Physics 228, Lecture 2
Monday, Jan. 24, 2005

Geometrical Optics. Ch 34:1–3

1 Flat mirrors

When you look in a mirror, you see an **image** of yourself behind the mirror. You are the **object** of which that image is the image. Light which bounces off you, the object, travels to the mirror, is reflected, and reaches the eye *as if* it had come from the image in back of the mirror. Other things are also imaged in the mirror. As a child you learned that the image is not a real object, but it appears to your eyes to be a real object, a person just like you except that she wears her watch on the wrong hand and parts her hair on the wrong side. We begin by carefully describing that image — where it is, and how big. Then we will do the same for images formed by more complex situations, curved mirrors and lenses.

S&BV5 Fig 36.1
object in plane
mirror
3 1/4" × 3 1/2"

Consider a plane, or flat, mirror with an object O a distance p in front of it, and consider several rays of light scattered off that point, being reflected off the mirror, and then entering your eye. The distance p that the object is in front of the mirror is called the **object distance**. If we extend the rays entering your eyes back behind the mirror, all of the rays will pass through a single point I. Because the angle of reflection is equal to the angle of incidence, we have a congruence which shows that the point I is exactly as far behind the mirror as O is in front, $q = p$. The distance q that the image is from[1] the mirror is called the **image distance**[2].

If the mirror is vertical, note that the heights of the point I and the point O are the same. If we look at an extended object with a height h, the image will also have a height h. This is true for the other coordinate parallel to the mirror as well. The image is neither enlarged or shrunk, so we say it has **magnification** equal to 1. Notice that these coordinates are not

[1]Later we will give signs to q, which in this case will be negative.

[2]The text calls the image distance s and the object distance s'.

reversed – if your mirror is in an East-West plane (so you are looking north perpendicular to it) your East hand is on the East side of the image just as your head is at the top end of the image. What is reversed is not left and right, it is front and back. Your bellybutton is north of your spine, but the image's bellybutton is south of her spine.

Why do we so persistently claim that mirrors reverse left and right? Suppose you could walk through the mirror and try to assume the position of your image (which we will imagine frozen in place). First you would walk the distance $p+q$ to get in the right location, but you would still be facing north while your image was facing south. Probably you would then turn around, more precisely rotate 180° about a vertical axis, so that now you too would be facing south, but in the process your hand with the watch, which previously had matched the images hand with the watch, is now on the wrong side. You cannot assume the position of the image without morphing like a cartoon character through a two dimensional flattening, but because we are so nearly symmetrical we tend to think we have matched the image, almost, after rotating, and then notice that we have reversed right and left. We could have rotated about a horizontal axis instead, but that would have left us with our feet in the air, and that doesn't seem very natural to us.

An image can be virtual or real, although for our flat mirror it is virtual. The distinction has to do with whether any light was really present at the image. The mirror probably has a plaster wall behind it — none of the light which appears to come from the image behind the mirror was ever really there, so this is a virtual image. We will see some real images when we consider curved mirrors and lenses.

Last time we discussed a light ray which was reflected off two mirrors. Light which leaves the object and bounces off the first mirror continues on a path which appears to originate from the image I_1, and this acts as an object for the second mirror, so the image of I_1 is I_3. Of course the object can also be seen at I_1 and at I_2. If O is a right hand, I_1 and I_2 are lefthanded, but I_3 is righthanded.

S&BV5 Fig 36.5
double reflection
at 90°
3" × 3 3/4"

One way to demonstrate that the image is behind the mirror at $q = p$ is to trace two convenient rays emanating from the object, rather than arbitrary ones as we did. One such ray is the one which goes perpendicular to the mirror, and bounces back along the same path, as in P–Q. This makes the congruence argument clearer, for angles $\angle PRQ$ and $\angle P'RQ$ are the complements of the angle of incidence and the angle of reflection respectively, but those are the same. Together with the fact that $\angle RQP = \angle RQP' = 90°$ and the common side RQ, we have congruent triangles, so $p = q$.

