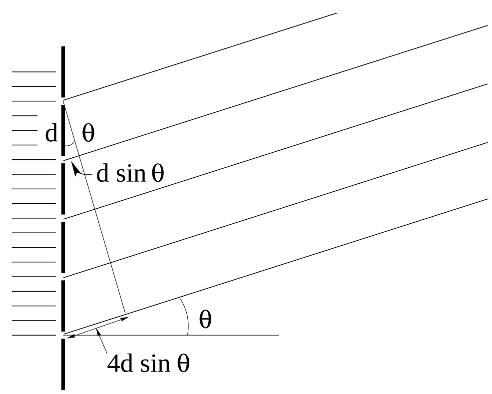
Physics 228, Lecture 6
Monday, Feb. 7, 2005

Diffraction. Ch 36:4-6, 33:5

1 Diffraction Grating

Last week we considered multiple slits, evenly spaced. We used phasors to understand the results. A practical device for analysing light is the diffraction grating, which can be thought of as a large number of evenly spaced slits. Consider the light coming to the grating perpendicular to it, and consider the light which goes off to very great distances.

The path length difference for light coming from neighboring slits is $d \sin\theta$ just as in the case with the double slit. If $d \sin\theta$ is an integer number of wavelengths, the two adjacent slits will interfere **constructively**, and so will each of the others, because they will differ by integer multiples also. So when



$$d \sin\theta = m\lambda, \quad m \text{ an integer}$$

all of the phasors point in the same direction, the field is N times that of a single slit, and the intensity N^2 times that of a single slit, and we get a strong "diffraction line".

It is not necessary for the slits to be in a totally opaque material. All that is needed is that the transmission or reflection properties are different in the "slits" from in between them. A diffraction grating is made by scratching fine lines very closely spaced into glass or into a mirror. It is quite possible to scratch 6000 lines per centimeter with good machinery.

We will then get maxima at angles with $d \sin\theta = m\lambda$. If $d = 1\,\text{cm}/6000 = 1.67 \times 10^{-6}$ m and $\lambda = 5 \times 10^{-7}$ m, we see that we get maxima for $\sin\theta = m/3.33$, for integer m. The undeflected light at $m = 0$ contains all colors, but the "first order maximum" at $m = \pm 1$ contains, for a given θ, light of only one visible frequency. There are also higher order diffractions, but for our example it stops at $m = 3$, for beyond that we would need angles with sines larger than one.

Note we get angles greater than 90°, because the grating works by reflection as well as transmission. Notice that if we use longer wavelength light, red, the angle of the first order diffraction is larger. Let's also see what happens if we use a grating with a larger distance between slits — this one has 100 lines per millimeter.

So we see that the grating only sends its light of a given wavelength off in a discrete set of directions. Of course, light of different wavelengths will have different angles for the bright spots. The red light will have a larger first order diffraction pattern angle than that of blue light. White light, which consists of a superposition of light of all wavelengths, will be decomposed by the grating. Here is a grating. I have a slide with a slit in the projector and have the white light pass through the grating. You can see the first order diffraction patterns on either side.

A diffraction grating together with some telescopes for focusing the light and mechanical apparatus to measure angles is a good way to measure the wavelengths of light. Fluorescing gases give off light at only certain frequencies, and a grating can be used to measure the wavelengths of these.

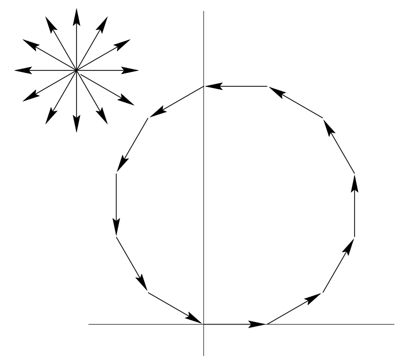
Some of these are close together. For example, sodium has a very bright "doublet" with wavelengths of 588.9950 nm and 589.5924 nm. To distinguish these lines we need a grating of **resolving power**

$$R = \frac{\lambda}{\Delta\lambda} = \frac{589}{0.597} \approx 1000,$$

where $\Delta\lambda$ is the minimum change in wavelength that we can distinguish with a given grating. We can use Rayleigh's criterion to evaluate $\Delta\lambda$. That would say that the maximum for $\lambda \pm \Delta\lambda$ should be in the same position as the closest *minimum* near the maximum for λ. So we need to ask about the intensity at angles other than the ones for which all slits are in phase.

