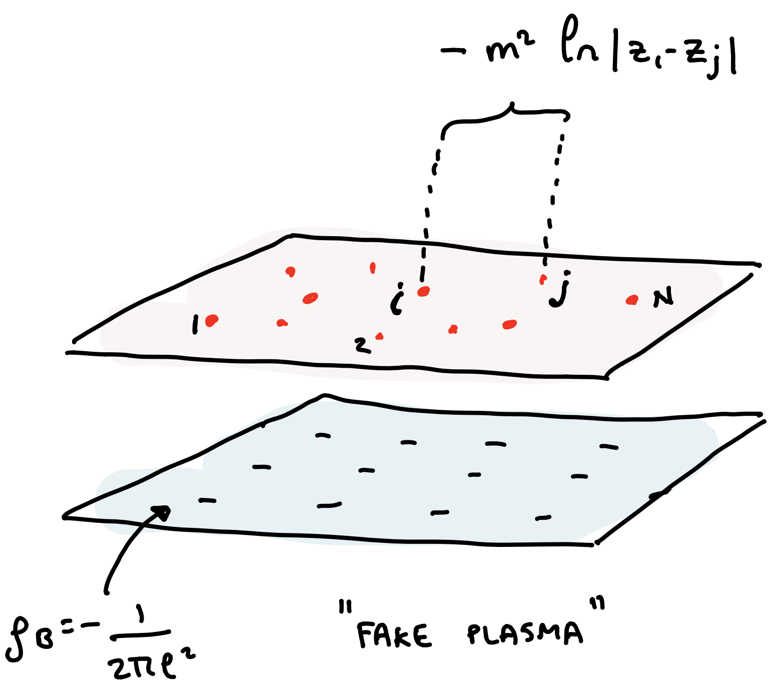
Coulomb Interaction Energy

$$m\left(\frac{1}{4}\sum_j |z_j|^2\right) = m\sum_j \phi(z_j)$$

Potential from constant background charge $\rho_B = -\frac{1}{2\pi \ell^2}$

$$-\nabla^2\left(\frac{1}{4}|z_j|^2\right) = -\left(\nabla_x^2 + \nabla_y^2\right)\left(\frac{1}{4}\frac{(x^2+y^2)}{\ell^2}\right) = -\frac{1}{\ell^2} = 2\pi\rho_B$$

$$\Rightarrow \rho_B = -\frac{1}{2\pi \ell^2}$$



$$\boxed{\begin{array}{l}\text{Neutrality} \Leftrightarrow \text{Largest } |\Psi|^2 \\ nm + \rho_B = 0\end{array}}$$

$$\Rightarrow n = \frac{1}{m}\left(\frac{1}{2\pi \ell^2}\right) = \frac{1}{m}\left(\frac{N_\phi}{A}\right)$$

$$\equiv \frac{1}{m}\text{th filled L.L.}$$

$$2\pi \ell^2 B = \phi_0$$

$$\Rightarrow \frac{1}{2\pi \ell^2} = \frac{B}{\phi_0} = \frac{\phi}{\phi_0 A} = \frac{N_\phi}{A}$$

$$\times\ \times\ \times\ \times\ \overbrace{\bullet\ \otimes}^{1}\ \times\ \times\ \times\ \bullet\ \times\ \times\ \times\ \times\ \bullet$$

$m = 5$