Physics 228, Lecture 8
Monday, Feb. 14, 2005

Consequences of the Relativity Principle. Ch 37:2–5

Strange conclusions; Gedanken Experiments

[Note: much of the material in this lecture is discussed in a presentation available at the MSLC on videodisk, section #42 of the Mechanical Universe. This is especially good on time dilation and the relativity of simultaneity, less good on length contraction and the Lorentz transformations, but excellent on space-time diagrams, which are barely touched in the book. There are also three other sections on relativity in The Mechanical Universe, but I am not as convinced that these are worth your time.]

So today we are going to place the two postulates of Einsteinian Relativity above our common sense. We are going to have to think imaginatively but not carelessly about what is going on. We have already seen that common sense reasoning in considering the Michelson-Morley interferometer is inconsistent with the postulates, so let us reconsider this situation.

Einstein explicitly denies that the frame we previously called the ether is special — from now on we simply refer to Esther's frame. We must have the same laws of physics apply to Esther's explanation of why there is no fringe shift as we have for Mick, the observer at rest with respect to the apparatus.

Now clearly according to Mick, the time it takes for the light to go from the half-silvered mirror to M_2 and back is just

$$T = 2L/c \tag{1}$$

where L is the length of the transverse arm. As we saw before, Esther sees the light travelling not only the length of the arm, which she calls L', but also in the x-direction as well, so that the time it takes is

$$T' = \frac{2L'}{c}\frac{1}{\sqrt{1-v^2/c^2}}. \tag{2}$$

Classically we would expect the two to agree both on how long it takes for the light to return to the half-silvered mirror and how long the arm is, but that is not consistent:

$$\text{(1) and (2) are inconsistent with} \qquad T = T' \quad \text{and} \quad L = L'.$$

So something wierd is definitely happening — Mick and Esther disagree, either on how long a time it was between two events (the first event is the point at which the light leaves the half-silvered mirror and the second is when it returns) or on the length of the transverse arm of the apparatus, or both.

To determine what is going on, we need to do what Einstein called a Gedanken experiment. Gedanken is german for thought — these are experiments which cannot be done in practice, because we can't send Esther off at half the speed of light, but could be done in principle, and furthermore, experiments for which we can figure out what would be observed by careful thought alone, (not guessing, careful thought!), without having to actually perform the experiment.

So first let's ask Esther to measure the transverse arm. As the arm is moving with respect to her, she can't just lay out metersticks in a leasurely fashion next to the arm to get the length. Instead she lays out some metersticks at rest (in her frame) perpendicular to Mick's motion, and asks Mick to place a piece of blue chalk on each end of the arm, so that when the apparatus goes by her meterstick, two blue marks will be left and she can measure the distance between them. Just to check her out, Mick asks Esther to put two pink pieces of chalk a distance L apart on her meterstick, and places a meterstick of his own along side the arm. The two pass each other, and we now ask

> Is the length of the arm as measured by Esther, L', less than L?
> If so, what does Mick measure as the length of Esther's meterstick?

If Esther sees Mick's transverse arm shrunken, she must see his meterstick shrunken as well, so the blue chalk marks are left inside the pink pieces of chalk. But then the pink lines left on Mick's meterstick must be outside his blue chalk, so Mick sees Esther's metersticks as expanded while Esther sees Mick's as shrunken. But this situation is completely symmetrical, so the laws of shrinking metersticks would have to say both shrunk or both expanded. The only consistent answer is that the metersticks don't contract or expand.

Now that may have seemed like an awful lot of fancy argument to reach the obvious conclusion, but remember that if you accept that they agree on how long the arm is, they **must disagree** on how much time elapsed between the same two events! Esther must find that it took longer than Mick does. So Esther says it took longer than is shown on the clock which Mick left at the half-silvered mirror to measure how long. So Esther says that Mick's clock runs slow! For Mick measures a time T which is shorter than Esther's:

$$T = T'\sqrt{1 - v^2/c^2} < T'.$$

This very surprizing result is known as time dilation, and is generally expressed by the imprecise language that moving clocks run slow. Let us be more precise. The time interval between two events, measured by a single clock in a reference frame for which these two events take place at the same spatial point, is called the **proper time** between those events. In our case Mick measures the proper time. If someone moving with respect to that clock wants to measure the time interval between these two events, she needs two clocks, one at the spatial point at which the first event takes place and one for the second.

