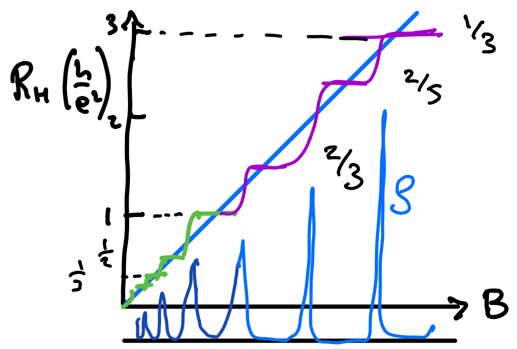
4. Fractionalization + the Quantized Hall Effect

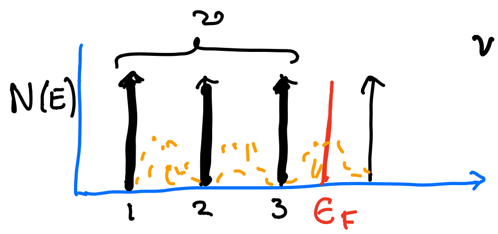
$\boxed{\text{Overview}}$

1 • Preliminaries. Hall effect, Drude theory. $\quad R_H = \rho_{xy} = \dfrac{V_y}{I_x}$

2 • Integer QHE $\quad R_H = \dfrac{h}{e^2}\left(\dfrac{1}{\nu}\right)$

$\nu \in \mathbb{Z}$





3 • Fractional QHE $\quad \nu = p/q$

Laughlin wavefunction,

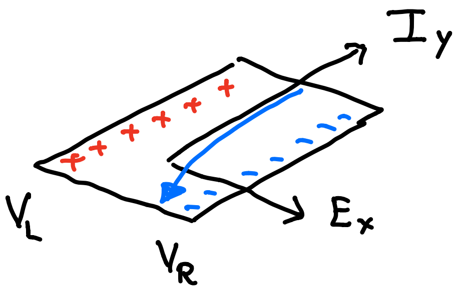
4 • Fractional charge + statistics, composite fermions

5 • Chern-Simons theory.

Integer QHE

Lorentz Force $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ $\boxed{q=-e}$

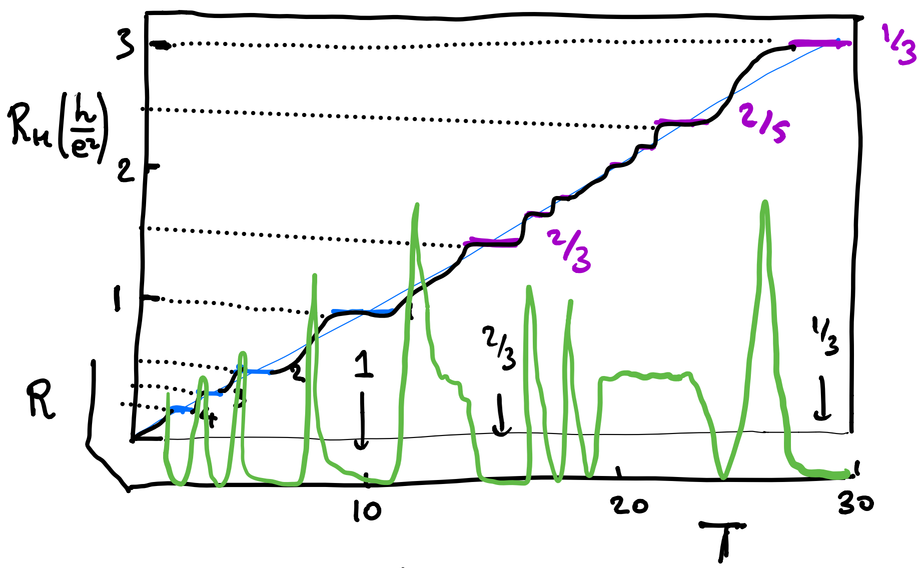
$$\boxed{\begin{aligned} J_\mu &= \sigma_{\mu\nu} E_\nu \\ E_\mu &= \rho_{\mu\nu} J_\nu \end{aligned}}$$



