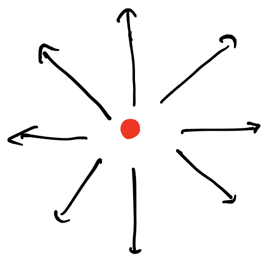
# 10. CHERN SIMONS THEORY

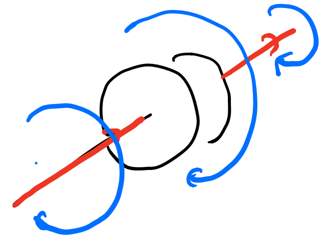
Sofar, we have looked at the microscopic wavefunction of the FQHE and its excitations. wever, we'd like to understand the long-wavelength properties: what is the electrodynamics, the long-wavelength action; is there an order parameter, is there an analog of the Ginzburg Landau theory of superconductors?

It turns out that to understand the quantum Hall effect, both the integer & the fractional variety, requires an emergent electrodynamics, Chern Simons Theory. The key idea, is that we integrate out the electronic degrees of freedom, to obtain an effective action

$$e^{i S_{eff}[A]/\hbar} = \int \mathcal{D}[\text{fields}]\, e^{i S[\text{fields}, A]/\hbar}$$

The conventional Maxwell action gives us Gauss' & Ampères law

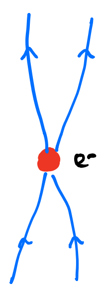


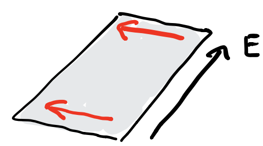


$\nabla . E = \rho / \epsilon_0$

$$\left( \nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} \right) = \mu_0 \vec{J}$$

We'll see that the new action — the CHERN-SIMONS action gives us the physics of <u>flux</u> attachment & the quantum Hall effect




$$\rho(x) = \frac{k}{2\pi} B(x)$$

FLUX ATTACHMENT

$$J_x = \frac{k}{2\pi} E_y$$

QUANTIZED HALL CURRENT

# Relativistic Electrodynamics

Before we jump into CS theory, let us remind ourselves about Maxwellian electromagnetism. Fundamental to the physics is the scalar & vector potential, it terms of which

$$E = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi \qquad \vec{B} = (\nabla \times \vec{A})$$

In terms of these fields, the classic Maxwell action is

$$S_M = \int dt\, d^2x \left( \left[ \frac{1}{2\mu_0} \left( \left(\frac{E}{c}\right)^2 - B^2 \right) \right] + \vec{J}.A - \rho \phi \right)$$

where I'm already preparing you for 2+1 dimensions.

This is most succinctly formulated in a relativistic formulation, using the electromagnetic field tensor

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

Now, it is very easy to get into a mess with minus signs & conventions in relativistic E.M, and the formalism is much akin to the theory of gravity.

Ok, but a few preliminaries. We denote $x^\mu = (ct, \vec{x})$ as space time co-ordinates, so that

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \equiv \left(\frac{\partial}{c\,\partial t}, \vec{\nabla}\right)$$

and

$$A^\mu = \left(\frac{V}{c}, \vec{A}\right), \quad A_\mu = \left(\frac{V}{c}, -\vec{A}\right)$$

where $A_\mu = g_{\mu\nu} A^\nu$, $g_{\mu\nu} \equiv \mathrm{diag}(1,-1,-1)$. Also

$$J^\mu = \rho(c, \vec{v}) = (c\rho, \vec{J}), \quad J_\mu = \rho(c, -\vec{v}) = (c\rho, -\vec{J})$$

You can check that in 2+1 D

$$\underset{\partial_\mu A_\nu - \partial_\nu A_\mu}{F_{\mu\nu}} = \begin{bmatrix} 0 & E^1/c & E^2/c \\ -E^1/c & 0 & -B \\ -E^2/c & B & 0 \end{bmatrix} = \begin{bmatrix} F_{00} & F_{01} & F_{02} \\ F_{10} & F_{11} & F_{12} \\ F_{20} & F_{21} & F_{22} \end{bmatrix}$$

e.g.

$$F_{12} = \partial_1\big(\overset{-\vec{A}_2}{A_2}\big) - \partial_2\big(\overset{-\vec{A}_1}{A_1}\big) = -B_z \equiv -B_z$$

$$F_{i0} = \partial_i A_0 - \partial_0 A_i = \frac{\nabla_i V}{c} + \frac{\partial A_i}{c\,\partial t} = -\frac{E_i}{c}$$

$$F^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta}\, g^{\beta\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 0 & E & E \\ -E & 0 & -B \\ -E & +B & \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$$= \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{pmatrix} 0 & -E/c & -E_y/c \\ E_x/c & 0 & -B_z \\ E_y/c & B_z & 0 \end{pmatrix}$$