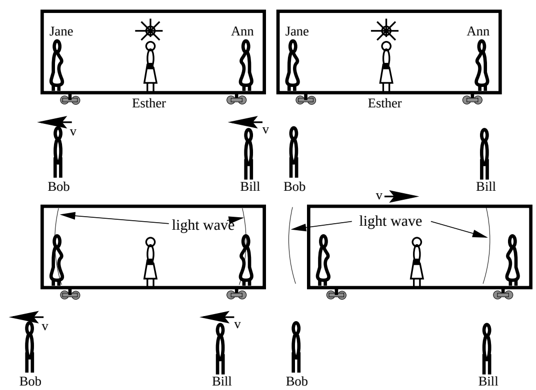
So Esther concludes that Mick's clock runs slowly because it is moving. But if moving clocks run slow, Mick must claim Esther's clocks run slow, not fast, so how come she measured too **large** a time interval for the light to travel?

Yes, according to Mick, when Esther synchronized her clocks, she did it wrong. Let's examine why

Let's suppose Esther is a friend and co-mover of Jane and Ann, and Mick is at rest with his friends Bill and Bob. In fact, when Esther did her timing she did it by having Ann present at the start, and Jane at the end, of the time the light took to make its trip. She needed to have a clock present at each event, which she did by giving one to Ann and one to Jane. When the clocks were in place, she needed to synchronize them. Here is how she did it.

Esther places herself in the middle of the car, equidistant between Jane and Ann, and sets off a flash of light. Each clock is set to reset to 0 when a bright flash is received. As the time it takes for the light to reach the two clocks is equal, they will be synchronized, says Esther.

What do Bob and Bill see? Suppose they were just next to Jane and Ann respectively when the flash went off, so the flash occurs equidistant from them too. Thus they will see the light at the same moment. But at this moment, Jane has moved forward, has already passed the flash, and her clock has already been reset and is running, while Ann, having moved away from the light, has not yet received it, so her clock won't be reset for another instant. When it is, clearly it will **not be synchronized** with Jane's. Say the men.



As seen by the women   As seen by the men

Notice that Jane has already reset her clock, and Ann has not yet done so.

Who is right? Are the ladies' clocks in synch or not? Well, not everything is relative, but this answer is. The ladies' clocks strike 12 simultaneously in the reference frame of the ladies, but not in the reference frame of the men. Whether or not two events are simultaneous is not a question with an absolute answer unless the two events also occur at the same position. Otherwise, it depends on the viewpoint of the observer. According to the men, the event "Jane's clock gets reset" occurs before the event "Ann's clock gets reset. According to the women, those events are simultaneous. If a monkey in a jet plane moving to the right faster than the ladies is watching, he says Ann's clock was reset before Jane's. Almost enough to make one say "it's all relative"!

All of the crazy stuff I'm telling you is real, logical, and can be understood. It is just not what we learn as children.

Can we show that this is really right? One thing we can check is time dilation, that moving clocks run slow. Cosmic rays come to the Earth from outer space, and when they hit the atmosphere there is a good chance that they will make unstable particles called muons. Muons live only a short time before decaying. If you have a bunch of muons at rest, half will die off in 1.5 microseconds. In that time, even moving at the speed of light they would travel less than a mile. They are produced high in the atmosphere, several miles up, yet most of them make it to the ground, because their "clocks", which determine their decay rate, are moving slow according to our reference system. We have some videotapes which show the experiment and explains it in some detail, available at the MSLC.