$$E_x = \rho_{xy} J_y = -\nabla_x V = \frac{\overbrace{(V_L - V_R)}^{V_H}}{L_x}$$

$$= \rho_{xy} \frac{I_y}{L_x}$$

- No need to know the dimensions of sample
- $\rho_{xy} = -\frac{1}{nq} = \frac{1}{n|e|}$ electrons.



$$\rho_{xy} = \frac{h}{e^2}\nu$$

$\nu \in \mathbb{Z}$ integer

$\nu \in \frac{p}{q}$ fractional

Von Klitzing — IQHE

Tsui, Störmer + Gossard. — FQHE.

$$R_H = \frac{1}{ne} = \frac{h}{e^2}\,\frac{Be}{nh} = \frac{h}{e^2}\left(\frac{BA}{h/e}\right)\frac{1}{nA}$$

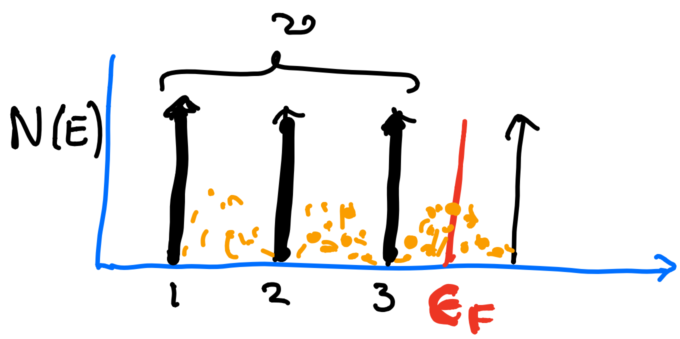
$$= \frac{h}{e^2}\,\frac{\Phi}{\Phi_0}\,\frac{1}{N}$$

$$= \frac{h}{e^2}\,\frac{N_\phi}{N} = \frac{h}{e^2 \nu}$$

$$\boxed{\Phi_0 = h/e}$$

Flux quantum

$$\boxed{\nu = N/N_\phi = \#\ e^-/\text{flux quanta} \equiv \text{filling factor.}}$$



Requires Disorder

Drude Theory

$$\dot{v} = -\frac{e}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right) - \frac{\vec{v}}{\tau}$$

relaxation time

$= 0$ in equilibrium

$$v_F \tau = \ell_{mfp}$$

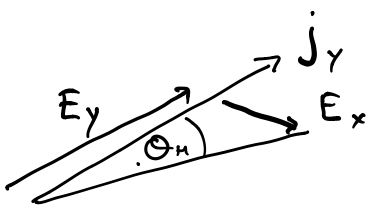
$$\frac{e}{m}\left(E + \vec{v}\times\vec{B}\right) = -\frac{\vec{v}}{\tau}$$

$$\Longrightarrow \quad \vec{v} + \frac{e}{m}\vec{v}\times\vec{B} = -\frac{e\vec{E}\tau}{m}$$

$$\boxed{\omega_c = \frac{eB}{m}}$$

$$\vec{v} - \omega_c\tau\, \hat{b}\times\vec{v} = -\frac{e\vec{E}\tau}{m}$$

$$\boxed{\vec{j} = -ne\vec{v}}$$



$$\tan\theta_H = \omega_c\tau$$

$$\boxed{\left(1 - \omega_c\tau\,\hat{b}\times\right)\vec{j} = \frac{ne^2\tau}{m}\vec{E}}$$

$$\boxed{\sigma = \frac{ne^2\tau}{m}}$$

$$\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \sigma\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\frac{\omega_c\tau m}{ne^2\tau} = \frac{\left(\frac{\cancel{e}B}{\cancel{m}}\right)\cancel{\tau}\cancel{m}}{ne^{\cancel{2}}\cancel{\tau}} = \frac{B}{ne}$$

$$\underbrace{\begin{pmatrix} \frac{1}{\sigma} & \frac{B}{ne} \\ -\frac{B}{ne} & \frac{1}{\sigma} \end{pmatrix}}_{\rho_{\mu\nu}}\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\rho_{xy} = \frac{1}{ne} = \frac{V_x}{I_y}$$

