# L23 LCR Circuits

Last lecture, we examined the response of resistors capacitors & inductors to an a.c. voltage. Whereas the voltage & current through a resistor are <u>in phase</u> & given by

$$V_R = IR \qquad \phi_R = 0$$

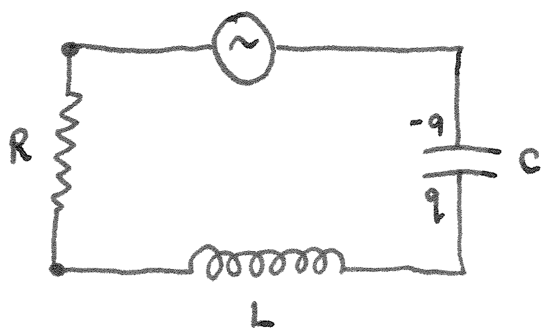
the voltage & current through an inductor are $90°$ out-of phase

$$V_L = IX_L \qquad \phi_L = 90°$$

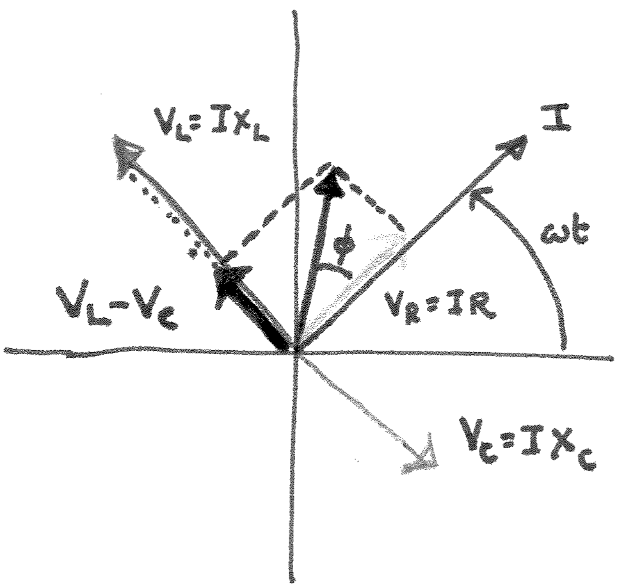
where $X_L = \omega L$ is the "reactance", whereas for a capacitor

$$V_c = IX_c \qquad \phi_c = -90°$$

& $X_c = 1/\omega C$. Now we wish to examine the behavior of a series combination of all three – the LCR circuit.




To represent the voltages through each element we will use "phasors". Lets assume $X_L > X_C$.




The voltage contains two parts, an "in phase" part $V_R = IR$ & a reactive part $V_L - V_c = I(X_L - X_c)$.

The resultant has magnitude

$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$

$$= \sqrt{(IR)^2 + (IX_L - IX_C)^2}$$

or

$$V = I\sqrt{R^2 + (X_L - X_C)^2}$$

We define

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

as the impedence of the L-R-C circuit, so that

$$\boxed{V = IZ}$$

Written out in full

$$Z = \sqrt{R^2 + (\omega L - (1/\omega C))^2}$$

Now the voltage leads the current with a phase angle $\phi$ where

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{I(X_L - X_C)}{IR} = \frac{X_L - X_C}{R}$$

or

$$\tan\phi = \frac{\omega L - 1/\omega C}{R}$$

The full expression for the current is then

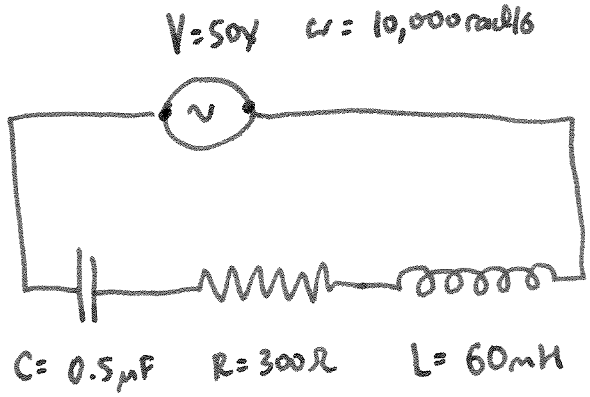
$$v = V\cos(\omega t + \phi)$$

Note that

$$V_{rms} = \frac{V}{\sqrt{2}} = \frac{I Z}{\sqrt{2}} = i_{rms} Z$$

This describes the <u>steady-state</u>. When the circuit is first connected there will in general be "transient" responses which die out to reveal the above steady state oscillation.

