Physics 228, Lecture 12
Thursday, March 3, 2005

Uncertainty Principle. Ch 39:2-5

1 The Double Slit Experiments

When we first discussed Young's double slit experiment as evidence that light is a wave, we discussed the conditions under which the idea that light moved in straight lines and casts shadows needs to be modified when slits comparable to the size of wavelengths are considered. When light passes through a slit, the area lit up on a screen in back of the slit is not precisely its geometrical shadow, but includes diffraction patterns beyond that. If it passes though two slits close to each other, so that the single slit interference pattern each one would make in the absence of the other overlap, there is interference between the two slits. The resulting intensity on the screen, for the light passing through the two slits, is **not** the sum of what you would get from each of the two slits. As we saw, what needs to be added is the electric field, and the sum of fields is then squared to get the intensity. The result is an interference pattern which clearly indicates that the light must be treated as a wave and not as independent particles, some of which pass through one slit and some through the other.

Now we have seen that de Broglie has suggested that even things we know are particles, like electrons, which can leave tracks in bubble chambers or photographic emulsions, have wave-like properties. Does that mean that if we shoot electrons at a wall with two holes in it, and have an electron-detecting screen behind that wall, we will get a two-slit interference pattern just as we did for light? The answer is YES!

Show Fig. 41.1 electron double slit
6 1/4" × 4 3/4"

This is really not easy to understand. When we think of a wave going through two slits and interfering with itself, it is not so hard to understand that the resulting amplitude could be different from the sum of what would happen with each slit separately open. If we left only slit 1 open for an hour with light incident, and exposed film on the screen, we would get a single slit pattern like the lower blue curve. If before we developed the film, we closed slit 1 and opened slit 2, and exposed that for an hour, that would add the intensity of the upper blue curve, and the result of developing the film would be the blue curve on the right. But instead, if we exposed the film for an hour with both slits open, we get an exposure given by the beige curve. We understand that happens with waves because the different parts of the wave can interfere with each other. But we have now seen that a light wave actually consists of discrete photons, and if we consider a very weak light wave, so that at any one time, there would be only one photon in the wave, surely that photon would have to pass through one of the slits or the other, so the chance of its getting though the slits and to any given location ought to be the sum of what you would get for each slit separately. And even more so for an electron. When an electron hits the screen, it does so at only one point. The pattern of exposure is built up of many hits. For any one electron's mark, if the electron came out of the electron gun and passed through the wall with the two slits and hit a detector on the screen, surely it passed through one of the slits or the other, and the electrons which hit the screen passing through slit A would have done so even if slit B was closed. That has to be right, doesn't it? **Actually, no — it is not right!**

Fig. 41.3
$|\psi_1|^2 + |\psi_2|^2$
vs. $|\psi_1 + \psi_2|^2$
6 1/4" × 4 1/4"

Fig. 41.2
dots in dbl slit distribution
2" × 4 1/2"

Here is what quantum mechanics says, and what experiment verifies, regardless of how nonsensical it sounds. The wave associated with the electron has the following meaning: the electron does not have a certain future but has a probability of being at certain positions, and that probability is given in terms of the square of the wave function. Just as the photon intensity, it has the same interference pattern. In fact, the way to understand the intensity for a light wave is to realize that it is just the rate at which photons are hitting various points on the screen, proportional to that probability. At points where there is complete destructive interference the probability is zero, and no photons or electrons hit that point, even if this is a point some would have reached if one or the other slits had been closed.

This is the strangest idea in all of physics, stranger than space being curved, stranger than the idea that there may be extra dimensions beyond the space and time we are familiar with, stranger than that the universe had a beginning and that not only was there no universe before that but that there was no before that. But this quantum mechanical idea that this set of electrons, which passed through the slits, does not consist of a set that passed through slit A and a set that passed through slit B, this crazy idea has been exhaustively verified in thousands of different applications, and has always proven correct. Philosophers have been fighting against this for 75 years, but the physics is relentless — our common sense ideas of reality are just wrong.