Classical Treatment of electron motion

$$H = \frac{1}{2m}(\underbrace{\vec{p}}_{\text{canonical}} + e\vec{A})^2 - eV \quad = \quad \frac{\overbrace{\Pi}^{\text{dynamical}}{}^2}{2m} - eA$$

$$\frac{\partial H}{\partial \vec{p}} = \vec{v} = \frac{\vec{p} + e\vec{A}}{m} \qquad \{x_i, p_j\} = \delta_{ij}$$

$$-\frac{\partial H}{\partial x_i} = \dot{p}_i = e\,\nabla_i V - \nabla_i A_j \overbrace{\left(\frac{p_j + eA_j}{m}\right)}^{v_j}$$

$$= e\,\nabla_i V - e\, v_j \nabla_i A_j$$

$$m\frac{dv_i}{dt} = \frac{dp_i}{dt} + e\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x^j} v_j\right)$$

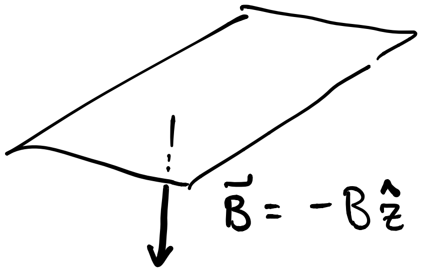
$$= e\left(\nabla_i V + \frac{\partial A_i}{\partial t}\right) + e\left(\frac{\partial A_i}{\partial x^j} v_j - \frac{\partial A_j}{\partial x^i} v_j\right)$$

$$v \times (\nabla \times A) = (v_j \nabla_i A_j - v_j \nabla_j A_i)$$

$$m\frac{d\vec{v}}{dt} = -e\left[\left(-\nabla V - \frac{\partial A}{\partial t}\right) + \vec{v} \times (\vec{\nabla} \times \vec{A})\right]$$

$$= -e\left[\vec{E} + \vec{v} \times \vec{B}\right]$$

LANDAU QUANTIZATION



$$\vec{A} = (0, -Bx, 0)$$

$$B_z = \partial_x A_y - \partial_y A_x = -B.$$

$$\hat{\mathcal{H}} = \frac{(p+eA)^2}{2m} = \frac{p_x^2}{2m} + \frac{(p_y - eBx)^2}{2m}.$$

No dependence on $y \Rightarrow [p_y, \mathcal{H}] = 0$ $\quad p_y$ is conserved.

$$\psi = e^{iky} f(x) \qquad p_y = \hbar k$$

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{(\hbar k - eBx)^2}{2m}$$

$$= p_x^2/2m + \frac{m\omega_c^2}{2}\left(x - \frac{\hbar k}{eB}\right)^2$$

$$\ell^2 = \frac{\hbar}{eB}$$

$$\boxed{\mathcal{H} = p_x^2/2m + \frac{m\omega_c^2}{2}\left(x - k\ell^2\right)}$$

$$2\pi\ell^2 = \frac{2\pi\hbar}{eB} = \frac{\Phi_0}{B}$$

$$\tilde{x} = x - k\ell^2 = x - X$$

$$\boxed{\mathcal{H} = \frac{p_x^2}{2m} + \frac{m\omega_c^2}{2}\tilde{x}^2}$$

$$p' = p\ell/\hbar \qquad [x', p'] = i$$
$$x' = \tilde{x}/\ell$$

$$H = \left(\frac{p'^2}{2} + \frac{(x')^2}{2}\right)\hbar\omega_c = h\omega_c\left(a^+ a + \frac{1}{2}\right)$$

