Physics 228, Lecture 9
Monday, Feb. 21, 2005

Energy, Momentum, Mass. Ch 37:6–8;

We first finish up Lorentz Transformations from Lecture 8

1 Relativistic Doppler Shift

One important consequence of relativity is a change in the formula for the Doppler shift, the effect that makes a locomotive whistle sound higher pitched when the locomotive is approaching you and lower pitched as it goes away. The formula relates the frequency f_0 emitted by the locomotive (the source) with the frequency f that the observer hears. For sound, there is one formula for when the observer is is at rest with respect to the air and the source is not, and another formula for the sound of a stationary alert siren as heard in a moving vehicle, where the source is at rest with respect to the air. But for light the medium would be the ether, and as we have seen it makes no sense to ask what the rest frame of the ether is.

I am going to derive the formula in a different way than the book does — seeing two derivations will only broaden your understanding

Let's work in the reference frame S of the source, and place the source at $x = 0$. If he is emitting light of frequency f_0, we can think of this as wave fronts emitted from the origin at times $t = n/f_0$, which then travel at speed c to the right (as well as other directions) so that the position of the n'th wavefront is $x(t) = c(t - n/f_0)$. If the observer S' is moving towards the source at speed u, the relative velocity to use in the Lorentz transformation is $v = -u$, because we defined v as the x component of the velocity of S' with respect to S. The worldline of S' is, according to S, $x_{S'} = L - ut$. Now S' receives each wavefront when it reaches him, so the coordinates of the event: S' recieves the n'th wavefront, is the solution of

$$x(t) = c(t - \frac{n}{f_0}) = x_{S'}(t) = L - ut \Longrightarrow t_n = \frac{L + cn/f_0}{c + u}, x_n = \frac{Lc - nuc/f_0}{c + u}.$$

The Lorentz transformation tells us what times those events occur to S':

$$\begin{aligned} t'_n &= \frac{t_n - v x_n/c^2}{\sqrt{1-v^2/c^2}} = \frac{t_n + u x_n/c^2}{\sqrt{1-u^2/c^2}} \\ &= \left(\frac{L + cn/f_0}{c+u} + \frac{u}{c^2}\frac{Lc - nuc/f_0}{c+u}\right)\frac{1}{\sqrt{1-u^2/c^2}} \\ &= \frac{L\left(1+\frac{u}{c}\right) + \frac{nc}{f_0}\left(1-\frac{u^2}{c^2}\right)}{(u+c)\sqrt{1-u^2/c^2}}. \end{aligned}$$

The time delay in receiving the first $n = 0$ wavefront is not what we are interested in here. Rather, the frequency received is the reciprocal of the time between successive wavefront receipts:

$$\begin{aligned} \frac{1}{f} &= t'_{n+1} - t'_n = \frac{c}{f_0}\frac{1-u^2/c^2}{(u+c)\sqrt{1-u^2/c^2}} = \frac{1}{f_0}\frac{c\sqrt{1-u^2/c^2}}{c+u} \\ &= \frac{1}{f_0}\frac{\sqrt{c^2-u^2}}{c+u} = \frac{1}{f_0}\frac{\sqrt{(c-u)(c+u)}}{c+u} \\ &= \frac{1}{f_0}\sqrt{\frac{c-u}{c+u}}, \end{aligned}$$

so

$$f = f_0\sqrt{\frac{c+u}{c-u}}.$$

Thus the received frequency is higher than the source, and visible light is shifted from the red side of the spectrum to the blue. This is known as a blue shift, even when the light is ultraviolet and may actually be shifting away from the lower frequency blue light.

If the observer is moving away from the source, we can use the same formula but consider u negative, or we can change the signs in front of the u's,

$$f = f_0\sqrt{\frac{c-u}{c+u}} \qquad \text{source and observer moving apart with speed } u.$$

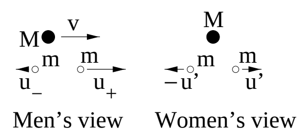
This is called a red shift.