1.1 Electron Microscope

Before we continue to try to come to grips with what this probability means, let's point out that electron interference isn't all bad. When we use a microscope to try to see small details, the resolution is limited by the wavelength of the light. As visible light has wavelengths about half a micron, we can't use visible light microscopes to see details much smaller than that. For some purposes one can use shorter wavelength light, but a photon with a wavelength of 1 nm, for example, has

$$p = \frac{h}{\lambda} \approx 7 \times 10^{-25}\,\mathrm{J\cdot s/m}, \quad E = cp \approx 2 \times 10^{-16}\,\mathrm{J} \approx 1000\mathrm{eV}.$$

That is enough energy to easily knock atoms out of position and destroy other small things. On the other hand, an electron with a wavelength of 1 nm and a momentum

$$p = 7 \times 10^{-25}\,\mathrm{J\cdot s/m} \quad \text{has } K = \frac{p^2}{2m_e} = \frac{49 \times 10^{-50}}{2 \times 9 \times 10^{-31}} = 3 \times 10^{-19}\,\mathrm{J} \approx 2\mathrm{eV},$$

much less likely to damage what you are trying to examine. Electron beams can easily be manipulated and focused with voltages and magnetic fields, and a fluorescent screen can convert the electrons into something humans can see. So electron microscopes are used everywhere to examine things on a scale between the atomic and a micron.

2 Uncertainty Principle

Newtonian mechanics made a great philosophical impact because it was manifestly and precisely a deterministic theory. If one (God perhaps) knew the laws of physics **and** the exact positions and velocities of every particle in the universe at one moment of time, one could in principle calculate the motion of every particle for the rest of eternity. We are familiar with this in simple situations were a single particle experiences no forces, in which case we can determine where it will be later, but only if we know both its position and its velocity at some initial time. Of course any means of measuring its position and velocity will have some error, but in principle, in classical mechanics, there is no limit to how accurately we could measure them.

Heisenberg warned us against assuming such precise measurements were in principle possible in a quantum mechanical world, and in fact he introduced the **Heisenberg uncertainty principle**, which states that any measurement of position and momentum of a particle at the same time has a minimum uncertainty

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

and thus it is **in principle** impossible to know a particle's position and velocity precisely at any instant of time.

One way to understand this is to imagine that we have a free particle for which, somehow, we have measured its momentum precisely. Now if we can measure where it is without messing up its momentum (which is constant as long as nothing interacts with it) we could defeat Heisenberg. But to measure where it is we have to interact with it somehow. For example, we can look at it, but only if at least one photon bounces off of it and reaches our eye. Now a photon that can measure where it is to an accuracy of Δx has a wavelength no larger than Δx, so $\lambda \leq \Delta x$, and the photon has a momentum $p = h/\lambda \geq h/\Delta x$. If we bounce such a photon off our particle, it will change the momentum by some fraction of that amount, so the final momentum of the particle now has an uncertainty of $\Delta p \geq h/\Delta x$, or $\Delta p \Delta x \geq h$. Actually the experimental methods can be refined somewhat, and the uncertainty is $1/4\pi$ times our estimate, but Heisenberg rigorously showed that

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

We see the particle nature of a particle when we focus our attention on where it is, and then its momentum is uncertain, so its wavelength is not clearly defined, and its wave nature is not so apparent. On the other hand, a wave with a well defined wavelength must have a large uncertainty in where it is (is very spread out), so its particle nature is less clear.

Wavelength has to position the same relation as the period, $1/f$ has to time. So wavelength is to period as position is to time, and also as momentum is to energy, according to de Broglie. So it should be no surprise that there is also an uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

but this relation has a slightly different meaning. It says that a state of a system which is unstable and lasts only some finite time Δt cannot have a precise energy, but has a spread of energies with width ΔE. This means excited states of atoms, which will decay by giving off a photon with some mean lifetime τ do not have a precise energy, despite the Bohr model, but have a certain spread of energies. This spread gives a width to the spectral lines, which have a small but not zero spread of wavelengths.

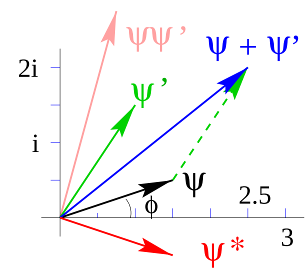
3 What is the wave?

For light, which we treated classically as a wave, we knew what physical properties the wave represented — the electric and magnetic fields are functions of position and time, and these functions satisfy the wave equation. For matter, that is for things we treated classically as particles, we do not have classically any physical property described by a wave. As we will develop, the quantum property which the wave describes is the **probability amplitude**. As we shall see, the state of a quantum mechanical system is generally described by having a known **wave function**, but not by having definite positions for the particles. The Bohr model tells us that, in the ground state, the electron is 5.29×10^{-11} m from the nucleus, but the Bohr model is not consistent with Heisenberg's uncertainty principle, which would state that if the electron's position were that well known it would have an uncertainty in its momentum that it would often have far too much energy to be bound inside the atom. So the true quantum mechanical picture of a hydrogen atom in its ground state is that there is a wave function for the electron which describes the probability that the electron is at any particular position at an instant. This probability function tells us that it is very likely that the electron is roughly 0.05 nm from the nucleus, and the most probable distance is 5.29×10^{-11} m, but not that it actually is at exactly that distance.

