L26. The Mathematics of Electromagnetism

In this course we have focussed on the physics of the electromagnetic field, and for reasons of expedience, we have deliberately relegated the mathematics to the back seat. In closing I'd like to say a few words about some of the math we have "skipped over".

We have seen Maxwell's equations in their "integral form". In a more advanced treatment it becomes equally important to rewrite these equations in their differential form. It is this formulation that most readily exposes the wave nature of electromagnetic waves.

The first two equations are Gauss' equations.

The quantities $\nabla . E$ & $\nabla . B$ are called the "divergence" of the electric & magnetic field respectively. The "divergence" is the outward flux per unit volume. If we normalize the first Gauss' equation per unit volume we get

$$\frac{1}{dV} \int \vec{E} . d\vec{A} = \frac{1}{\epsilon_0} \left(\frac{dQ_{encl}}{dV} \right)$$

For a tiny volume V $dQ_{encl}/dV = \rho$ is the density of charge, whereas $\frac{1}{dV} \int \vec{E} . dA = \nabla . E$ is

I'd like to give you a flavor of the differential form of Maxwell's equations. We write them as follows

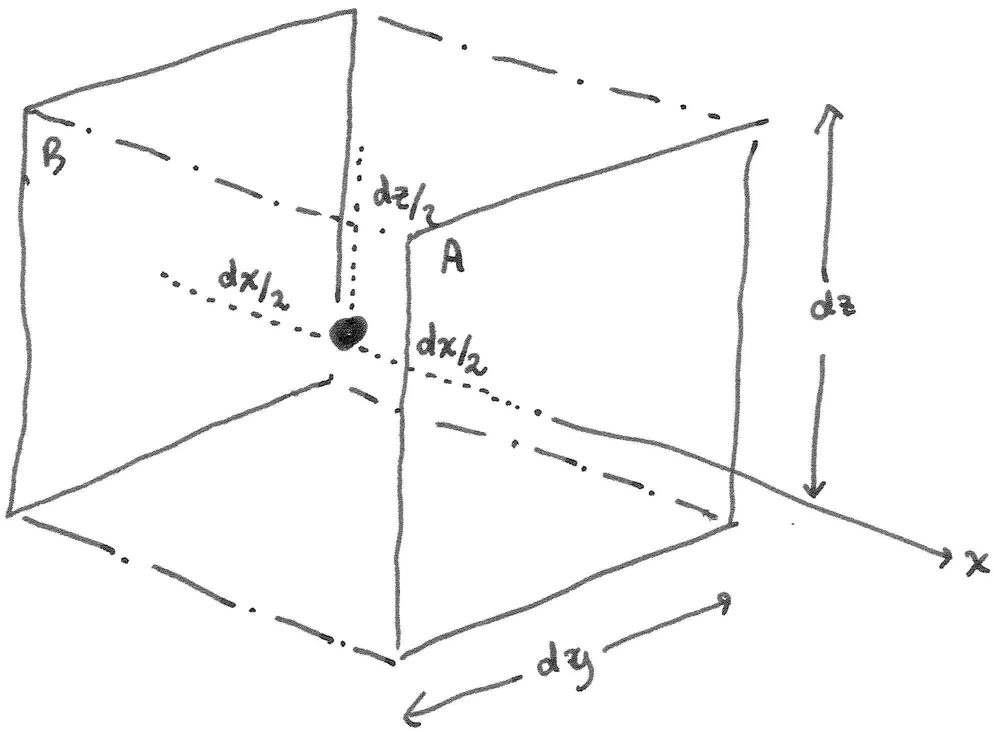
$$\int \vec{E}.d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \longrightarrow \nabla . E = \rho / \epsilon_0$$

$$\int \vec{B}.d\vec{A} = 0 \longrightarrow \nabla . B = 0$$

$$\oint \vec{E}.d\vec{\ell} = -\frac{d\Phi_B}{dt} \longrightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint \vec{B}.d\vec{\ell} = \mu_0 \left(I_{encl} + \epsilon_0 \frac{d\phi_E}{dt} \right) \longrightarrow \nabla \times B = \mu_0 \left(\vec{j} + \frac{\partial \vec{E}}{\partial t} \epsilon_0 \right)$$