In terms of this notation

$$\frac{1}{2\mu_0}\left(E^2 - B^2\right) = \frac{1}{4\mu_0}\left(F_{\mu\nu} F^{\nu\mu}\right)$$

and the full action is

$$\left(-\partial_\nu \delta A_\mu + \partial_\mu \delta A_\nu\right)$$

$$S = \int d^3x \left[\frac{1}{4\mu_0}\left(F_{\mu\nu} F^{\nu\mu}\right) - J_\mu A^\mu\right]$$

$$\left.\begin{array}{l} J^\mu = \left(\rho c, \vec{J}\right) \\ A^\mu = \left(\phi/c, \vec{A}\right) \end{array}\right\} \quad J^\mu A_\mu = \left(\rho\phi - \vec{A}\cdot\vec{J}\right)$$

When you take variations with respect to $A^\mu \to A^\mu + \delta A^\mu$, one obtains

$$\delta S = \int d^3x \left(\frac{1}{\mu_0}\, \partial^\nu F_{\nu\mu} - J_\mu\right)\delta A^\mu$$

which gives Maxwells equations (Ampère & Gauss)

$$\partial_\nu F^{\nu\mu} = \mu_0 J^\mu \quad \left\{ \begin{array}{l} \text{Gauss} \\ \text{Ampère} \end{array} \right.$$

e.g

$$\partial_i F^{i0} = \frac{\nabla . E}{c} = \mu_0 c \rho \Rightarrow \nabla . E = \mu_0 c^2 \rho$$

$$= \rho / \epsilon_0$$

$$\partial_i F^{ij} = \mu_0 J^j \Longrightarrow (\nabla \times B) - \frac{1}{c^2} \frac{\partial E}{\partial t} = \mu_0 \vec{J}$$

We will often adopt $\mu_0 = 4\pi$, $c = 1$ in our work.

# CHERN SIMONS THEORY : IQHE

$$S_{cs} = \frac{k}{4\pi}\int d^3x \left(\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho\right) \qquad \left(\text{FRADKIN } \frac{k}{\pi} = \theta\right)$$

$$= \frac{k}{8\pi}\int d^3x \left(\epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}\right)$$

$$= \frac{k}{4\pi}\int d^3x \left(A_0 F_{12} + A_1 F_{20} + A_2 F_{01}\right)$$

$$= \frac{k}{4\pi}\int d^3x \left(\frac{\phi}{c} B - \left(A_1 E_2 - A_2 E_1\right)\right)$$

$$= \frac{k}{4\pi}\int d^3x \left(\frac{\phi}{c} B_z - (A \times E)_z\right)$$

Very strange! Breaks time reversal + Mirror symmetry.

Not obviously gauge invariant because it involves a naked vector potential.

$$\psi \to e^{i\omega}\psi$$

## Gauge invariance

$$A_\mu \longrightarrow A_\mu + \partial_\mu \omega$$

$$F_{\mu\nu} \longrightarrow F_{\mu\nu}$$

$$S_{cs} \to S_{cs} + \frac{k}{4\pi}\int d^3x \left(\epsilon^{\mu\nu\rho}\, \partial_\mu \omega\, \partial_\nu A_\rho\right)$$

$$= S_{cs} + \frac{k}{4\pi}\int d^3x \left[\partial_\mu\left(\epsilon^{\mu\nu\rho}\, \omega\, \partial_\nu A_\rho\right) - \omega\, \underbrace{\epsilon^{\mu\nu\rho}\, \partial_\mu \partial_\nu A_\rho}_{=0}\right]$$

Total Derivative

→ Vanishing surface term

= 0

$$= S_{cs}$$

## "Maxwell" equations

$$A_\mu \to A_\mu + \delta A_\mu$$

$$\delta S = \int d^3x \left[\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\left(\delta A_\mu\, \delta_\nu A_\rho + A_\mu\, \partial_\nu \delta A_\rho\right) - J^\mu \delta A_\mu\right]$$

$$- \epsilon^{\rho\nu\mu}\, \partial_\nu A_\rho\, \delta A_\mu$$

$$= \int d^3x \left[\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\left(\delta A_\mu \partial_\nu A_\rho - \partial_\nu A_\mu\, \delta A_\rho\right) - J^\mu \delta A_\mu\right]$$

$$= \int d^3x \left[\frac{k}{2\pi}\epsilon^{\mu\nu\rho}\, \partial_\nu A_\rho - J^\mu\right]\delta A_\mu$$

$$\Rightarrow \qquad J^\mu = \frac{k}{2\pi}\epsilon^{\mu\nu\rho}\, \partial_\nu A_\rho$$