e.g




$$X_C = \frac{1}{10^4 \times 0.5 \times 10^{-6}} = 200\,\Omega$$

$$X_L = 10^4 \times 60 \times 10^{-3} = 600\,\Omega$$

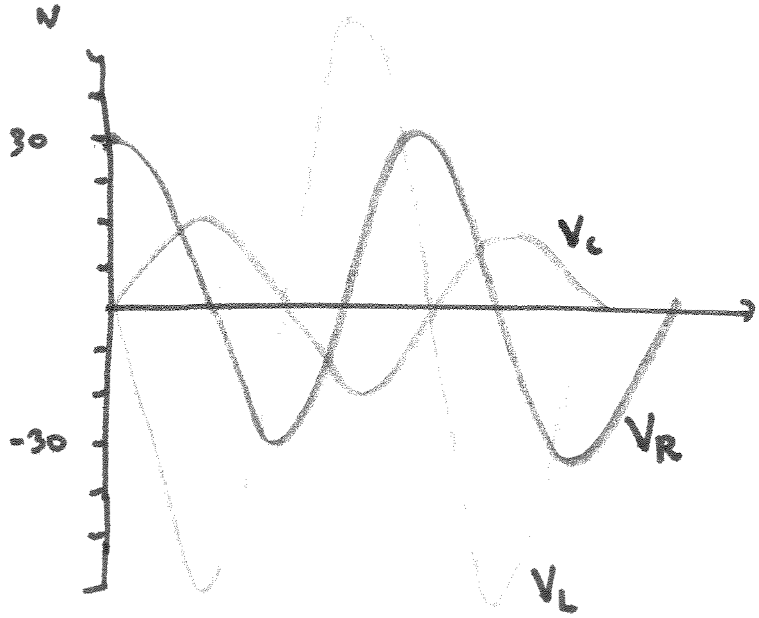
$$Z = \sqrt{(300)^2 + (600-200)^2} = 500\,\Omega$$

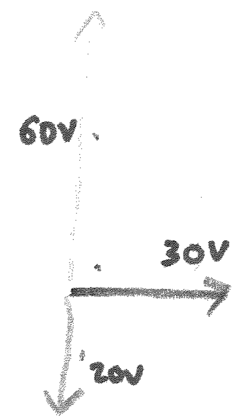
$$I = \frac{V}{Z} = \frac{50V}{500\Omega} = 0.1\,A$$

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{400\Omega}{300\Omega} = \cancel{0.8}\ 1.25$$

$$\phi = 53^\circ$$

$V_R = IR = 30V$ $V_L = IX_L = 60V$ $V_C = IX_C = 20V$


V
30
-30
$V_C$
$V_R$
$V_L$


60V
30V
20V

# 31.4 Power in A.C circuits

$$p = vi$$

Pure resistance $\quad v = V\cos\omega t \;, \quad i = I\cos\omega t$

$$p = VI\cos^2\omega t$$

$$P_{av} = \frac{VI}{2} = V_{rms} I_{rms} .$$

$$P_{av} = I_{rms}^2 R = V_{rms}^2 / R .$$

More generally

$$vi = \left[V\cos(\omega t + \phi)\right] I\cos\omega t$$

$$= \left[V \cos\omega t \cos\phi - \sin\omega t \sin\phi\right] I\cos\omega t$$

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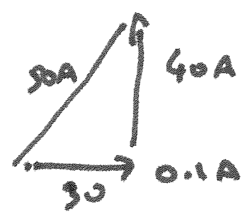
$$vi = VI\cos\phi\,\underbrace{\cos^2\omega t}_{\text{average}=\frac{1}{2}} - VI\sin\phi\,\underbrace{\cos\omega t\,\sin\omega t}_{=\frac{1}{2}\sin 2\omega t}$$

average $= 0$.

$$P_{av} = \frac{1}{2} VI\cos\phi$$

$$\boxed{P_{av} = V_{rms}\, i_{rms} \cos\phi}$$

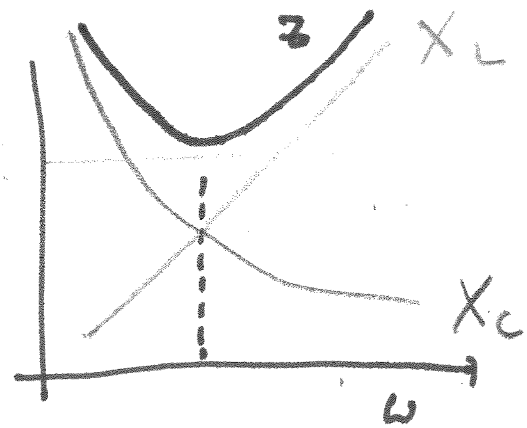


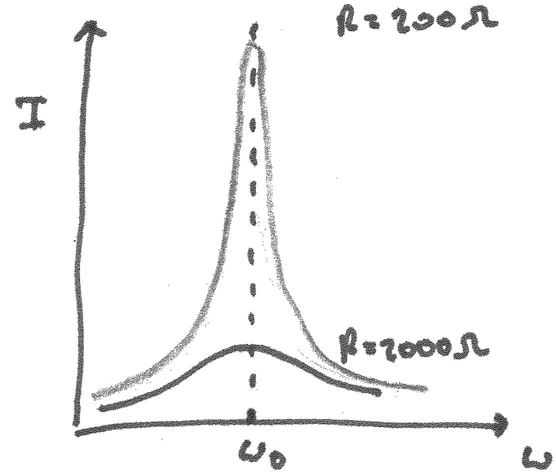

e.g. earlier example. $I = 0.1A$ $V = 50V$. $\cos\phi = 3/5$.