In our discussion of the multiple slit interference, we found that there are local maxima at other angles, but they are very small compared to the maxima for which all slits contribute constructively, which have intensities N^2 times what comes from a single slit. The width of this maximum is determined by what happens if the phase shift differs by a small amount from an integer multiple of 2π.

A phase shift of $2\pi m + \Delta\phi$ is the same as a phase shift of $\Delta\phi$, because the phase shift only occurs within sine and cosine functions, so we drop the $2\pi m$ and consider what $\Delta\phi$ needs to be to get the first minimum. The phasor from the second slit is at an angle $\Delta\phi$ with respect to the phasor from the first. That from the third will be at an angle $2\Delta\phi$, and so on. If the phase shift from one end of the grating to another is 2π, then the phase shift between adjacent slits is $\Delta\phi = 2\pi/N$. Then the N phasors are as shown, and sum up to zero, giving the minimum. Thus we see that the first minimum occurs when



$$d \sin\theta = m\lambda \pm \lambda/N \quad \text{(minimum)}.$$

Now the m'th order maximum for λ' occurs when $d\sin\theta = m\lambda'$ and the first minimum for λ occurs when $d\sin\theta = m\lambda + \lambda/N$, one extra wavelength across all the slits. So a wavelength λ' will have its m'th maximum at the position of minimum next to the m'th maximum for λ, which is what Rayleigh asks us to consider, if $Nm\lambda + \lambda = Nm\lambda'$, or

$$\Delta\lambda = \lambda' - \lambda = \frac{\lambda}{mN},$$

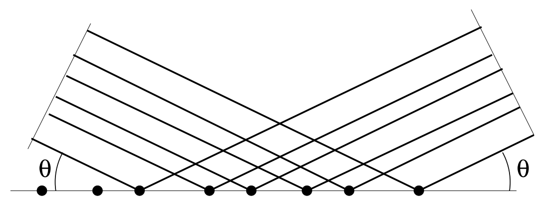
so we see that

$$R = mN.$$

2 Diffraction of X-rays by Crystals

One very important use of diffraction comes not from the diffraction of visible light but from X-rays, which have wavelengths roughly of the size of the spacing between atoms in a solid or liquid. Many solids are crystals, which means they are a regular repeating structure of atoms arranged in a lattice. X-rays scatter off the nuclei in this crystal, so a reasonable approximation is that the light is scattered from the balls in the figure. Now that is awfully many discrete sources to have to add up, and we can simplify the problem if we focus first on a bunch of atoms lying in a plane.

Consider a plane of atoms, looking edge on, on which some light is incident. If we examine the rays which come off at the same angle to the plane, we see that the distance each travels from the first wavefront shown to the last is the same — that is, the light scattered by all the atoms in this plane interferes constructively. Thus we find that the plane of atoms acts like a mirror, with a strong reflection for which the angle of incidence equals the angle of reflection.

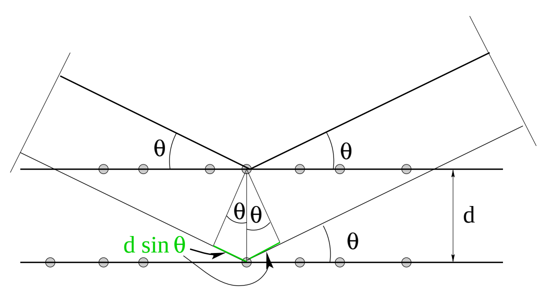


Plane of atoms, not necessarily evenly spaced, still constructively interferes to reflect like a mirror.

Important: Notice that the angle θ *used here is different for all the others* we have been using — it is the angle to the plane rather than to the normal to the plane, so it is not the angle of incidence but its complement. I can give you an excuse for doing this, but basically it is a bad idea that is set historically and can't be fixed!

Now X-rays are quite penetrating and one plane of atoms is not very much material, so not all the light will be reflected. In a crystal, the arrangements of atoms repeats, so there will be another plane of atoms some distance d away from the first. It will also reflect at the angle θ, and all these atoms will interfere constructively with each other, but how about their interference with the light reflected from the first plane?

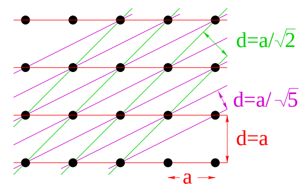
The light which is reflected off the bottom plane has had to travel an extra distance along the two short green segments each of length $d \sin\theta$. The light from the two planes will interfere constructively if that extra distance is an integral multiple of the wavelength,



$$2d\sin\theta = m\lambda$$

and all the subsequent evenly spaced planes will then also interfere constructively. This famous equation is known as *Bragg's scattering law*.