S&BV5 Fig 36.2 Image in plane mirror

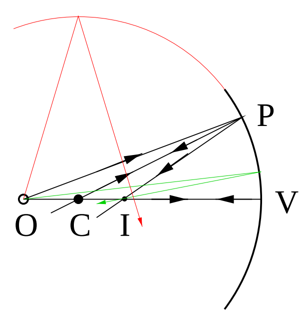
3" × 2 3/8"

We will see that this method of finding two convenient rays is more essential for more complex situtations.

2 Spherical mirrors

Not all mirrors are flat. We will consider spherical mirrors. They are not complete spheres, but parts of a spherical surface. Mirrors generally work from one side only, so we distinguish concave mirrors from convex ones. If the mirror works from the inside, it is a concave mirror, while a convex mirror works from the outside of the sphere.

The mirror has a radius of curvature R and a center point C. Any light ray emerging from C will hit the mirror perpendicular to it and be reflected to C. If there is a point O further from the sphere, one ray easy to trace is the one that goes through the center and is bounced straight back. Another ray from O will intersect that ray at the point I after being reflected from P with $\angle OPC = \angle CPI$.



When considering objects which lie close to the line OCV, we call this line the **principal axis**, and call rays which make only small angles with this axis **paraxial** rays. All paraxial rays from O will pass through the image point I, as they did (virtually) for the plane mirror, though here I is a real image. But this is not precise, and rays which make larger angles, such as the red one, will not quite pass through the right point. This will cause a lack of good imaging called **spherical aberration**.

Now suppose O is the bottom of an object of height h. Let us find the image of the top of the object. One ray passes through the center of curvature of the mirror C, and back along the same line. Another hits V at an angle θ to the principal axis, and is reflected, again at angle θ, to the tip at the bottom of the small arrow h'. The two right triangles with bases O–C–V and I–V respectively are similar, The object distance (from V, $|OV|$) is called p and the image distance $|IV|$ is called q, so

S&BV5 Fig. 36.11 real image, concave mirror

6 1/2" × 3 1/2"

$$\frac{h}{p} = \frac{|h'|}{q}.$$

I have placed absolute value signs on the height h' of the image because we will consider h' to be negative to reflect the fact that the image in **inverted**. We will also define the magnification to be negative:

$$M = \frac{h'}{h} = -\frac{q}{p}.$$

Note that the right triangles with bases CO and CI are also similar, so

$$\frac{h}{|OC|} = \frac{|h'|}{|CI|} \quad \text{or} \quad \frac{h}{p-R} = \frac{-h'}{R-q},$$

so

$$M = \frac{h'}{h} = -\frac{R-q}{p-R} = -\frac{q}{p},$$

$$\text{so} \quad \frac{R-q}{q} = \frac{p-R}{p}, \quad \text{or} \quad \frac{R}{q} - 1 = 1 - \frac{R}{p}, \quad \text{or} \quad \frac{R}{q} + \frac{R}{p} = 2,$$

or finally

$$\frac{1}{q} + \frac{1}{p} = \frac{2}{R}.$$

This is the mirror equation.

If an object is very far away, $p \gg R$, $1/p$ becomes negligible in comparison to $2/R$, and the image approaches a fixed distance,

$$q \to \frac{R}{2} = f,$$

where f is called the focal length. The rays emerging from a point of an object infinitely far away approach the mirror in parallel, so the focal point

F is the point to which incoming parallel rays (parallel to the principal axis) are focussed. We may use f instead of R to rewrite the mirror equation in a form which will also hold for lenses,

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{f}.$$

Note also, by reversibility of the paths, that an object at the focal point is focussed at infinity, and an object closer than the focal point has a *negative* q, which means its image is on the other side of the mirror.

Show ball on pendulum in concave mirror

2.1 Convex Mirror

A spherical mirror can be made the other way too, so it reflects light on the outside. Consider two rays from the tip of the object. One ray simple to investigate is the one directed towards the center of curvature, which will hit the mirror perpendicular to it and be reflected back along its path.