At the end of this semester we shall see that all far-away stars and galaxies are moving away from us, and we can tell that, and just how fast, by observing how the identifiable spectra are red-shifted by the Doppler effect.

Notice that relativity assures us that the same formula applies whether the source is moving away from the observer, or the observer is moving away from the source, for which one is moving is not a meaningful question.

2 Relativistic Momentum and Force

Suppose the ladies have a bomb of mass M at rest, which explodes into two equal mass pieces, each of mass m and moving at $\pm u'$ in the x direction. Momentum is conserved for the ladies. What do the men say?



According to Newtonian formulas, $M = 2m$ and $p_i^{(N)} = Mv = 2mv$ initially, because the bomb at rest with respect to the ladies is moving in the x direction with velocity v, according to the men. The final momentum is

$$\begin{aligned} p_f^{(N)} &= mu_+ + mu_- = m\left(\frac{u'+v}{1+u'v/c^2} + \frac{-u'+v}{1-u'v/c^2}\right) \\ &= m\left(\frac{(u'+v)(1-u'v/c^2)+(v-u')(1+u'v/c^2)}{1-u'^2v^2/c^4}\right) \\ &= 2mv\left(\frac{1-u'^2/c^2}{1-u'^2v^2/c^4}\right) \neq 2mv = p_i^{(N)}. \end{aligned}$$

Thus we do not have conservation of mass times velocity.

Conservation of momentum is too valuable to give up without a fight. Instead, we modify the definition of the momentum of a particle of velocity $\vec{u}$

$$\vec{p} = m\frac{\vec{u}}{\sqrt{1-u^2/c^2}}.$$

This formula agrees with Newton's for $u^2 \ll c^2$, which is very good even for the fastest near-Earth macroscopic object around, a satellite going 7500 m/s, for which u^2/c^2 is less than 10^{-9}. With this formula, we can find that momentum is conserved, but only if we give up another assumption, that $M = 2m$. Instead, we need to assume

$$M = 2\frac{m}{\sqrt{1 - u^2/c^2}}.$$

Thus some mass is lost in the explosion, not in gases which got away (we have made an ideal explosion with no gases) but just disappeared. We will soon see how to understand what this means.

3 Relativistic Energy

Newton tells us that the kinetic energy of a moving particle of mass m is $\frac{1}{2}mu^2$, which can be understood as the work it takes to accelerate a particle from rest to speed u,

$$W = \int \vec{F} \cdot d\vec{r} = \int \frac{d\vec{p}}{dt} \cdot \vec{u}\, dt.$$

Let's consider only motion in the x direction. For Newton, $p = mu$, so $W = \int m(du/dt)udt = \frac{1}{2}mu^2$. Now, however, the force is not $m(du/dt) = ma$, so

$$\begin{aligned} \frac{dp}{dt} &= m\frac{d}{dt}\frac{u}{\sqrt{1-u^2/c^2}} = m\frac{du}{dt}\left(\frac{1}{\sqrt{1-u^2/c^2}} - \frac{1}{2}\frac{u \times (-2u/c^2)}{(1-u^2/c^2)^{3/2}}\right) \\ &= m\frac{du}{dt}\left(\frac{1 - u^2/c^2 + u^2/c^2}{(1-u^2/c^2)^{3/2}}\right) = m\frac{du}{dt}\left(\frac{1}{(1-u^2/c^2)^{3/2}}\right) \end{aligned}$$

Using x for u^2/c^2, we find the kinetic energy of a particle with velocity u is

$$\begin{aligned} W &= m\int_0^u \frac{u}{(1-u^2/c^2)^{3/2}}\frac{du}{dt}dt = \frac{1}{2}mc^2\int_0^{u^2/c^2}(1-x)^{-3/2}dx \\ &= mc^2 \left.\frac{1}{\sqrt{1-x}}\right|_0^{u^2/c^2} = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2. \end{aligned}$$

If we talk about the energy of a particle in Newtonian physics, we assume a particle at rest has no kinetic energy. It would make no difference if we assumed it had some additional energy proportional to the mass, because as mass is conserved in Newtonian physics, the extra contributions would not affect energy conservation. But in relativistic (Einsteinian) physics mass[1] is not conserved, so it does make a difference, and we find that what is conserved is the total energy

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = K + mc^2,$$

the sum of the kinetic energy and a **rest mass energy** mc^2.