the divergence. To see this consider a small volume



The electric flux out of the "x" faces is

$$\Phi_x = \Phi_A + \Phi_B = dy\,dz\left(E_x\left(x+\frac{dx}{2},\dots\right) - E_x\left(x-\frac{dx}{2},\dots\right)\right)$$

$$= dx\,dy\,dz\left(\frac{E_x\left(x+\frac{dx}{2},y,z\right) - E\left(x-\frac{dx}{2},y,z\right)}{dx}\right)$$

$$= dV\left(\frac{\partial E_x}{\partial x}\right)$$

The total flux is then

$$\phi_x + \phi_y + \phi_z = dV \overbrace{\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)}^{\text{FLUX PER UNIT VOLUME.}}$$

The quantity

$$\nabla . E = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \left(\frac{\text{E Flux out}}{\text{unit volume}} \right)$$

is the divergence of the electric field. Written out longhand the Gauss' equations look scary

$$\left.\begin{aligned} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) &= \rho/\epsilon_0 \\ \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) &= 0 \end{aligned}\right\} \text{Gauss' equations}$$

But you should not be scared — these equations simply tell us the amount of flux per unit volume

The last two Maxwell's equations tell us about the "circulation" of the electric & magnetic fields about changing magnetic fields. The quantity $\vec{\nabla} \times \vec{E}$ is the circulation per unit area, and it is called the "curl" of $\vec{E}$.

If we take the circulation of E around a small rectangular loop we find that

$$\int \vec{E} \cdot d\vec{l} = dA_x \cdot (\nabla \times E)_x + dA_y \cdot (\nabla \times E)_y + dA_z \cdot (\nabla \times E)_z$$

The expressions for $(\nabla \times E)_{x,y,z}$ are even more scary

$$(\nabla \times E)_x = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)$$

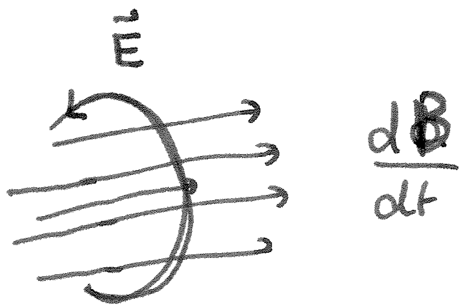
$$(\nabla \times E_y) = \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)$$

$$(\nabla \times E_z) = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

but all you need to internalize is that each of these measures the x, y & z components of circulation per unit area. Faradays law becomes

$$\underbrace{\frac{\text{circulation of } E}{\text{unit area}}}_{\vec{\nabla} \times \vec{E}} = -\frac{1}{\text{area}} \frac{d\Phi_B}{\partial t} = -\frac{\partial B}{\partial t}$$

where the minus sign tells us that the E-field circulates anticlockwise around the increasing flux



Likewise, Amperes law becomes

$$\underbrace{\frac{\text{Circulation of } B}{\text{area}}}_{\nabla \times B} = \mu_0 \left(\frac{I}{\text{area}} + \epsilon_0 \frac{1}{\text{area}} \frac{\partial \Phi_E}{\partial t} \right)$$

$$\nabla \times B = \mu_0 \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right)$$

current density

Together with Darwin's theory of evolution,

Maxwell's four equations are the greatest

triumph's of Victorian — 19th century science, and

they provide the pillars on which 20th century

science & 21st century technology & science

are being constructed.

We'll see next semester how Maxwell's equations

led onto the discovery of space-time & relativity, and

to the discovery of the quantum nature of matter.

For the moment however — I would like you to rest

content – for you have all learnt, what only
a tiny fraction of our world will ever know –
the inner workings of the theory of radiation,
electricity & light – and that's no mean feat.