Length Contraction

Let's return once again to the Michelson-Morley experiment. Mick has no difficulty explaining why, as the two arms are equal, the light beams interfere constructively. We have already seen that Esther thinks it takes longer for the light to make its trip up the transverse arm and back

$$T'_T = \frac{2L}{c}\frac{1}{\sqrt{1 - v^2/c^2}}$$

where she agrees that the transverse arm has length L. What about the longitudinal arm? We have already done the calculation:

$$T'_L = \frac{2L'_L}{c}\frac{1}{1 - v^2/c^2}$$

where now I am allowing for the possibility that she measures different times and a different length for the longitudinal arm.

But if the two beams always interfere constructively, the two times must be the same

$$T'_T = T'_L \Longrightarrow \frac{2L}{c}\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{2L'_L}{c}\frac{1}{1 - v^2/c^2}$$

or

$$L'_L = L\sqrt{1 - v^2/c^2}.$$

Thus she sees the longitudinal arm **contracted** by the Lorentz-Fitzgerald contraction.

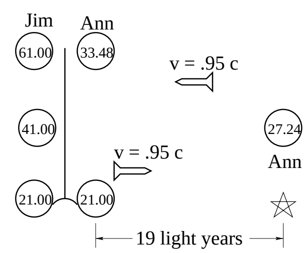
The length of an object in a coordinate system in which it is at rest is called the **proper length**. For other observers for whom the object is moving, the length of the object will be smaller by the factor[1] $\sqrt{1 - v^2/c^2}$.

The twin paradox

These counter-intuitive effects, time dilation, length contraction, and the relativity of simultaneity, produce many startling conclusions which are (until one explains them) paradoxes. One that is often discussed is called the twin

[1]This only applies to the lengths in the direction of the relative motion. As we have already seen, transverse lengths are unaffected.

paradox. Once again, our politically correct twins Ann and Bob will have Bob stay home, while his adventurous sister takes off at high speed on her 21st birthday. She flies off at 95% of the speed of light to travel to a star 19 light-years away. This, according to Bob, takes 20 years. Then she turns around and comes back, travelling at the same speed. Bob goes to the spaceport to meet her, complaining a bit about the arthritis he has acquired now that he is 61, but his sister steps off the ship without a grey hair, looking only 33 1/2. There was no room on her spaceship for hairdye — she looks only 33 1/2 because she is only 33 1/2 — her aging clock has been running slow, so she aged only



$$40 \times \sqrt{1 - .95^2} = 12.48 \text{ years}$$

during the trip. Each half of her trip took only 6 years and 3 months.

Now this is a pretty strange result, two twins whose ages differ by 27 1/2 years, but that is not the paradox — that comes when we ask how Anne explains how her brother got so old. After all, to her, her brother's clock ran slow, so in the 6.24 years she was outbound, her brother should have only aged

$$6.24 \times \sqrt{1 - .95^2} = 1.95 \text{ years},$$

and another 1.95 years on the way back. Why can't Ann conclude that her brother ought to be only 24.9 years old?

Well, it is confusing! The reason that Ann cannot use the argument Bob used in reverse is that she did not spend all her time in an inertial coordinate system. It is true that the events

1) Jim celebrates his 22.95^{th} birthday

2) Ann hits the breaks to turn around

are simultaneous in the inertial system of her rocket before the breaks were hit. But that system keeps going out, while it is a different inertial system she finds herself in once she has turned around and headed home.

This paradox could use a lot more explanation than I have time for here, but if you are intrigued, it is well worth while to investigate it in greater detail by reading James H. Smith, Introduction to Special Relativity, pp 88-102.

