Physics 228, Lecture 18

Monday, April 5, 2004

Nuclei Ch 44:1–4

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1 The Nucleus

We resume our trip to smaller and smaller distance scales, shifting our focus from atoms to the nucleus of the atom. Nuclear physics began with the observation of various kinds of radiation of much higher energy than the atomic spectra that come from atomic energy levels. We now understand that these come from decays of unstable atomic nuclei.

The three observed kinds of radiation were originally called alpha, beta, and gamma, because their nature was unknown. We now know that they are

- alpha rays: the emission of a particle identical to a helium nucleus, or twice ionized helium atom, He^{++}.

- beta rays are high energy electrons.

- gamma rays are high energy photons, higher energy than X-rays.

By scattering alpha rays from other atoms, Rutherford showed that the atomic nucleus was very small, contained most of the mass of the atom, and a positive charge. The charge of the nucleus is a multiple Z of the fundamental charge e, with Z a positive integer, and an atom of that nucleus will include Z bound electrons outside the nucleus. The mass of the nucleus in general grows with Z, but not proportional, and in fact there are nuclei of different masses but the same Z, which are known as isotopes.

In 1919 Rutherford found that an alpha ray hitting a nitrogen nucleus can convert it to oxygen and emerge as a proton,

$$\alpha + N \longrightarrow O + p,$$

thus discovering the proton and creating the first transmutation of elements. A proton is the same as the nucleus of a hydrogen atom. Clearly there might be protons inside a nucleus. But most nuclei are much heavier than Z times the proton mass, so it didn't seem like they were just a collection of protons.

At first physicists thought some nuclei might have electrons inside, explaining why the charge might be less than indicated by the mass, but we now know this is not a viable explanation, not least because an electron confined to such a small volume would have a very high zero-point energy. Rutherford proposed there must be neutral particles as well, and in 1932 Chadwick identified the neutron as the particle emitted in the interaction in which an alpha particle hitting a beryllium nucleus turns it into a carbon nucleus and a neutron:

$$ {}^{4}_{2}\alpha + {}^{9}_{4}\text{Be} \rightarrow {}^{12}_{6}\text{C} + {}^{1}_{0}\text{n}. $$
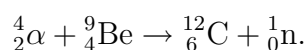
Here is what the notation means:

Every nucleus has an **atomic number**, Z, which is its charge divided by the fundamental charge e. It is always positive, except for the neutron, which is neutral ($Z = 0$), and for antiparticles. Each nucleus has a mass which is roughly, but not exactly, an integer number of atomic mass units — the integer is called its **mass number** A. A nucleus is then described by its atom's chemical symbol, which for beryllium is Be, prefixed with a subscript Z and a superscript A. The nucleus can be thought of as composed of Z protons and $A - Z$ neutrons bound together by a force we have not yet met, known as the strong interaction. So ${}^{9}_{4}\text{Be}$ is a beryllium nucleus consisting of 4 protons and five neutrons. Notice there are a total of 6 protons and $13 - 6$ neutrons in the initial state and also in the final state in Chadwick's reaction above.

By the way, neutrons and protons are referred to as the two kinds of **nucleons**. So A is the number of nucleons in the nucleus.

The most common form of iron has a nucleus with 30 neutron and 26 protons, so $Z = 26$ and $A = 56$. We write this as ${}^{56}_{26}\text{Fe}$. But only 92% of natural iron nuclei have those values; 6% have only 28 neutrons and $A = 54$, 2% have 31 neutrons and $A = 57$, and 1/4% have 32 neutrons and $A = 58$. All have 26 protons, because otherwise they would not be iron nuclei. Giving the Z value and the chemical name is redundant, and often the Z is left off, and then we might talk of these four different nuclei as ${}^{54}\text{Fe}$, ${}^{56}\text{Fe}$, ${}^{57}\text{Fe}$, and ${}^{58}\text{Fe}$. Different nuclei with the same Z value are called **isotopes**. Atoms with different isotopes behave almost identically, at least in their chemical properties, but there are very slight differences in their physical properties, as for example in the infrared spectrum of HCl I displayed last time, due to the different masses of the atoms.

Nuclear masses are often measured in atomic mass units rather than in kilograms. An atomic mass unit is defined as $1/12$ of the mass of a carbon **atom** with a $^{12}_{6}$C nucleus, and is represented by the unit u. The masses of the three particles that build up atoms are:

proton	m_p	1.007 276	u
neutron	m_n	1.008 665	u
electron	m_e	0.000 548 6	u

A $^{12}_{6}$C atom consists of 6 protons, 6 neutrons and 6 electrons. Adding up those masses gives 12.098 938 u, considerably more that the 12.000 000 u the atom weighs, by definition. Why? Because there is binding energy holding the atom together and holding the nucleus together. It would take energy to split up a carbon atom into its 18 constituents, and that energy has an equivalent mass, as relativity tells us.