S&BV5 Fig 36.13
Convex mirror
real object
7" × 3 1/2"

Another ray starts parallel to the principal axis and will be reflected. Of course the angle of incidence is equal to the angle of reflection, which means the angles the light makes with the line through C is the same. For paraxial rays this implies that it appears to have come from the focal point F which is still halfway between the center of the mirror, V and the center of curvature C. That is because the triangle AFC is isosceles so the orange and purple angles are the same. These two rays intersect at the head of the arror at I, the image. We see that the image is **behind the mirror**, and so it is virtual, not real — no light actually reaches the point I. Furthermore we see the image is upright. Notice that the image is on the opposite side from the case of the concave mirror, and so are the focal points and centers of curvature. So we might think of letting q, f and R take on negative values to indicate this, rather than insisting they be actual *distances*, which of course are always positive.

Using similar triangles again, and assuming the angles of the rays are paraxial enough to approximate $\tan\alpha \approx \sin\alpha$, we can show[3] that

$$\frac{1}{p} - \frac{1}{|q|} = -\frac{2}{|R|} = -\frac{1}{|f|}. \tag{1}$$

Now if we decide to consider q, f and R to be negative, this is

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R} = \frac{1}{f},$$

so we have exactly the same equation as for the concave mirror. We also have the magnification is

$$M = -\frac{q}{p} = \frac{|q|}{p} > 0,$$

which is natural as the image is upright, not inverted. Note that the image is virtual, as there is, of course, no light actually present behind the mirror.

We can use the same formula for both convex and concave mirrors, we just need to remember the sign conventions:

Sign conventions for mirrors

Object in front of mirror	$p > 0$	real object
Object behind mirror	$p < 0$	virtual object
Image in front of mirror	$q > 0$	real image
Image behind mirror	$q < 0$	virtual image
Concave mirror	$R > 0$, $f > 0$	C and F in front of mirror
Convex mirror	$R < 0$, $f < 0$	C and F behind mirror
M positive	$M > 0$	image upright
M negative	$M < 0$	image inverted

3 Images by refraction

While mirrors are certainly very useful, and the most important optical parts of high end telescopes, most practical optical imagine is done with refraction

[3]By similar triangles, $h'/(R-|q|) = h/(R+p)$, and $h'/(\frac{1}{2}R - |q|) = \tan\angle AFO \approx \sin\angle AFO = h/(\frac{1}{2}R)$. Thus

$$\frac{h'}{h} = \frac{R-|q|}{R+p} = \frac{R-2|q|}{R},$$

which gives Eq. 1.

rather than reflection. We will consider a spherical interface between two transparent media with indices of refraction n_1 and n_2, one of which may be, but doesn't have to be, air, which is almost vacuum. Our example will have the center of curvature to the right, opposite the location of the object. We will assume the image is on the right, but we will see that is not always the case.

Consider a ray of light leaving the object and passing through the center of curvature. As it hits the surface perpendicular to it, the angle of incidence is zero and it is undeflected, and forms the principal axis. The image has to include this ray, so it is on the principle axis. Consider another ray leaving O at an angle α, hitting the surface at P and refracted to I, with the angles β at C and γ at I as shown. We are considering only paraxial rays, so we may approximate

S&BV5 Fig 36.18 Refraction convex surface

6 1/4" × 2 1/4"

$$\sin\alpha \approx \alpha \approx \tan\alpha, \qquad \cos\alpha \approx 1,$$

and the same for β and γ, and also for the angles of incidence $\theta_1 = \alpha + \beta$ and refraction, $\theta_2 = \beta - \gamma$ The side d is the altitude of three right triangles whose bases are approximately p, R, and q respectively. For the first and last, the bases actually differ by an amount $R(1 - \cos\beta)$, but as β is small this is negligible. Thus

$$d = R\sin\beta \approx R\beta \quad \approx p\tan\alpha \approx p\alpha \quad \approx q\tan\gamma \approx q\gamma,$$

or

$$\alpha \approx \frac{d}{p}, \qquad \beta \approx \frac{d}{R}, \qquad \gamma \approx \frac{d}{q}.$$

