Motivation : Hidden Off-diagonal Long Range Order in the FQHE

A superfluid, or Bose Einstein condensate displays Off-Diagonal Long Range order:

$$\boxed{\langle \hat{\psi}^{+}(x)\,\hat{\psi}(y)\rangle \xrightarrow[|x-y|\to\infty]{} \langle \psi(x)\rangle^{*}\langle \psi(y)\rangle \qquad \text{ODLRO}} \qquad \boxed{\Psi = \sqrt{\rho_s}\,e^{i\alpha}}$$

In 1987 Girvin + MacDonald proposed a tentative order parameter, but it could not be expressed as a local quantity. In 1989, Nicholas Read refined their ideas and proposed a composite field

$$\hat{k}(z) = \psi^{+}(z)\big(U(z)\big)^{m}$$

where $\hat{U}(z)$ creates a quasihole at z. This composite of m anyons and an electron behaves as a neutral boson. Read was able to show that this operator exhibits ODLRO. Specifically, with N particles in the mth Laughlin state, $|\phi_m;N\rangle$

$$\boxed{\begin{aligned}\langle \phi_m;N|\,\hat{k}^{+}(z)\,k(z')\,|\phi_m;N\rangle &= \frac{1}{\rho_0}\langle O_m;N+1|\,\hat{\rho}(z)\,\hat{\rho}(z')\,|O_m;N+1\rangle \\ &\xrightarrow[|z-z'|\to\infty]{} \rho_0 \qquad \text{N.Read '89}\end{aligned}}$$

FQHE: Chern-Simons Theory.

Key idea (Zhang, Hansson, Kivelson, 1989; building on earlier work of Read & of Girvin + Macdonald): think of fluxes in the FQHE as an EMERGENT GAUGE FIELD a^μ. Treat the problem as bosons with m flux tubes attached by a C.S theory with $k = \frac{1}{m}$.

$$\boxed{S_B = \int d^3x \left\{ \psi^*(x) \left[i D_0 + \mu \right] \psi(x) - \frac{1}{2M} |D\psi|^2 + \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right\} + S_{Coulomb} + S_{local}}$$

$$D_0 = \partial_0 - i\left(\frac{e}{\hbar} A_0 + a_0\right)$$

$$\boxed{D_\mu = \partial_\mu - \frac{ie}{\hbar} A_\mu - i a_\mu}$$

$$D_i = \vec{\nabla}_i + i\left(\frac{e}{\hbar} \vec{A} + \vec{a}\right) \quad (\vec{A}_i = A^i = -A_i)$$

$$D_\mu = \partial_\mu - i(A_\mu + a_\mu)$$

$$S_{Coulomb} = -\frac{1}{2} \int d^3x \left(|\psi(x)|^2 - \rho_0\right) V(x-x') \left(|\psi(x')|^2 - \rho_0\right)$$

$$S_{Local} = - \int d^3x \ \lambda |\psi(x)|^4.$$

expect a Meissner effect that expels

$$[0 = (A_\mu + a_\mu)] \Rightarrow a = -A_\mu^{ext}$$

Ginzburg - Landau Theory.

$$Z = \sum_{\text{configs}} e^{-S_E}$$
$$Z = \sum_{\text{configs}} e^{iS/\hbar}$$

We seek a mean field theory where S is stationary with respect to variations in the fields.

$$\boxed{\begin{aligned} S = \int d^3x \Big[\psi^*(iD_0+\mu)\psi + \tfrac{1}{2m}\psi^* D^2\psi + \tfrac{k}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \Big] \\ -\tfrac{1}{2}\int d^3x\, d^3x' \left(|\psi(x)|^2-\rho_0\right) V(x-x') \left(|\psi(x')|^2-\rho_0\right) - \int d^3x\, \lambda|\psi|^4 - \mu\rho_0 L^2 T \end{aligned}}$$

$$D_\mu = \partial_\mu - i(A_\mu + a_\mu) \qquad \left(e=1,\ \hbar = 1 \Rightarrow \frac{h}{e} = 2\pi\right)$$

$$\delta S = \int d^3x\ \delta\psi^*(x)\Big[(iD_0+\mu)\psi + \frac{1}{2M}D^2\psi - \psi(x)\int d^3x'\, V(x-x')\left(|\psi(x')|^2-\rho_0\right) - 2\lambda\psi|\psi|^2\Big] + \text{h.c}$$
$$+\ \delta a_0\Big(-\frac{k}{2\pi}\overbrace{b}^{\nabla\times\vec{a}} + |\psi(x)|^2\Big)$$