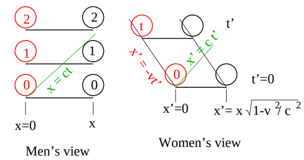
Lorentz Transformation

We have now derived all the effects we need to get the general relationship which gives the coordinates in one inertial coordinate system in terms of that of another reference frame (reference frame and coordinate system are used pretty much interchangably). The statement that the laws of physics apply equally to both is a statement about symmetry of the laws of physics. For a moment before we get into the serious problem of finding Ann's coordinates in terms of Bob's, let us observe that there is a very simple kind of change of coordinate system so trivial we have not mentioned it — that of going from Ann's to Jane's. That is, nothing specifies where the origin of one's coordinates should be or when his clock should say zero. So one trivial set of coordinate transformations is to take $\vec{r}\,' = \vec{r} + \vec{k}$, $t' = t + b$, where $\vec{k}$ and b are constants. These are boring changes of coordinates which are included in both Galilean and Einsteinian relativity. But we are not interested in that trivial change of coordinates, so we will use it to arbitrarily make sure the that event the men call $x = y = z = t = 0$ is the same as the event the women call $x' = y' = z' = t' = 0$. Now we will find the transformation of coordinates which tell us, given the coordinates Bob uses to describe an event, (x, y, z, t), what the coordinates are for Ann, (x', y', z', t'). This is known as the **Lorentz transformation**, and it is the Einsteinian modification of the Galilean transformation we discussed earlier.

We will again assume Ann is moving with uniform velocity v in x-direction according to Bob. Of course there is a more general possibility of moving in an arbitrary direction, but that has no new physics and a lot of additional complication in the algebra, so we will not consider that. We have already shown that transverse lengths are not affected by the relative motion, so the two agree on the y and z coordinates,

$$y' = y, \qquad z' = z.$$

The x and t coordinates are more difficult. To help us, the men lay out a measuring rod of length x with one end at the origin and the other at x, and they also place clocks at both ends. What we would really like to know is what x' and t' are for the event that the clock at x strikes t. The men, of course, carefully synchonize the clocks. Both clocks are moving, according to the ladies, at speed v in the $-x$-direction, so the clock at the origin obeys the equation



$$x'(t') = -vt' \quad (\text{for } x = 0)$$

while the other clock is at

$$x'(t') = x\sqrt{1 - v^2/c^2} - vt' \tag{3}$$

How do I know that? Because the proper length between the clocks is x, and the women see that length contracted to $x\sqrt{1 - v^2/c^2}$ because the meterstick between them is in motion. So the distance between the clocks, at any one instant t', must be $x\sqrt{1 - v^2/c^2}$. To simplify the notation a bit, we define the standard factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

which occurs repeated in relativity problems. With this notation, we can solve (3) for Bob's x in terms of Ann's coordinates:

$$x(x', t') = \gamma x' + v\gamma t' \tag{4}$$

Now the clock at the origin is seen by the women as running slow, so the event that this clock strikes t occurs at a later time $t' = t/\sqrt{1 - v^2/c^2} = \gamma t$, and of course at $x' = -vt' = -v\gamma t$. We are more interested in the other end. That clock is also running slow, so $\Delta t' = \gamma \Delta t$, but for this clock we don't expect that $t = 0$ corresponds to $t' = 0$. Instead, we have some as yet unknown offset, which could depend on x, so

$$\text{Right hand clock:} \qquad t'(x, t) = \gamma t + b(x)$$

Now consider a light pulse leaving the origin. It traces out a path which Bob describes as the line $x = ct$ and Ann as $x' = ct'$ Along this path we have, from (4), $x = \gamma(c + v)t'$, so

$$t' = \frac{x}{\gamma(c + v)} = \gamma t + b(x) = \gamma \frac{x}{c} + b(x),$$

and thus

$$\begin{aligned}
b(x) &= \frac{x}{\gamma(c+v)} - \gamma\frac{x}{c} = x\gamma\left(\frac{1}{\gamma^2(c+v)} - \frac{1}{c}\right) \\
&= \frac{x\gamma}{c}\left(\frac{1-v^2/c^2}{1+v/c} - 1\right) = \frac{x\gamma}{c}\left(1 - \frac{v}{c} - 1\right) \\
&= -\frac{v\gamma x}{c^2}
\end{aligned}$$