The masses of nuclei and especially of elementary particles are often expressed in terms of the equivalent rest energy mc^2, measured in MeV, millions of electron volts, with

$$1\,\text{u} \equiv 931.494\,\text{MeV}/\text{c}^2.$$

So the total binding energy in the $^{12}_{6}$C atom is $0.098938 \times 931.494 \times 10^6 \approx$ 92 MeV, of which only a few KeV is due to the atomic binding of the electrons, nearly all the binding is due to the strong interactions holding the nucleus together.

How big is a nucleus? You will recall that Rutherford was surprised that alpha particles shot at heavy nuclei were sometimes scattered directly backwards. What else could happen if a positive charge is aimed directly at another positive charge? The initial kinetic energy will be converted into Coulomb repulsive potential energy at the point where the alpha particle turns around[1], so

$$E = \frac{1}{2}mv^2 = k_e\frac{q_1 q_2}{r}, \quad \text{so } r_{\text{turnaround}} = 2Zk_e e^2/E.$$

So the higher the energy of the incident α, the closer the point of turnaround, but there always is one, so why was Rutherford surprised?

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[1]This argument ignores the recoil energy of the gold nucleus, but the gold is much heavier than the alpha particle. Also the discussion can take the recoil into account without significantly changing the results.

Rutherford expected that an energetic alpha particle would penetrate close enough so that it would penetrate the nucleus, which he thought of as a spherical region roughly the size of an atom with the charge spread out throughout. As we have considered before, this should lead to a force which decreases (proportional to r for a uniform charge) and a potential which approaches a finite limit as $r \to 0$, probably less than the energy of the alpha particle. In fact, he calculated after the experiment that the nucleus had to be less than 10^{-14} m in radius, compared to about 5×10^{-11} for an atom.

In fact, nuclei have been shown to all have about the same density, so the volume is proportional to A, the number of nucleons, and therefore the radius is

$$r = r_0 A^{1/3}, \quad \text{with } r_0 = 1.2 \times 10^{-15}\,\text{m} = 1.2 \text{ fm}.$$

The prefix f or femto, means 10^{-15}, though most old-timers pronounce femtometer as "fermi".

2   Nuclear Binding

If the nucleus is so small and filled with positive charge, what holds it together? It turns out there is a strong but short range attractive force between all nucleons.

This is a very strong force. Consider a proton and a neutron. They can bind together to form a deuteron, the nucleus of deuterium, ^2_1H, one of the isotopes of hydrogen. We can find the binding energy of a deuteron by examining the masses. The mass of an atom of hydrogen plus a free neutron is 2.016 490 u, which exceeds the mass of the deuterium atom that results from combining them by 0.002 388 u. The mass of the composite is smaller because the total energy of the composite is smaller by the binding energy

hydrogen	^{1}H	1.007 825	u
neutron	m_n	1.008 665	u
sum		2.016 490	u
deuterium	^{2}H	2.014 102	u
binding E/c^2		0.002 388	u
binding energy		2.22 MeV	

$$E(^2\text{H}) = E(^1\text{H}) + E(^1\text{n}) - \text{binding energy},$$

with the binding energy

$$E_b = 0.002388\,\text{u} \times \frac{931.494\,\text{MeV}}{1\,\text{u}} = 2.22\text{MeV}.$$

Compare this with the binding energy of the electron in the ground state of hydrogen, $E_b = 13.6\,\mathrm{eV}$, and we see that the nuclear binding is more than 100,000 times stronger.

While the strong force is certainly strong when the nucleons are close together, it is a very short range force. The coulomb force drops off with distance, but only as the square of the distance. The strong force, on the other hand, drops off exponentially fast, so that when two nucleons get more than a few femtometers apart, the force is negligible.

The short range interaction has an important effect. If you surround a nucleon by others, it will get more negative potential energy as long as the additional nucleons can get close, but after that there is no additional benefit. Thus in a big nucleus each nucleon in the middle gets a certain benefit, a fixed binding energy, 15.7 MeV in fact, but that does not continue to increase as the size of the nucleus gets bigger. As this is true for each of the A nucleons, we expect the total binding energy of a large nucleus to grow like A,

$$E_b^{(1)} = C_1 A, \quad \text{with } C_1 = 15.7\,\mathrm{MeV}.$$

This is to be distinguished from the situation with the Coulomb potential. There, a sphere with charge Q gets a potential proportional to Q^2, not Q, because every part of the charge interacts with every other part.