$$a = \frac{1}{\sqrt{2}}(p' - ix') \qquad a^+ = \frac{1}{\sqrt{2}}[p' + ix'] = \frac{1}{\sqrt{2}}\left(-i\frac{\partial}{\partial x'} + ix'\right)$$

$$|k_y, n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|k_y, 0\rangle$$

$$\psi_0 = e^{iky}\exp-\left[\frac{(x-k\ell^2)^2}{2\ell^2}\right]$$

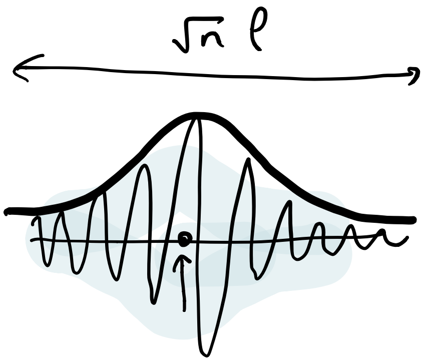
$$\psi_n = \frac{e^{iky}}{\sqrt{n!}}\left(-i\frac{\partial}{\partial x'} + ix'\right)^n e^{-\frac{(x-k\ell^2)^2}{2\ell^2}}$$

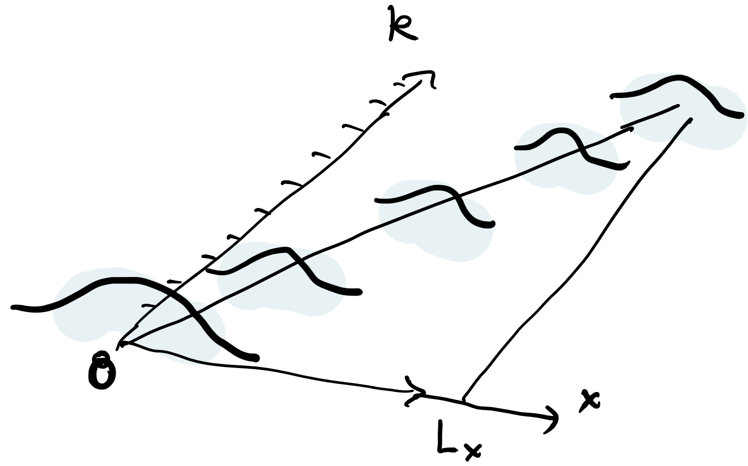
$$\boxed{\psi_{kn} = e^{iky}\,\mathcal{H}_n\left(\frac{x}{\ell} - k\ell\right) e^{-\frac{(x-k\ell^2)^2}{2\ell^2}}}$$

$$\boxed{E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)}$$

- $\frac{\partial E_{kn}}{\hbar\partial k} = 0 \Rightarrow v_y = 0$
- centered at $x = k\ell^2$
- width in x direction is $\sqrt{n}\,\ell$

$\left(n\hbar\omega_c \sim \frac{m\omega_c^2}{2}x^2 \Rightarrow x^2 \sim n\frac{\hbar}{m\omega_c} = n\frac{\hbar}{eB} = n\ell^2\right)$





Notice that $\quad X_n = k_n \ell^2 = n\left(\frac{2\pi}{L_y}\right)\ell^2 \ll \sqrt{n}\,\ell.$

In the nth Landau level there are.

$$N = \sum_k 1 = L_y \int_0^{L_x/\ell^2} \frac{dk}{2\pi} = \frac{L_x L_y}{2\pi \ell^2}$$

$$= \frac{A\,eB}{2\pi\hbar} = \frac{AB}{(h/e)}$$

$$N = \left(\frac{AB}{\Phi_0}\right) = \frac{\Phi}{\Phi_0} = N_\Phi$$

For ν levels $\quad N = \nu N_\Phi$.