The wave function is generally denoted by the greek letter psi, $\Psi(x, y, z, t)$. Unlike the electric field, it is a scalar, not a vector, but it is not a real-valued scalar but instead takes on complex values. That is, for any given position and time, the value of the function Ψ is a **complex number**.

We are going to postpone the discussion of Schrödinger's equation and wave packets until next time, because right now is when we will really examine what we mean by a complex number.

A complex number z is like a phasor in that it can be represented as a two dimensional vector, and complex numbers are added as two dimensional vectors are. Thus the complex number Ψ and the complex number Ψ' can be added as shown.



Complex numbers are a generalization of real numbers, and one component of the the complex number is called its **"real part"**. By convention, these vectors are drawn with the horizontal (x) component representing the real part of the complex number and the vertical (y) component is called the **"imaginary part"**. But these are not vectors in ordinary space — they do not point towards or away from any physical locations. The number shown as Ψ has a real part of $3/2$ and an imaginary part equal to $1/2$. The complex number with zero real part and imaginary part equal to one is given a special name, i. A complex number with no imaginary part is written without any unit vector, so the complex number $\Psi = 3/2 + i/2$ and $\Psi' = 1 + (3/2)i$.

Complex numbers are more than just two dimensional vectors written funny, however, because they have a special rule for multiplication. There are two ways to state this rule. First, we can specify that the product of two complex numbers Ψ_1 and Ψ_2 is a vector Ψ_3 with length equal to the product of the lengths, $|\Psi_3| = |\Psi_1||\Psi_2|$ and an angle with the x-axis which is the sum of the angles $\phi_3 = \phi_1 + \phi_2$. Thus $\Psi\Psi'$ is the complex number shown.

Complex numbers can also be multiplied by the ordinary rules of scalar multiplication with the additional rule

$$i^2 = -1.$$

Thus

$$\Psi\Psi' = \left(\frac{3}{2} + \frac{1}{2}\,i\right)\left(1 + \frac{3}{2}\,i\right) = \frac{3}{2} + \frac{1}{2}\,i + \frac{9}{4}\,i + \frac{3}{4}\,i^2 = \frac{3}{2} + \frac{11}{4}\,i - \frac{3}{4} = \frac{3}{4} + \frac{11}{4}\,i.$$

The vector obtained by reversing the imaginary part of a complex number

Ψ is called its **complex conjugate** and written Ψ^*. Thus if

$$\Psi = x + yi, \qquad \Psi^* = x - yi,$$

and

$$\Psi\Psi^* = (x + yi)(x - iy) = x^2 + xyi - xyi - y^2i^2 = x^2 - y^2i^2 = x^2 + y^2 = |\Psi|^2.$$

What does any of this have to do with the meaning of the wave function? In quantum mechanics, if a particle is in a state represented by the wave function[1] $\Psi(x, y, z, t)$, the probability that it is in an infinitesimal box of volume ΔV centered on the point (x, y, z) at time t is

$$P = |\Psi|^2 \Delta V.$$

We will discuss what equation determines the wave function and what the wave function is for some particular physical situations in the next two lectures.

4 Summary

- Ordinary particles as well as photons exhibit interference, for example in a double slit experiment, as predicted by their de Broglie wavelength $\lambda = h/p$.

- The wave function that has this wavelength (and frequency), generally called Ψ, is the **probability amplitude**, a complex number whose length squared, evaluated at some position and time, is proportional to the probability of finding the particle in a small volume around that position at that time.

- Because positions are uncertain to the extent the wave function is spread out, the Heisenberg uncertainty principle tells us that the position and velocity can never be simultaneously determined to an accuracy better than

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

[1] Strictly speaking, we need to assert that Ψ is normalized, which we will discuss next time.

- A state which lasts a finite time Δt does not have a precise energy but has an energy uncertainty

  $$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

  This also applies to any measurement of energy which is done within a time interval Δt.

- Complex numbers are a form of two-dimensional vector with an additional multiplication rule (which is not a dot product or a cross product). The ordinary real numbers are one line in this two dimensional space, and negative numbers have square roots which are on the other axis in this space.