In fact, this coulomb repulsion does affect the binding energy. Each pair of protons contributes a positive potential energy (and therefore a negative contribution to the binding energy) inversely proportional to the distance they are apart, which on the average is r, the radius of the nucleus, which we saw is proportional to $A^{1/3}$. As there are $Z(Z-1)/2$ pairs, this contributes

$$E_b^{(3)} = -C_3 \frac{Z(Z-1)}{A^{1/3}}, \quad \text{with } C_3 = 0.71\,\mathrm{MeV},$$

to the total binding of the nucleus.

You may notice in my numbering I skipped 2. The first term considered that each nucleon contributed a full amount due to the attraction of the neighboring nucleons with which it is surrounded. But the nucleons on the surface are not surrounded, so they don't contribute as much to the binding energy. We need to subtract some amount for each surface nucleus, and as the number on the surface is proportional to r^2 and therefore to $A^{2/3}$, we have a surface effect

$$E_b^{(2)} = -C_2 A^{2/3}, \quad \text{with } C_2 = 17.8 MeV.$$

The effects I have just described are consequences of a **liquid-drop model** for the nucleus, in which we treat the nucleus much as we would a drop of water, with a cohesive force and a surface tension. There is, however, one further term we need to consider in our understanding of the binding energy, which can be understood only quantum mechanically, using the **independent particle model**, also known as the **shell model**. The strong binding we have considered so far treats the protons and neutrons indestinguishably, but there is a penalty to be paid for having unequal numbers.

Let us consider each nucleon as finding itself confined to a spherical box, and otherwise free, as we did for a particle in a box and also for the electrons in a metal. There is a set of energy levels for the neutrons, and they will fill up as we add neutrons, and there is a set of energy levels for the protons, which will similarly fill up, each to the Fermi level, which is proportional to the density to the 2/3 power. The total energy of the neutrons is therefore proportional to[2] $N \times (N/V)^{2/3} \propto A(N/A)^{5/3}$. Exactly the same thing happens for the protons, so the energy of the two sets of free particles is

$$E = KA \left[\left(\frac{N}{A} \right)^{5/3} + \left(\frac{Z}{A} \right)^{5/3} \right].$$

We are interested in nuclei for which N and Z are roughly equal, so let us write $N = \frac{1}{2}[A + (N - Z)]$ and $Z = \frac{1}{2}[A - (N - Z)]$, and expand the 5/3 power in a power series,

$$(1+x)^{5/3} + (1-x)^{5/3} = 1 + 5x/3 + 5x^2/6 + ... + 1 - 5x/3 + 5x^2/6 + ... = 2 + 5x^2/3,$$

so

$$E = KA2^{-5/3} \left[2 + \left(\frac{N-Z}{A} \right)^2 \right] = K_1 A + C_4 \frac{(N-Z)^2}{A}.$$

The first term, proportional to A, has already been included in C_1, but we now see how an inbalance between neutrons and protons costs energy and therefore lowers the binding energy. All together, we get the **semiempirical binding energy formula**,

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[2]Problem 43.29 shows that the average energy of each particle is $3/5\ E_F$, but we are only looking for proportionality and can ignore the 3/5.

$$E_b = C_1 A - C_2 A^{2/3} - C_3 \frac{Z(Z-1)}{A} - C_4 \frac{(N-Z)^2}{A}$$

C_1	15.7 MeV
C_2	17.8 MeV
C_3	0.71 MeV
C_4	23.6 MeV

with $C_4 = 23.6$ MeV. It is called semiempirical because each term has a theoretical explanation but the constants are not calculated theoretically but are instead determined by a fit to the actual nuclear masses.

Two other things the independent particle model gives us, if we consider it a bit more closely than a fermi sea analysis. Firstly, protons and neutrons are, like electrons, spin 1/2 particles. That is, they have a spin quantum number which can take on only two values, $\pm 1/2$, and have a spin angular momentum of $\pm\hbar/2$ along any axis. Thus each energy level has two states, and so states with an even number of protons are a bit better bound than the formula might say, and those with an odd number a bit less well bound. This has a considerable effect on which nuclei are stable; If you look closely at the stable nuclei in Fig. 44.3, you see many more blue dots for even values of N and Z then for the odd values. Secondly, as you can imagine, the energy levels of a spherical well have, just as the hydrogen potential does, shells and subshells, and so there are **magic numbers** of neutrons, or of protons, corresponding to full subshells. Unlike the hydrogen atom, however, the states with different ℓ but the same n do not have the same energy, so the mixing of subshells from different shells is more noticible than for atomic physics.