$$P_{av} = \frac{1}{2}\, 50 \times 0.1 \times \frac{3}{5} = \underline{\underline{1.5W}}$$

# 31.5 Resonance

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$




Z is minimum when $X_L = X_C$, $\omega = \omega_0$

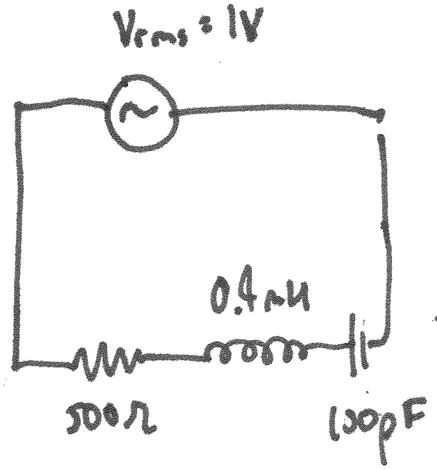
$$\omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

This is the natural oscillation frequency.

$I = V / Z(\omega)$

23.11

e.g




a) $\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{0.4\times10^{-3}\times 100\times10^{-12}}$

$= 5\times10^{6}$ rad/s.

$f = \omega_0/2\pi = 8\times10^{5} = 800$ kHz

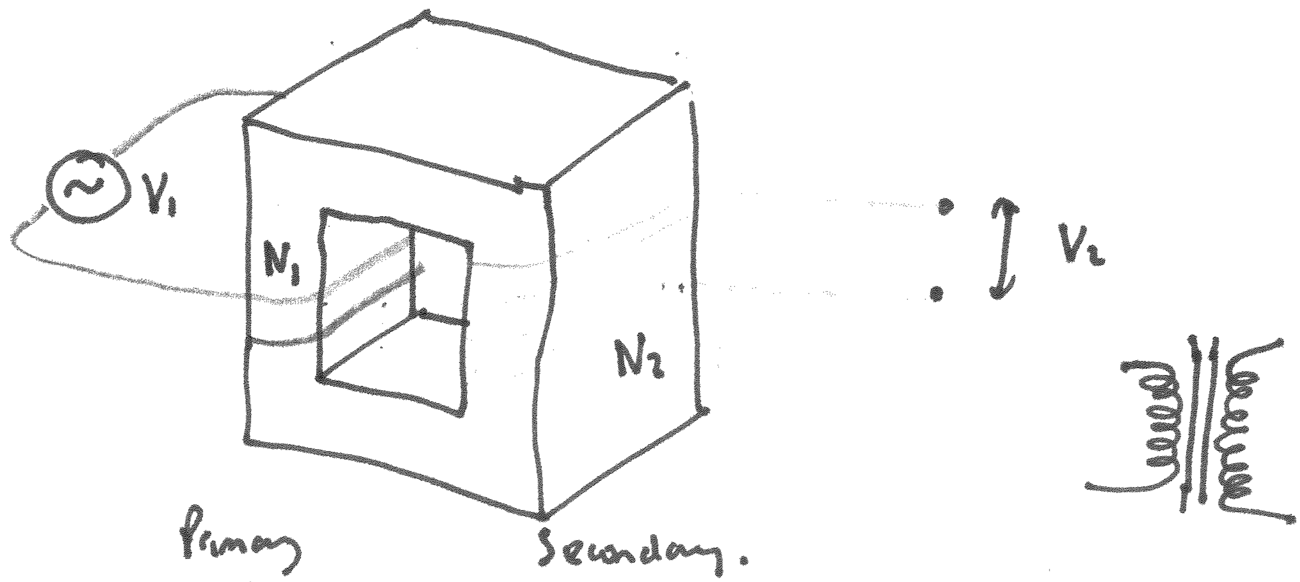
b) $X_C = \frac{1}{\omega_0(100\times10^{-12})} = \frac{1}{5\times10^{6}\times10^{-10}} = 2000\,\Omega$

$X_L = 5\times10^{6}\times 0.4\times10^{-3} = 2000\,\Omega$.

$Z = R = 500\,\Omega$.

c) $V_{rms} = 1V$ $\quad I_{rms} = \frac{V_{rms}}{Z} = \frac{1V}{500\,\Omega} \equiv \underline{2mA}$

## 31.6 Transformers




$$\mathcal{E}_1 = -N_1 \frac{d\phi_B}{dt} \qquad \mathcal{E}_2 = -N_2 \frac{d\phi_B}{dt}$$

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1} \Rightarrow \boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}}$$

$$V_1 I_1 = V_2 I_2$$ (energy conservation).