If we understand γ for a particle to represent

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}},$$

(which is the same expression we used before for a reference frame, but using the particle's velocity u), we may write the expressions for momentum and energy more simply:

$$\vec{p} = m\gamma\vec{u}, \quad E = mc^2\gamma.$$

If we form the combination

$$E^2 - c^2p^2 = m^2c^4\gamma^2 - m^2c^2u^2\gamma^2 = m^2c^2\frac{c^2 - u^2}{1 - u^2/c^2} = m^2c^4,$$

which is often a convenient relativistic relationship among m, p and E. In particular, we can let the mass go to zero in this relationship without necessarily having E and p become zero:

$$\text{As } m \to 0, \qquad E^2 = c^2p^2.$$

Note that this is precisely the relationship we found between the energy density and the momentum density in an electromagnetic wave! Note also that $\vec{u} = c^2\vec{p}/E$ in general, and for the massless case this gives $|u| = c^2p/cp =$

[1]Some introductory physics textbooks define mass differently than we, and our text, do. What we are using is called the rest mass, and is a fixed property of a particle, such as an elementary particle, independent of its velocity. Some introductory texts, and many popular articles about relativity, define relativistic mass m_r to be $m\gamma$, which varies with velocity and is really just the energy divided by c^2. That definition is motivated by pretending that $\vec{p}$ is still $m_r\vec{u}$, but it is not a concept used by people who do relativistic physics and it is misleading in many ways. You need not worry about this unless you are reading one of those textbooks or articles.

c, the velocity of light. So this is a hint that light might also be **massless** particles.

If we had asked before relativity to explain where the energy of the bomb fragments in the ladies explosion came from, we would have said there was chemical energy inherent in the bomb before explosion, and would not have expected it to have anything to do with the bomb's mass. But now we see that that energy should be included in the mass, and that the total mass of the bomb before hand is more than the mass of the fragments and burnt gases afterwards. In fact, could we measure the masses accurately enough, we could have evaluated the chemical energy.

Here is an example of a very small bomb. One atom of uranium 238 has a mass[2] of $238.050784 \times 1.6605402 \times 10^{-27}\,\text{kg} = 3.95292897 \times 10^{-25}\text{kg}$. It can decay into one atom of thorium 234 and one atom of helium, with masses of $234.043593 \times 1.6605402 \times 10^{-27}\,\text{kg} = 3.88638795 \times 10^{-25}\text{kg}$ and $4.002602 \times 1.6605402 \times 10^{-27}\,\text{kg} = 6.646482 \times 10^{-27}\text{kg}$ respectively. Thus when the atom explodes (decays) the mass lost is 7.62×10^{-30} kg. This lost mass is converted into kinetic energy of the fragments, $K = \Delta mc^2 = 6.8 \times 10^{-13}$ J. That might not seem like a lot of energy, but it is a lot to come out of one atom. One gram of ${}^{238}_{92}\text{U}$ contains 2.5×10^{21} atoms, so if they all decay they will release 1.7×10^9 J.

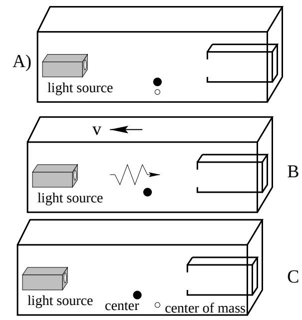
This is a little bomb. Real atomic bombs do not work off this "alpha decay" mode of uranium, but instead from fission into two roughly equal nuclei plus a few neutrons. These tend to release more energy per decay.

4 Mass Energy Equivalence

Einstein considered as a *gedankenexperiment* a closed box with a pulsed light source at one end an a perfect absorber at the other. Suppose the center of mass is originally in the center of the box and the box is at rest, totally isolated from all external forces. Thus its momentum is zero and it is conserved.

