GRADUATE QUANTUM MECHANICS: 501 Fall 1999

Final exam. Monday, Dec 20th. 9:00am-

1. Short questions.
   (a) If $|i\rangle$ and $|j\rangle$ are eigenkets of Hermitian operator A. Under what conditions is $|i\rangle + |j\rangle$ an eigenket of A?
   (b) A beam of intensity I carrying spin 1/2 atoms polarized in the $+z$ direction passes through two Stern Gerlach type measurements. The first measurement only accepts atoms with $S_n = \hbar/2$, where $S_n = \mathbf{S} \cdot \hat{\mathbf{n}}$ is the spin component along an axis $\hat{\mathbf{n}}$ at an angle β to the z-axis. The second measurement only accepts "down-spin" atoms with $S_z = -\hbar/2$. What is the final beam intensity?
   (c) If $f(A)$ is a function of an operator A with eigenkets $|a\rangle$ where $A|a'\rangle = a'|a'\rangle$, write down an expression for the matrix elements $\langle b'|f(A)|b''\rangle$ of $f(A)$ in a new basis where the matrix elements relating the $|a'\rangle$ and $|b'\rangle$ basis are known.
   (d) The states $|1\rangle$ and $|2\rangle$ are energy eigenstates with energies E_1 and E_2. The operator A has eigenkets $|+\rangle$ and $|-\rangle$ given by $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$, where A has values a_+ and a_- respectively. Calculate the time dependence of the expectation value $\langle A(t)\rangle$.
   (e) An electron moves in one dimension a potential $V(x) = -\lambda x$. Give an approximate mathematical form for the wavefunction and sketch it, taking care to show how the amplitude varies with position. Is the spectrum bounded or unbounded?

2. A particle moves in a sperically symmetric potential, and has wavefunction
$$\psi(x, y, z) = \langle x, y, z|\psi\rangle = f(r)(x + y + 3z). \tag{1}$$
   (a) Is this state an eigenstate of $\mathbf{L}^2$? Explain your answer.
   (b) If the component of angular momentum in the z direction is measured, what values can be obtained, and what will their probability be?
   (c) If $|\psi\rangle$ is an energy eigenstate with energy E_0, use the wavefunction to derive the corresponding potential $V(r)$.

3. This is a question about positronium, a bound-state of an electron and a positron. Since positrons and electrons have the same mass, we have to take into account the motion of both particles.
   (a) For two particles of mass m_1 and m_2 show that the total momentum $\mathbf{P} = \mathbf{p_1} + \mathbf{p_2}$and the center of mass position
$$\mathbf{X} = (m_1\mathbf{x_1} + m_2\mathbf{x_2})/M, \tag{2}$$
   (where $M = m_1 + m_2$), are canonically conjugate.
   (b) Show that the relative position $\mathbf{x} = \mathbf{x_2} - \mathbf{x_1}$ and relative momentum
$$\mathbf{p} = \mu\left(\frac{1}{m_1}\mathbf{p}_2 - \frac{1}{m_2}\mathbf{p}_1\right), \tag{3}$$
   where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass, commute with $\mathbf{P}$ and $\mathbf{X}$ and are also canonically conjugate.
   (c) Show that the Hamiltonian for the system
$$H = \frac{(\mathbf{p}_1)^2}{2m_1} + \frac{(\mathbf{p}_2)^2}{2m_2} + V(|\mathbf{x_2} - \mathbf{x_1}|) \tag{4}$$
   decouples into two terms $H = H_{CM} + H_{internal}$ where
$$H_{CM} = \frac{1}{2M}\mathbf{P}^2, \qquad H_{internal} = \frac{1}{2\mu}\mathbf{p}^2 + V(|\mathbf{x}|) \tag{5}$$

(d) Use the equations of motion for $\mathbf{P}$ and $\mathbf{p}$ to show that (i) the center of mass momentum is conserved and (ii) the internal motion is equivalent to a *single* single particle of mass μ moving about a fixed potential $V(r)$, where $r = |\mathbf{x}|$.

(e) Working by analogy with the hydrogen atom, give expressions for (a) the bound-state energies and (b) the "Bohr" radius associated with the ground-state wavefunction of positronium. What are the approximate numerical sizes of these quantities?

(f) When a positron encounters an electron, they annihilate into two photons. Which angular momentum states of positronium will be the most unstable? Explain your answer carefully.