X-rays are a bit hard to work with in a lecture situation, so we've enlarged things a bit. Here I have a lattice of "atoms" (which are really steel balls) with a distance between them of $a = 3.7$ cm instead of, perhaps, 100 million times smaller than that. I have a source of microwaves which claims to have a frequency of 10.5 GHz, which by $\lambda = c/f$ gives 2.85 cm. So the only Bragg angles we can expect to get is $m = 1$ or $m = 2$, $\theta = \sin^{-1}(2.85/(2 * 3.7)) = 22.7°$ or $\sin^{-1}(2 * 2.85/(2 * 3.7)) = 50.4°$. That means in the first case, the beam will be bent through 45.4° and the crystal planes will be at 22.7° to the beam.



The planes from which X-rays can bounce are not only the ones formed by the sides of the cubes, but also those from various diagonals. For example, the (green) planes connecting each sphere not to its nearest neighbor but to its next nearest, on a diagonal, are separated by $d = a/\sqrt{2} = 2.62$ cm. Thus the $m = 1$ scattering is at 33°.

3 Polarization

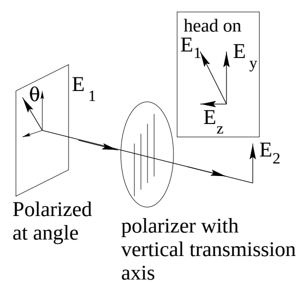
When we derived the existence of electromagnetic waves from Maxwell's equations, we saw that the electric field had to be transverse, that is, perpendicular to the direction of motion. That still leaves it free to point in an arbitrary direction in the transverse plane. If the wave is travelling in the x direction, the $\vec{E}$ can point anywhere in the yz plane, or in other words, it can have arbitrary y and z components. When light is emitted, say by a hot filament, each atom vibrates in an arbitrary direction, so the light is an incoherent mixture of waves with $\vec{E}$ in all transverse directions. Such light is called **unpolarized**.

We saw, however, that a radio wave produced by a dipole antenna has its electric field oscillating in the direction which lies in a plane including the direction of travel of the wave and the direction of the antenna. So it is clearly possible to have an electromagnetic wave where the electric field is, at all times, in one direction, or its opposite, at a given point in space. Such a wave is called **linearly polarized**. The plane including the direction of the electric field and the direction of propagation is called the **plane of polarization**.

Electromagnetic waves of the frequency of light can also be polarized, if we can arrange things so electric charges oscillate in one direction rather than the perpendicular one. One way to do that is to use long-chain hydrocarbon molecules which are aligned in one direction. That produces a material invented by Land called *polaroid*, which absorbs light polarized in one direction but not light polarized in the perpendicular direction. Your sunglasses almost certainly contain such material. Light with its E field along the **transmission axis** passes through the material, but light with its E field perpendicular to that is absorbed.

Maxwell's equations are linear, which means electromagnetic waves in empty space, and indeed in most transparent materials, obey a superposition principle, which says that two waves can be added together. This is only useful if they are coherent. If in addition two waves with the same direction of propagation and frequency are in phase, the E fields add up the same way at all points and times. That is, if we have two plane waves moving in the x direction differing only in that one in polarized in the y direction with amplitude $E_y = E_a$, and the other is polarized in the z direction with $E_z = E_b$, then the two waves combine into one, with a polarization in the direction $E_a\hat{\jmath} + E_b\hat{k}$.

If light is initially polarized at an angle θ as shown, and it passes through a polarizer with a vertical transmission axis, the light will be decomposed into a part with polarization vertical and part horizontal, and only the part with polarization vertical will get through.



The initial polarized beam might have been made with a polaroid filter. Suppose we have an unpolarized beam in the x direction and it passes through a polaroid film with its transmission angle at θ to the y direction. Then half the light will be absorbed, and half the light, with its polarization axis in the direction $\cos\theta\hat{\jmath}+\sin\theta\hat{k}$, will get through. Thus the electric field has an amplitude $E_1(\cos\theta\hat{\jmath} + \sin\theta\hat{k})$. If this linearly polarized light then passes through another polaroid film with transmission axis in the y direction, the light which gets through will have amplitude $E_2 = E_1\cos\theta\hat{\jmath}$, and an intensity reduced to $\cos^2\theta$ of what it was before the second filter:

S&BV5 38.28,
Y&F 33.22
Polarizer and analyzer

$$I = I_{\text{incident}}\cos^2\theta.$$

This is known as **Malus's law**.