Flux addition

$$+\ \delta a^i\Big[\frac{k}{2\pi}\epsilon_{i\mu\nu}\partial^\mu a^\nu \ + \frac{i}{2M}\left(\psi^* D_i\psi - (D_i\psi)^*\psi\right)\Big]$$

Quantization of Hall current

$$+\ \delta\mu\left(\int d^3x\, |\psi(x)|^2 - \rho_0 L^2 T\right)$$

Static MFT.

$$\begin{cases} |\psi(x)|^2 = \rho_0 \\ \rho_0 - \frac{k}{2\pi} b_z = 0 \\ \mu - 2\lambda \rho_0 = 0 \\ \langle a_\mu \rangle + A_\mu = 0 \end{cases}$$

$\left(\Leftrightarrow D_\mu \psi = 0 \right)$ $\boxed{b_z = -B_z}$

Fields cancel.

Thus $|\rho_0| = \frac{k}{2\pi} B$

$= \frac{k}{2\pi e^2}$

Recall $e^2 = \frac{\hbar}{eB} \equiv \frac{1}{B}$

But $\rho_0 = \frac{\nu}{2\pi e^2} \Rightarrow \boxed{\nu = k = \frac{1}{m}}$

$$\boxed{\psi(x) = \sqrt{\rho(x)}\, e^{i\theta(x)}}$$

Low energy Fluctuations

$$\psi = \sqrt{\rho(x)}\, e^{i\theta(x)}$$

$$\psi(x) \to \psi(x)\, e^{i\phi(x)}$$
$$a \to a + \partial_\mu \phi$$
$$\theta \to \theta + \phi$$

$$\partial_\mu \psi = \left(\frac{1}{2\sqrt{\rho}} \partial_\mu \rho + i(\partial_\mu \theta) \sqrt{\rho} \right) e^{i\theta}$$

$$D_\mu \psi = \left(\frac{1}{2\sqrt{\rho}} \partial_\mu \rho + i\left(\partial_\mu \theta - (A_\mu + a_\mu)\right) \sqrt{\rho} \right) e^{i\varphi}$$
$$= \left(\frac{1}{2\sqrt{\rho}} \partial_\mu \rho + i D_\mu \theta \sqrt{\rho} \right) e^{i\varphi}$$

$$|D_\mu \psi|^2 = \left(\frac{1}{2\sqrt{\rho}} \partial_\mu \rho \right)^2 + |D_\mu \varphi|^2 \rho$$

$$D_\mu \theta = \partial_\mu \theta - (A_\mu + a_\mu)$$

$$S = \int d^3x \left[\frac{1}{4} (i\partial_0 + \mu)\rho - \frac{1}{8M\rho} (\nabla \rho)^2 - \lambda \rho \right]$$
$$- \frac{1}{2} \int d^3x\, d^3x' \,(\rho(x) - \rho_0)\, V(x-x')\, (\rho(x') - \rho_0)$$
$$+ \int d^3x \left[-\rho D_0 \theta - \frac{\rho}{2M} (D\theta)^2 \right] + \frac{k}{4\pi} \int d^3x\, \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

$$\delta S = \int d^3x \left[(\delta a_0 - \partial_0 \theta)\delta\rho + \frac{1}{8M\rho} \delta\rho\, \nabla^2 \delta\rho - \lambda \delta\rho^2 \right] - \frac{1}{2} \int d^3x\, d^3x'\, \delta\rho(x)\, \delta\rho(x') V(x-x')$$
$$+ \int d^3x \left[-\frac{\rho_0}{2M} |\nabla\theta - \delta\vec{a}|^2 \right] + \frac{k}{4\pi} \int d^3x \left(\delta a^\mu \partial^\nu \delta a^\lambda \right) \epsilon_{\mu\nu\lambda}.$$

$$\delta\rho(x) = \sum_q \delta\rho_q\, e^{iq\cdot x}$$

$$\sum_q \left[(\delta a_0 - \partial_0\theta)_{-q}\,\delta\rho_q - \frac{1}{2}\overbrace{\left(\frac{q^2}{4M} + V(q) + 2\lambda\right)}^{\chi^{-1}(q)}|\delta\rho_q|^2\right]$$