But Snell's law tells us

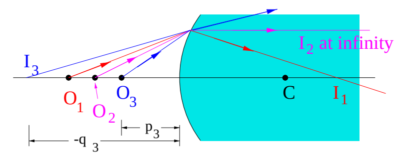
$$\begin{aligned} n_1 \sin\theta_1 &\approx n_1\theta_1 = n_1(\alpha+\beta) \approx dn_1(\frac{1}{p} + \frac{1}{R}) \\ &= n_2\sin\theta_2 \approx n_2\theta_2 = n_2(\beta - \gamma) \approx dn_2(\frac{1}{R} - \frac{1}{q}). \end{aligned}$$

So

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}.$$

This is the equation for refracting surfaces. We need to examine what happens as we vary the parameters.

Let's assume $n_2 > n_1$, which is implied by the shading, so the right hand side is a positive number independent of where the source is. As I bring the source closer, eventually n_1/p will equal the right hand side, $(n_2 - n_1)/R$, so n_2/q goes to zero and the image will move out to infinity $q \to \infty$. This is shown in magenta. If we bring it still closer, n_2/q will have to become negative to balence the equation, and so q will be as well. I have shown this in blue. Note that the light continues to the right and diverges, appearing to come from the image I_3, but there is no light at the image, so I_3 is a **virtual** image, while I_1 is a real image.



Suppose we imagine being able to continually adjust the curvature of the interface. Notice that making the surface less curved means making R bigger. If we let R go all the way to infinity, we have a flat surface, and then we see

$$\frac{n_1}{p} = -\frac{n_2}{q} \quad \longrightarrow \quad q = -\frac{n_2}{n_1} p.$$

If you look down into a pool and see a fish a distance $|q|$ under the surface, it is really a distance $p = \frac{n_1}{n_2}|q|$ under the surface. As the light is originating in the water, $n_1 = 1.33$ and it emerges into air, $n_2 = 1$, you must aim your spear 33% below where you see the fish in order to get it.

If we continue to deform the spherical interface further, we get a spherical surface with its center of curvature on the left. We have been deforming $1/R$ continuously, so we should now consider the radius of curvature negative. This leads us to

Sign conventions for refracting surfaces

Object in front of surface	$p > 0$	real object
Object behind surface	$p < 0$	virtual object
Image in back of surface	$q > 0$	real image
Image in front of surface	$q < 0$	virtual image
Convex (from "front")	$R > 0$	C in back of surface
Concave (from "front")	$R < 0$	C in front of surface

4 Summary

- Optical components form images of objects. They deflect light so that to an observer tracing back the path the light is taking entering the eye, the light seems to come from the image. Images may be real or virtual, depending on whether the light rays actually pass through the image point of not.

- Images can be magnified by a ratio M, which is considered positive if transverse dimensions are not inverted.

- The object distance p is the distance the object is away from the optical element, and the image distance q is the distance of the image from the same point. But each of these may be considered negative if they are not on the "natural" side of the optical element.

- $M = -q/p$.

- The rays from the object, after deflection, appear to come from the image, but that is only true, in general, for paraxial rays, which make small angles with the principal axis. For larger angles spherical aberration describes the fact that the rays do not come from precisely the point of the image they ought to.

- For spherical mirrors, $\dfrac{1}{q} + \dfrac{1}{p} = \dfrac{2}{R}$.

- For spherical interfaces between transparent media, $\dfrac{n_1}{p} + \dfrac{n_2}{q} = \dfrac{n_2 - n_1}{R}$.

- q, p, R, f, and M can be positive or negative, with sign conventions given in the tables, one for mirrors and one for interfaces.