We can try that out. I have here two large circular polaroid filters. Half of the unpolarized light from the projector will pass through one filter no matter what orientation I place it at. If I place the second polaroid with the transmission axes aligned, it will not remove any additional light, but if I rotate it, you see that more and more light is blocked, until, when the transmission axes are 90 degrees apart, no light gets through.

Now let me show you something curious. If I place a third polarizer between the two, and orient it properly, some light now does get through the two polarizers with perpendicular transmission axes.

3.1 Other Polarizing Processes

In most transparent materials, whatever lack of rotational symmetry there is in the structure of the material does not affect light travelling within it, so whatever polarization there is in the light is unaffected and has no affect on the travel in the material. There are exceptions, however, called **birefringent** materials, in which two directions of polarization behave with different indices of refraction. Such a material can be used to separate unpolarized light into two polarized components. Another exceptional property is called **optical activity**. When polarized light passes throough such a material, the plane of polarization is rotated as the light passes through the material. One very interesting example is the common sugar dextrose. A solution of dextrose is optically active, so that light passing through it has its polarization axis twisted clockwise. Why clockwise and not counterclockwise? That is because the sugar itself has an asymmetrical shape, and in fact there is a mirror image to dextrose called levulose, which rotates the polarization counterclockwise. If you make the sugar inorganically in a chemistry lab, you get a 50/50 mixture of dextrose and levulose, but all creatures on Earth produce the right handed version and not the left handed version. This is evidence that we are all descended from a single bacterium which just happened to be right handed instead of left-handed.

3.1.1 Polarization by scattering

There is one situation, however, where there is an asymmetry between the two polarizations even if the media involved are normal. That is when light is incident on a surface at an angle. The electric field of the incident light could be polarized in the plane which includes the light ray and the normal, or it could be polarized perpendicular to that plane. When the light hits the surface, in one case the electric field will not lie along the surface, while in the other case it will, so it is understandable that the reflection properties might be different. If the angle of incidence is zero, there is no distinction, but if the angle of incidence is such that the reflected light and the refracted light are perpendicular to each other, only light with polarization parallel to the surface will be reflected! This angle of incidence is called the **Brewster angle**[1]

S&BV5 38.30, Y&F 33.24-5 polarization at surface 7" × 4 3/4"

$$\tan\theta_p = \frac{n_2}{n_1}$$

Then the reflected ray is totally polarized.

We can demonstrate this by reflecting the beam from the projector off the front surface of this transparent block at the Brewster angle. You can see the reflected beam on the screen. The light from the projector is not polarized, but the light reflected off the block is, as you can see when I pass it through the polarizer, and rotate the polarizer.

For angles between 0 and θ_p, or greater than θ_p, the reflected light will be partially polarized.

Light will also be polarized if it is scattered through some angle, for the polarization in the plane of scattering faces different physics than the polarization perpendicular to that plane. This is why sunglasses are polarized — if they can block the light scattered directly from the sun more than light which has been multiply scattered, some of the glare will be removed relative to the solid objects you wish to make out.

[1]For the reflected and transmitted rays to be perpendicular, $\theta_2 + \theta_1 = 90°$, but by Snell's law we also have $n_1 \sin\theta_1 = n_2 \sin\theta_2 = n_2 \sin(90° - \theta_1) = n_2 \cos\theta_1$. Dividing by $n_1 \cos\theta_1$ gives $\tan\theta_1 = \frac{n_2}{n_1}$. In this case we call the incident angle θ_p, as it is the polarizing angle.

4 Summary

- Diffraction gratings with N "slits" a distance d apart will produce maxima for $d \sin\theta = m\lambda$ for integer m. The resolving power of such a grating, using its m'th order maxima, is

$$R := \frac{\lambda}{\Delta\lambda} = Nm.$$

- Multiple planes of atoms can act as a reflection grating, giving bright spots when there is a plane of atoms at angle θ with respect to the beam, where $2d \sin\theta = m\lambda$. Notice that this θ is, confusingly, not the angle of incidence but its complement.

- Light, being a *transverse* wave, can be polarized with two independent linear polarizations. Polaroid filters, reflection at an angle, and scattering of light can cause one polarization to behave differently than the other. This can also happen in birefingent materials and in optically active materials.

- If linearly polarized light is passed through a polarizer with transmission axis at an angle θ with respect to the initial polarization direction, the intensity of the transmitted light is multiplied by $\cos^2\theta$.

- Light incident on an interface (between normal transparent materials) at the Brewster angle will have one polarization, that with $\vec{E}$ in the plane including the normal to the surface, entirely refracted with no reflection, so the reflected beam will be 100% polarized along the opposite axis.