Plugging back into the last equation for t' and (3),

$$\begin{aligned}
t' &= \gamma t - \frac{v\gamma}{c^2}x \\
x' &= \gamma^{-1}x - v\left(\gamma t - \frac{v\gamma}{c^2}x\right) \\
&= \gamma\left(\gamma^{-2} + \frac{v^2}{c^2}\right)x - v\gamma t \\
&= \gamma x - v\gamma t
\end{aligned}$$

Let me repeat this crucial equation, the **Lorentz Transformations**

$$x' = \gamma x - v\gamma t \tag{5}$$
$$t' = \gamma t - \frac{v\gamma}{c^2}x \tag{6}$$

which gives Ann's coordinates in terms of Bob's, or the inverse Lorentz Transformation

$$x = \gamma x' + v\gamma t' \tag{7}$$
$$t = \gamma t' + \frac{v\gamma}{c^2}x' \tag{8}$$

giving Bob's in terms of Ann's. The only difference is the sign of v, which is indeed the only thing that changes if we interchange Ann and Bob.

This discussion was rather algebraic without a good intuitive feel, so we need to check that we get something worthwhile here. Well, one thing we need is to derive the formula for relative velocities. Let some object be moving as a function of time, which Bob says is $x(t)$ and Ann says is $x'(t')$. What does Ann say the velocity is?

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma dx - v\gamma dt}{\gamma dt - \frac{v\gamma}{c^2}dx} = \frac{\gamma u_x - v\gamma}{\gamma - \frac{v\gamma}{c^2}u_x} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}.$$

This is called the velocity addition formula. Notice that if the object in question is a light beam moving in the x direction, so that $u_x = c$, we find

$$u'_x = \frac{c - v}{1 - \frac{vc}{c^2}} = c,$$

so that the light beam is **also travelling at** c according to Ann.

We also need to examine the transverse velocities. Although the transverse coordinates are unchanged by the Lorentz transformation, time is not If we have an object moving, according to Bob, with velocity (u_x, u_y, u_z), in a very small time interval we will have $dx = u_x dt, dy = u_y dt$. Thus the value Ann gets for the y component is

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma dt - v\gamma dx/c^2} = \frac{u_y dt}{\gamma(1 - vu_x/c^2)dt} = \frac{u_y}{\gamma(1 - vu_x/c^2)}.$$

Summary

- An event happens at a particular point in space and at a particular time, but the coordinates which describe that point in space and time depend on the reference frame.

- The time interval between two events in a reference frame (S) in which the two events take place at the same point is called the **proper time interval**. In any other frame, moving with speed v with respect to S, the time interval between the events is longer by a factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

- Lengths perpendicular to the relative motion of two reference frames appear the same.

- The length of a physical object as measured in a reference frame (S) in which it is at rest is called the **proper length** of the object. If another frame S' is moving with velocity v in the direction of the length of the object, relative to S, the length of the object as measured by S' is contracted by a factor of γ,

$$L' = L/\gamma = L\sqrt{1 - v^2/c^2}.$$

- Simultaneity is relative: two events which are simultaneous in one reference system need not be simultaneous in another. Even which event came first differs from one system to another.

- The Lorentz transformation describes the way to determine the coordinates of an event in one reference system given the coordinates of the event in another. If the S' coordinate system is moving in the $+x$ direction relative to S with velocity v,

$$x' = \gamma x - v\gamma t, \quad y' = y, \quad z' = z, \quad t' = \gamma t - \frac{v\gamma}{c^2}x,$$

$$x = \gamma x' + v\gamma t, \quad t = \gamma t' + \frac{v\gamma}{c^2}x'.$$

- And the velocities (for $\vec{v}$ in the x direction) are related by

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}, \quad u'_y = \frac{u_y}{\gamma(1 - vu_x/c^2)}, \quad u'_z = \frac{u_z}{\gamma(1 - vu_x/c^2)}.$$