Complete the square, integrate out $\delta\rho_q$ fluctuations: $\quad \delta\rho_q^* = \delta\rho_{-q}$

$$\delta\tilde{\rho}_q = \delta\rho + \chi(\delta a - \delta_0\theta)_q$$

$$-\frac{1}{2}\chi^{-1}\left(\delta\rho_{-q} + \chi(\delta a_0 - \partial_0\theta)_{-q}\right)\left(\delta\rho_q + \chi(\delta a - \partial_0\theta)_q\right)$$

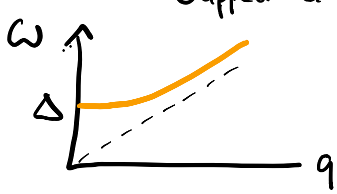
$$+\frac{\chi}{2}\left|(\delta a_0 - \partial_0\theta)_q\right|^2$$

$$\Rightarrow \boxed{\mathcal{L}_{\text{eff}} = \frac{\chi}{2}(\partial_0\theta - \delta a_0)^2 - \frac{\rho_s}{2}(\nabla\theta + \delta\vec{a})^2 + \frac{k}{4\pi}\epsilon_{\mu\nu\lambda}(a^\mu\partial^\nu a^\lambda)}$$

$$c_s^2 = \rho_s/\chi$$

$$\chi = \frac{1}{2\lambda + V(0)} \qquad \rho_s = \frac{\rho_0}{M} = \frac{k}{2\pi}\frac{B}{M} = \frac{\nu}{2\pi}\frac{B}{M} = \frac{\nu\hbar\omega_c}{2\pi}$$

- Gapped excitations $\quad c^2 = (\rho_s/\chi) \qquad \Delta = \rho_s\left(\frac{4\pi}{k}\right)$



$$\omega^2 = c^2q^2 + \Delta^2.$$

- $\delta a_\mu = a_\mu - \langle a_\mu\rangle = a_\mu + A_\mu$ is a massive field

$\therefore$ $a_\mu = -A_\mu$ in the ground state.

(Meissner effect.)

- θ can be absorbed into δa $\qquad \delta a_\mu \to \delta a_\mu - \partial_\mu\theta$

"Anderson - Higgs" effect.

$$\delta S = \frac{1}{2}\sum \left(\delta a_0, \delta a_1, \delta a_2\right)_{-q} \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \chi & 0 & \frac{-ik}{4\pi} iq_x \\ 0 & -\rho_s & \frac{ik}{4\pi}\omega \\ \frac{ik}{4\pi} q_x & \frac{-ik}{4\pi}\omega & -\rho_s \end{pmatrix} \end{array} \begin{pmatrix} \delta a_0 \\ \delta a_1 \\ \delta a_2 \end{pmatrix}_q$$

$$\chi\left(\rho_s^2 - \frac{k^2\omega^2}{(4\pi)^2}\right) + \frac{k^2}{(4\pi)^2}\rho_s\left(q_x^2 + q_y^2\right)$$

$$\Rightarrow \omega = \sqrt{(cq)^2 + 0^2}$$

$$\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\,\delta a_\mu \partial_\nu \delta a_\lambda$$

$$= \sum_q \frac{k}{4\pi}\left[\epsilon^{\mu\nu\lambda}(-iq_\nu)\right]\delta a_\mu(q)\,\delta a_\lambda(q)$$

$$\frac{k}{4\pi}\epsilon^{012}(+iq^1)\,\delta a_0(-q)\,\delta a_2(q)$$

$$\frac{k}{4\pi}\epsilon$$

Fractionalization

→ Break up of microscopic degrees of freedom

→ Possibility of fractional statistics + topological action.

Hall Conductance

$A^{ext} \to A^{ext} + \delta A^{ext}$

$$\boxed{\mathcal{L}_{eff} = \frac{\chi}{2}\left(\partial_0\theta - (a_0 + e\,\delta A_0)\right)^2 - \frac{\rho_s}{2}\left(\nabla\theta + (\vec{a} + e\,\delta\vec{A})\right)^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda.}$$

$$e^{iS_{eff}[\delta A]} = \int \mathcal{D}[\theta, a] \exp\left[i \int d^3x\, \mathcal{L}_{eff}[\theta, a_\mu, \delta A_\mu] \right]$$

Can choose the "London" gauge $\theta = 0$ (Anderson – Higgs).