Write in co-ordinates $(c=1)$

$$\rho(x) = J^0(x) = \frac{k}{2\pi}\left(\partial_1 A_2 - \partial_2 A_1\right) = -\frac{k}{2\pi} B_z$$

$$\int \rho \, d^2x = eN_p$$

$$J^i(x) = \frac{k}{2\pi} \epsilon^{ij}\left(\partial_j A_0 - \partial_0 A_j\right) = -\frac{k}{2\pi}\epsilon^{ij} E_j$$

i.e. $$N_p = \frac{k}{2\pi}\frac{\Phi}{e} = \frac{k\Phi_0}{2\pi e}\overbrace{\left(\frac{\Phi}{\Phi_0}\right)}^{N_\Phi}$$

$$\Rightarrow \frac{N_p}{N_\Phi} = \frac{k\Phi_0}{2\pi e} = \nu$$

$$\Rightarrow \nu = \frac{k\hbar}{e^2} \Rightarrow \boxed{k = \frac{\nu e^2}{\hbar}}$$

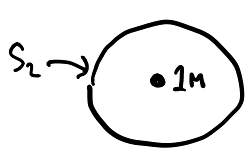
FIXED DENSITY ; FIXED # particles/flux tube

$$\boxed{S_{cs} = \frac{\nu e^2}{2h}\int d^3x \, e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho}$$

$$\boxed{\sigma_{xy} = \frac{k}{2\pi} = \frac{\nu e^2}{h}}$$

IQHE.

It turns out, that if you quantize the C.S theory on a sphere (containing one monopole), then one is only allowed $\nu = \mathbb{Z}$ integers. Explains **IQHE**.




$$\int B.dS = 1 \quad + \text{Periodicity in time.}$$


# CHERN SIMONS AND FLUX ATTACHMENT

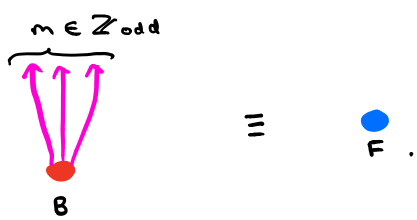
We found before, that if you attach a flux of strength $\Phi$ to a boson, it acquires a statistical angle

$$\boxed{\theta = \pi\left(\frac{q}{e}\right)\left(\frac{\Phi}{\Phi_0}\right)} \quad \sim \pi \times \#\text{ flux tubes}.$$

$$\eta = e^{i\theta} = -1$$

$$\phi/\Phi_0 = \text{odd integer } (m)$$

So if $q = e$, attaching $m = \Phi/\Phi_0$ odd flux quanta to a boson produces a Fermion.




But the C.S action leads to $\quad N_\phi / N = \dfrac{2\pi e}{k\Phi_0} = \dfrac{1}{k}$

Since in units where $e = \hbar = 1 \quad \Phi_0 = h/e = 2\pi\dfrac{\hbar}{e} = 2\pi.$

Thus if $k = 1/m$, the Chern Simons Theory attaches $m$ flux quanta. In this way we can TRANSMUTE bosons into Fermions.

$$D_\mu = \partial_\mu + i\frac{e}{\hbar}A_\mu + ia_\mu$$

$$iD_0 = \left(i\partial_0 - \frac{e}{\hbar}A_0 - a_0\right)$$

# FQHE: Chern-Simons Theory.

Key idea (Zhang, Hansson, Kivelson, 1989; building on earlier work of Read & of Girvin + Macdonald): think of fluxes in the FQHE as an EMERGENT GAUGE FIELD $a^\mu$. Treat the problem as bosons with $m$ flux tubes attached by a C.S theory with $k = \frac{1}{m}$.

emergent gauge field

$$\boxed{S_B = \int d^3x \left\{ \psi^*(x)\left[iD_0 + \mu\right]\psi(x) - \frac{1}{2M}|D\psi|^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right\} + S_{Coulomb} + S_{local}}$$

$$S_{Coulomb} = -\frac{1}{2}\int d^3x \left(|\psi(x)|^2 - \rho_0\right) V(x-x') \left(|\psi(x')|^2 - \rho_0\right)$$

$$S_{Local} = -\int d^3x \, \lambda |\psi(x)|^4.$$