[2]The masses are given in atomic mass units in Table A.3. The conversion factor is on the front cover, 1 amu = $1.6605402 \times 10^{-27}$ kg.

Now suppose a pulse of light with total energy E is emitted from the light source. As we know for a light wave, this carries momentum $p = E/c$, and as the total momentum must still be zero, the box will have momentum $p_x = -E/c$, and must be moving to the left. When the pulse of light gets to the absorber at the right end of the box, the total momentum of the system is now just the momentum of the box, so it is again at rest, but moved some distance to the left.



But we know that no external force has acted on the system, so its center of mass cannot have moved, even though the center of the box has moved. The explanation can only be that the mass of the light source decreased when it emitted the light pulse, and the mass of the absorber increased when it absorbed the light pulse, so the center of mass of the box is no longer at the geometrical center. We can work out how much mass moved from one side to the other quantitatively, and we find that the mass moved by the pulse from one side to the other is E/c^2. But it is important to note that this is **not the mass of the light pulse!** The light pulse has no mass, but the mass of the box at B and the pulse together, as a system, is not the sum of the masses of the box and the pulse. Only the energies sum up the way we are used to.

5 E & M, General Relativity

Later in his life, after his special relativity paper had become generally acknowledged, Einstein stated that he had not been aware of the Michelson-Morley experiment at the time. Einstein was motivated by observing that whether a loop of wire approached a magnet or the magnet approached a fixed loop of wire, the EMF produced in the loop was the same, though in one case the explanation involved the magnetic force on moving charges, while in the other it involved the electric field produced by a changing magnetic flux. But the relativity of electromagnetic fields is a bit complex, with what one observer calls a magnetic field looking like an electric field to another observer. This is very pretty but beyond what we need to discuss in this course.

Having once been rewarded by insisting that differing explanations of the same effects muct actually be equivalent once, Einstein considered the fact that the physics inside an elevator box at rest on the surface of the Earth is the same as the physics inside an "elevator box" out in empty space which is accelerating upwards at $a = g$. All the effects are the same, even though in one case the accelerations of unsupported objects are due to the gravitational force and in the other are due to working in a non-inertial reference frame. Einstein insisted on finding a theory in which the causes are the same — that is, he insisted that gravity is simply an indication that we are working in a noninertial frame. This led to General Relativity, a truly mind bending theory in which space and time are not only mixed but curved. This is the theory (not special relativity) of which it was once said that only half a dozen people in the world understood it. But that is not true, lots of people now have a pretty good understanding of general relativity. Still, it is a hard theory to understand.

One of the consequences is that it makes a definite prediction for how light should be effected by a gravitational field, which is not really clear in Newtonian physics. The prediction by Einstein of exactly how much light from stars should be bent when passing by the sun, verified during the total solar eclipse in 1919, made him the most famous physicist of modern times.[3]

6 Summary

- An light signal emitted at frequency f_s is perceived by an observer to have a different frequency f_o given by

  $$f_o = f_s \sqrt{\frac{c-u}{c+u}},$$

  if the observer and source are moving away from each other with velocity u.

- The momentum of a particle of (rest) mass m and velocity $\vec{u}$ is not $m\vec{u}$ but

  $$\vec{p} = m\gamma\vec{u}, \quad \text{with } \gamma = \frac{1}{\sqrt{1-u^2/c^2}}.$$

[3]Newton was probably even more famous in his day — poets wrote poetry extolling him, and he was given significant credit for the enlightenment.

- And the energy of such a particle (including the previously ignored "rest energy" mc^2) is
$$E = mc^2\gamma$$

- mass is **not** conserved, but the energy and momentum of an isolated system is. When mass is lost, the equivalent energy Δmc^2 appears, often in the form of kinetic energy.

- In particular, if an atomic nucleus of mass M_I decays into fragments with masses which sum up to M_F, the fragments will have kinetic energies which sum up to $Q = (M_I - M_F)c^2$. Because c^2 is a big number in everyday units, even a conversion of 1/20 of one percent of the mass of a kg of uranium, say, is a large amount of energy.