As you see from the plot of the stable nuclei, they lie in a narrow band of N versus Z. For fairly small nuclei the coulomb repulsion term in not very important, and the fermi-level term proportional to $(N-Z)^2$ dominates,

Show
Fig. 44.3   3
4 1/2" × 6 12"

keeping the stable nuclei close to the line $N = Z$. But as the charge gets larger, the Coulomb term becomes important even though its coefficient is smaller than the others. This favors neutrons over protons, and we see that the larger stable nuclei have considerably more neutrons than protons. For large enough nuclei these two terms conflict with each other enough to make all large nuclei unstable. Bismuth ($Z = 83$) is the heaviest stable nucleus, although some of the heavier nuclei live a very long time before decaying.

The competing effects in the semiempirical mass formula explain the gen-

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[3]Note that the web page listed in the book is obsolete, replaced by `http://www2.bnl.gov/ton`.

eral tendencies for the strength of nuclear binding. If we plot binding energy per nucleon for the most stable partition into protons and neutrons, as a function of mass number A, we see that the very light nuclei have less binding, due to the surface term. Then comes a middle region in which both the surface term and the coulomb term are small, and the binding energy per nucleon is pretty much constant. For heavier nuclei, the coulomb repulsion term begins to significantly diminish the binding.

Show
Fig. 44.8
6" × 5"

3 Nuclear Spin and $\vec{\mu}$, NMR and MRI

As we mentioned, the neutron and proton are each particles of spin $1/2$, like the electron. All such particles are called, collectively, fermions, because the obey the Fermi-Dirac distribution function (and Fermi-Dirac statistics) due to the Pauli exclusion principle. The independent particle model also encourages us to imagine that the protons and neutrons within the nucleus may have various orbital angular momentum. Unlike in atomic physics, the orbital angular momentum and the spin angular momentum of the nucleons interact with each other strongly, and we can only distinguish the total angular momentum, described by the quantum number I [4]. This is the spin of the nucleus. As for the atom, the nucleus' total angular momentum and its z component are quantized,

$$
\begin{aligned}
|L| &= \sqrt{I(I+1)}\,\hbar, \\
L_z &= m_I \hbar, \qquad -I \le m_I \le I.
\end{aligned}
$$

Because I includes both spin and orbital angular momentum, it can be either an integer, for nuclei with an even number of nucleons, or half an odd integer, for nuclei with an odd number. The same is true for m_I.

As a charged object with a spin, we would expect a nucleus to have a magnetic moment. The analogue of the Bohr magneton for the electron's spin, $\mu = e\hbar/2m_e$, is the **nuclear magneton**,

$$
\mu_n \equiv \frac{e\hbar}{2m_p} = 5.05 \times 10^{-27}\,\mathrm{J/T}.
$$

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[4]The use of I for the spin of a nucleus is somewhat old-fashioned, and I would have preferred to use J, because nuclei, and particular nucleons, also have a concept called isotopic spin, which is generally represented by I. Isospin has considerable resemblence to angular momentum but is definitely not the same thing.

Actually no nucleus or nucleon actually has that magnetic moment. As we shall see, these are all complex composite objects which don't have the simplicity of an electron. The magnetic moments of the proton and neutron are

proton	2.7928 μ_n
neutron	-1.9135 μ_n

More complex nuclei have different proportionality constants between the magnetic moment and the spin, but they are always proportional, possibly, as for the neutron, with a negative constant.

When a nucleus is in a magnetic field, it has an energy $-\vec{\mu} \cdot \vec{B}$, just as for an electron, and this will cause an energy difference between the states according to what the projection of the spin in the direction of the field is. For a spin $1/2$ nucleus, as for example hydrogen ^{1}H, there are only two states, one with the spin aligned with the field and one with the spin in the opposite direction. So the energy difference is $\Delta E = 2\mu B$. This has a very practical use.

Show
Fig. 44.5
6 1/2" × 3 3/4"

As we discussed for atomic transitions, when light of the frequency corresponding to the energy difference between energy levels is incident, it can cause transitions between the levels. If we have a substance which is largely water, H_2O, there are a lot of protons with magnetic moment, μ_p. If we place this substance in a strong magnetic field, most of the protons will align their magnetic moments with the magnetic field. If we also expose them to light with $hf = 2\mu_p B$, this will cause transitions to the higher energy states, absorbing the light. If we measure the absorption, we can tell the density of water. If the magnetic field is not uniform, only the water molecules which are on the surface for which $B(\vec{r}) = hf/2\mu_p$ can absorb the light. So one slice at a time can be examined. To extract an actual image is a marvel of mathematical engineering, the result of which is called MRI, or nMRI, nuclear magnetic resonance imaging. Here are a few slices through my lower abdomen. Despite the taking of these pictures, as you can see I am still in one piece.

Show
abdominal
MRI slices.