The resulting path integral is dominated by configurations where $a_\mu + e\,\delta A_\mu = 0$. Evaluating the action then gives

$$S_{eff}[A] = \int d^3x\, \mathcal{L}_{eff}[\theta = 0, a_\mu = -e\,\delta A_\mu, \delta A_\mu]$$

$$= \frac{e^2 k}{4\pi} \int d^3x\, \epsilon^{\mu\nu\lambda}(\delta A_\mu \partial_\nu \delta A_\lambda)$$

QHE CS $\longleftrightarrow$ Meissner S.C.

$\mu = i \quad \nu\lambda = j0 \text{ or } 0j$

$$j^\mu = \frac{\delta S}{\delta\,\delta A_\mu} = \frac{k e^2}{2\pi}\epsilon^{\mu\nu\lambda}(\partial_\nu \delta A_\lambda)$$

$$\boxed{j^\mu = \frac{k e^2}{4\pi}\epsilon^{\mu\nu\lambda} F_{\nu\lambda}}$$

$$j^i = \frac{k}{2\pi} e^2 \epsilon^{ij}\,\delta E_j \quad \Rightarrow \quad \sigma_{xy} = \frac{k}{2\pi}\left(\frac{e^2}{\hbar}\right) = \frac{1}{m}\left(\frac{e^2}{h}\right)$$

$$j^\mu = \chi_r\left(\partial^\mu\theta - (A^\mu + a^\mu)\right)$$

$\mu_0 j = (\nabla \times B)$

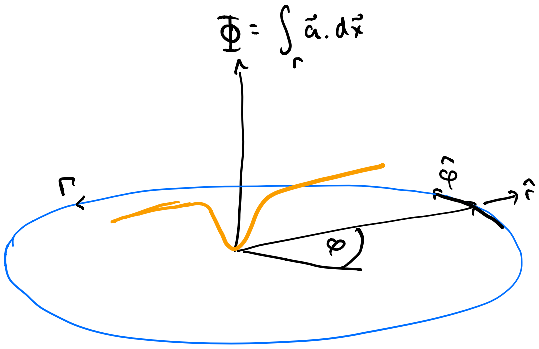
$\mu_0 \rho_s A = -\nabla^2 A$

$\Rightarrow$ Meissner

Vortices: Anyons.

$$\boxed{\mathcal{L}_{eff} = \frac{\chi}{2}\left(\partial_0\theta - \delta a_0\right)^2 - \frac{\rho_s}{2}\left(\nabla\theta + \delta\vec{a}\right)^2 + \frac{k}{4\pi}\epsilon_{\mu\nu\lambda}\left(a^\mu\partial^\nu a^\lambda\right)}$$

Static solution $\left(\nabla\theta + (\vec{a}+\vec{A})\right) = 0$. Supercurrent vanishes



$$\boxed{\Psi = \sqrt{\rho_0}\, e^{i\theta(x)}}$$

$$\theta = \pm\varphi$$

$$\nabla\theta = \pm\frac{1}{r}\hat{\varphi}$$

$$\delta\vec{a} = -\vec{\nabla}\theta = \mp\frac{1}{r}\hat{\varphi}$$

At large distances $\lim_{|x|\to\infty}\left|\left(i\nabla - \vec{a}\right)\Psi(x)\right|^2 \to 0$, ∞

the cost of the vortex is finite. The net flux of the gauge field is

$$\oint \vec{a}.d\vec{x} = \mp 2\pi$$

The induced charge density is

$$\rho(x) \equiv J^0(x) = -e\frac{\delta S_{eff}}{\delta a_0(x)} = -e\frac{k}{2\pi}\epsilon_{ij}\partial_i a_j(x)$$

$$= -e\frac{k}{2\pi}\left(\nabla\times a\right)$$

Thus the total induced charge is

$$Q = \int d^2x \, \rho(x) = -\frac{k}{2\pi} e \int d^2x \, (\nabla \times a)_z$$

$$= \frac{ek}{2\pi} \oint_{\Gamma} d\vec{x} \cdot \vec{a}(x)$$

$$= ek$$

$$Q_{\text{Anyon.}} = e/m. \quad \equiv \underline{\text{LAUGHLIN QUASIHOLE.}}$$